PHY411. PROBLEM SET 4

November 13, 2023

Relevant python notebooks are available here `https://astro.pas.rochester.edu/~aquillen/phy411/lectures.html` Specifically cobweb.ipynb, logistic.ipynb, lyap.ipynb

1. On Pitch-Fork Bifurcations

Consider the dynamical system

$$\dot{x} = rx - 2\sinh x$$

Show that a pitchfork bifurcation occurs and find the value of r and x at which it occurs. Classify the bifurcation as super-critical or sub-critical.

2. On Classifying Bifurcations

In each case find the value of r and x^* at which a bifurcation occurs and classify the type of bifurcation. Draw a bifurcation diagram.

(a) $\dot{x} = 4 - re^{-x^2}$

(b) $\dot{x} = rx - \frac{x}{1+x^2}$

(c) $\dot{x} = rx - \frac{3x}{1+x}$

This problem is much easier to do if you plot the function for different values of r and look at where it crosses the x axis.

3. On Cobweb Plots

See cobweb.ipynb for code examples showing cobweb plots.

Consider the map

$$x_{n+1} = \mu \sin(\pi x_n)$$

that has a similar shape to the logistic map.

A cobweb plot that shows a period two orbit is shown in Figure 1.

(a) Explore what the cobweb plot looks like for smaller values of μ than 0.81.

(b) Increase μ until you find a value where the two cycle becomes unstable and you can see a period four orbit in the cobweb plot.

(c) Increase μ further until you see a chaotic orbit.

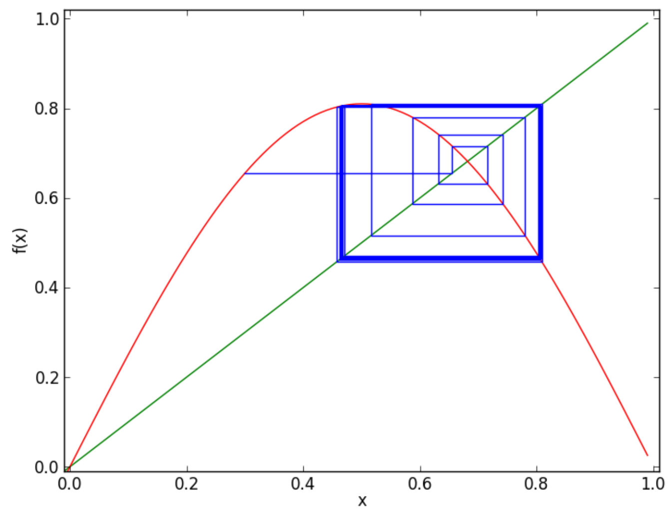

FIGURE 1. Graphical analysis of $x_{n+1} = \mu \sin(\pi x_n)$ with $\mu = 0.83$. The orbit is a two-cyle. The green line shows $y = x$, the red-line $f(x) = \mu \cos(\pi x)$.

4. The logistic map, period doubling and self-similarity

The logistic map,

$$f_\mu(x) = \mu x(1 - x)$$

You can modify the python code "logistic.ipynb' to make plots of the attracting orbits as a function of μ. See:

http://astro.pas.rochester.edu/~aquillen/phy411/pylab/logistic.py

Consider the properties of $f^3_\mu(x)$ that is $f(f(f(x)))$ for a given μ value.

(a) For which parameter values of μ is there an attracting prime period 3 point for f_μ^3? Illustrate the range graphically.

(b) Find a box where the attracting points of $f_\mu^3(x)$ resembles that of $f_s(x)$ for some s. By box I mean a range of xy values on a plot of μ (in the x-axis) vs $f_s(x)$ or $f_\mu^3(x)$ where these are on the y axis.

(c) Identify a region (in μ or s) where there is period doubling in one of these maps.

5. Conjugacy between the Logistic and Tent maps and computing a Lyapunov Exponent

The logistic map (for $0 \le x_n \le 1$) is

$$x_{n+1} = r x_n (1 - x_n)$$

The tent map (for $0 \le y_n \le 1$) is

$$y_{n+1} = \begin{cases} R y_n & 0 \le y_n \le \frac{1}{2} \\ R(1 - y_n) & \frac{1}{2} \le y_n \le 1 \end{cases}$$

In this problem we relate the logistic map with $r = 4$ to the tent map with $R = 2$.

Consider the transformation

$$x_n = \sin^2\left(\frac{\pi y_n}{2}\right)$$

that takes $0 \le y_n \le 1$ to $0 \le x_n \le 1$.

(a) Show that the logistic map with $r = 4$ transforms into the tent map with $R = 2$.

(b) Compute the Lyapunov exponent of the tent map with $R = 2$.

(c) Using part b, find the Lyapunov exponent of the logistic map with $r = 4$.

6. Numerical Computation of a Lyapunov exponent

Consider the map

$$x_{n+1} = r \sin(\pi x_n)$$

that is similar to the logistic map.

Show the periodic attracting orbits and compute the Lyapunov exponent as a function of r.

You can modify the routines in the notebook lyap.ipynb

7. The perturbed cat map

a) Use the generating function

$$S(q_{n+1}, q_n) = q_{n+1}^2 - q_n q_{n+1} + q_n^2 + \frac{\kappa}{(2\pi)^2}\sin(2\pi q_n)$$

to find an area preserving map $p_n, q_n \to p_{n+1}, q_{n+1}$ with p_n, q_n mod 1. For κ sufficiently small ($\kappa \lesssim 0.33$), this map, known as the *perturbed cat map*, does not have any stable islands and fills the torus (unit interval with periodic boundary conditions on both sides).

b) Choose a value for κ less than 0.33. What is the Lyapunov exponent of the map? This can be estimated numerically or analytically.

8. A chaotic map on the circle

Delaney's definition of a chaotic map is the following: Let D be a metric space. The function $f : D \to D$ is *chaotic* if

(a) The periodic orbits of F are dense in D.

(b) The function f is topologically transitive. In other words f mixes the set really well.

(c) The function f exhibits extreme sensitivity to initial conditions.

Recall that a function $f : D \to D$ is *topologically transitive* if for all open sets U, V in D there is an x in U and a natural number n such that $f^n(x)$ is in V.

Consider the map on the circle $\theta \in [0, 1]$ with

$$g(\theta) = 2\theta \mod 1$$

a) Show that the map $g(\theta)$ is chaotic. (In particular show a,b, you need not show c).

b) Consider the map on the complex plane $h(z) = z^2$. Show that for $|z| = 1$, the map $h(z)$ is chaotic.

9. On Newton's method - Stable sets

Given a function $f(x)$, the mapping

$$N_f(x) = x - \frac{f(x)}{f'(x)}$$

can give an efficient way to iteratively find a root of f from a starting initial x that lies near the root. This is known as Newton's method. However, N_f does not always converge to the nearest root and N_f can also have chaotic orbits. The function N_f is particularly badly behaved if $f'(x)$ has roots.

Consider the function

$$f(x) = \frac{1}{x} - 1$$

that has a single root at $x = 1$.

(a) What is the map $N_f(x)$?

(b) What are the fixed points for N_f?

(c) To which point (or ∞ or $-\infty$) does Newton's method converge for $x \in (0, 1)$?

(d) To which point (or ∞ or $-\infty$) does Newton's method converge for $x \in (1, 2)$?

(e) To which point (or ∞ or $-\infty$) does Newton's method converge for $x < 0$?

(f) To which point (or ∞ or $-\infty$) does Newton's method converge for $x > 2$?

(g) How does N_f behave for $x = 0, 1, 2$?

(h) Over what region does Newton's map converge to the root of f?

Hint: It helps to construct cobweb plots for N_f.

10. On Topological conjugacy for a map on the complex plane

Consider the map from the complex plane to the complex plane

$$g(z) = az + b$$

where a, b are nonzero complex numbers (and $a \neq 1$).

a) Show that $g(z)$ is topologically conjugate to

$$f(z) = az$$

In other words find an invertible map $h(z)$ such that

$$f(h(z)) = h(g(z))$$

Hint: try $h(z) = \alpha z + \beta$ and find α, β.

When two maps are topologically conjugate, then periodic orbits in one map correspond to periodic orbits in the other map.

b) Find the fixed point for the map and its basin of attraction.

11. Some complex quadratic maps with chaotic orbits

The map $f(z) = z^2$ with $z \in \mathbb{Z}$ is chaotic on the unit circle. Consider a topological conjugacy map $h(z) = \alpha z + \beta$.

a) Find the inverse map h^{-1}.

b) Compute the map $g(z) = h^{-1}(f(h(z)))$.

c) Find a set of quadratic maps on the complex plane that contain chaotic orbits. In other words for $g(z) = az^2 + bz + c$, find conditions on a, b, c that allow $g(z)$ to exhibit chaotic orbits.

12. Box dimension of the Baker map's attractor

The Baker map

$$(x_{n+1}, y_{n+1}) = \begin{cases} (cx_n, 2y_n) & \text{for} \quad y_n \leq 1/2 \\ (1 + c(x_n - 1), 1 + 2(y_n - 1)) & \text{for} \quad y_n > 1/2 \end{cases} \tag{1}$$

For $c < 1/2$ the map is not area preserving and there is an attractor.

Minkowski dimension or box-counting dimension is a way of measuring the fractal dimension of a set. Suppose that $N(\epsilon)$ is the number of boxes of side length ϵ required to cover the set.

The box dimension is:

$$\dim_{box} = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$$

Compute the box dimension for the attractor of the Baker's map with $c = 1/3$.

If the limit for the box dimension does not exist, one may still take the limit superior and limit inferior, which respectively define the upper box dimension and lower box dimension. The upper box dimension is sometimes called the entropy dimension, Kolmogorov dimension, Kolmogorov capacity, limit capacity or upper Minkowski dimension, while the lower box dimension is also called the lower Minkowski dimension. The upper and lower box dimensions are strongly related to the more popular Hausdorff dimension.

13. Choose your own problem to work on!