PHY411. PROBLEM SET 3

October 17, 2023

1. Drifting the standard map

Consider a modification of the standard map

$$\begin{aligned} p_{n+1} &= p_n + K \sin \theta_n + b \\ \theta_{n+1} &= \theta_n + p_n + K \sin \theta_n \end{aligned} \tag{1}$$

The parameter b can be considered a drift rate.

Note: You can consider this map or you can consider the map

$$\begin{aligned} p_{n+1} &= p_n + K \sin \theta_n + b \\ \theta_{n+1} &= \theta_n + p_n + K \sin \theta_n + b \end{aligned} \tag{2}$$

These two maps are the same if you substitute $p'_n = p_n + b$ (which is a canonical transformation).

a) Is this map area preserving?

b) Find a time dependent Hamiltonian model that creates this map when mapping points to points in phase space (p, θ) at equal time intervals.

Hint: You can use a Dirac comb $\sum_n \delta(t-n)$ and Hamiltonian in the form $H(p, \theta, t) = p^2/2 + K \cos \theta \sum_n \delta(t-n) + f(\theta)\delta(t-n)$ for some function $f(\theta)$. Depending upon whether you apply the extra kick at the beginning or the end of the time interval interval gives the different forms of the map.

c) Find a stationary Lagrangian model for this map. This is equivalent to finding a generating function $F(\theta_n, \theta_{n+1})$ that satisfies

$$\frac{\partial F}{\partial \theta_n} = -p_n \qquad \frac{\partial F}{\partial \theta_{n+1}} = p_{n+1}$$

Hint: $F(\theta_n, \theta_{n+1}) = \frac{1}{2}(\theta_n - \theta_{n+1})^2 - K \cos \theta_n + ?$ The unknown would depend upon $b\theta_n$ and $b\theta_{n+1}$.

d) Consider a sequence of angles $\theta_0, \theta_1, ...\theta_n$. The sum of the generating function

$$S = \sum_{i=0}^{n-1} F(\theta_i, \theta_{i+1})$$

is like an action. Here F is the generating function found in the previous problem. Show that this sum is minimized on orbits of the map. In other words show that $\frac{\partial S}{\partial \theta_i} = 0$ if the sequence of angles is part of an orbit of the map.

e) Is the map in equation 2 a twist map? A twist map should satisfy $\left.\frac{\partial \theta_{n+1}}{\partial p}\right|_{\theta_n, p_n} \neq 0$. If the derivative is always above for some positive ϵ then the twist map is monotone. Is the map in equation 2 a monotone twist map?

f) Consider the map

$$\begin{aligned} p_{n+1} &= p_n + K \sin\theta_n \\ \theta_{n+1} &= \theta_n + p_n + K \sin\theta_n + b \end{aligned} \tag{3}$$

which looks similar to the map in equation 2. Show that this map becomes the standard map via the canonical transformation $p' = p + b$ with θ unchanged.

2. Drifting the standard map - numerically

Again consider the modification of the standard map discussed in the previous problem.

$$\begin{aligned} p_{n+1} &= p_n + K \sin\theta_n + b \\ \theta_{n+1} &= \theta_n + p_n + K \sin\theta_n \end{aligned} \tag{4}$$

Compare two maps with same K value but one has $b = 0$ and the other one has a small b.

a) Numerically plot some orbits and show that orbits with rational winding numbers are not necessarily destroyed by the drift.

To understand why this is true, consider the map $H(p, q) = \frac{p^2}{2} + \frac{q^2}{2} + bq$. This is a harmonic oscillator with an extra term. Hamilton's equation gives $-\dot{p} = \frac{\partial H}{\partial q} = q + b$, so b causes a drift in momentum. The transformation $Q = q + b$ (and preserving p) is canonical and gives $H(p, Q) = \frac{p^2}{2} + \frac{Q^2}{2} +$constant. The harmonic oscillator is recovered but with origin shifted.

b) What happens if you increase b?

c) Are KAM tori in the standard map destroyed by a small drift? KAM tori are 1 dimensional orbits that extend from $\theta = 0$ to 2π.

d) Try plotting orbits after shifting $p_i \to p_i - ib$ for each iteration i of the map. This corrects for the drift in momentum caused by the parameter b.

I find that if K is not large, then KAM tori look like they are destroyed in the map. However, if I plot them after shifting the momentum, they look chaotic but are still fairly coherent. I find two types of orbits: 1) Orbits that are associated with periodic orbits (and they have rational winding numbers). These don't contain all possible θ values. These orbits look like lines (or tori) in the regular map and they look like a mess when you shift them to correct for the drift. 2) Orbits that cover all possible θ values. These look like a mess in the regular map but look like they connect a few different separatrices and are remarkably coherent in the shifted map.

e) Do you think that the drift causes chaotic regions to expand? Are there orbits with rational winding numbers in the standard map that are destroyed with the drift?

3. On Poisson bracket's of conserved quantities

Given Hamiltonian $H(p, q)$, and functions f, g that satisfy $\{f, H\} = \{g, H\} = 0$. Here $\{\}$ is the Poisson bracket. The quantities f, g are conserved by the flow generated by the Hamiltonian. Show that $h = \{f, g\}$ is also a conserved quantity.

Hint: use the Jacoby identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

4. Conserved Quantities; the Runge-Lenz Vector

The Hamiltonian for the Kepler system is

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2} - \frac{GM}{r}$$

where $\mathbf{p}$ is momentum, $\mathbf{L}$ is angular momentum per unit mass, and $\mathbf{r}$ is the coordinate of a massless particle in motion about a massive body of mass M.

The Runge-Lenz vector is

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - GM\hat{\mathbf{r}}$$

a) Using a Poisson bracket, show that $\mathbf{A}$ is a conserved quantity.

b) Show that $A^2 = (GM)^2 + 2EL^2$.

(Hint: Use vector identities and recall that p^2 can be subdivided into a sum of components parallel to r and parallel to L.)

Remarks

The seven scalar quantities $E, \mathbf{L}, \mathbf{A}$ actually only contribute 5 independent conserved quantities. Because $\mathbf{L} \cdot \mathbf{A} = 0$ there are only 5 degrees of freedom among $\mathbf{L}, \mathbf{A}$. Furthermore the length of $\mathbf{A}$ depends on the energy and angular momentum (as shown above) reducing the number of conserved quantities to 5.

An orbit is specified by six initial conditions (three positions and three velocities). For Hamiltonian system in six dimensional phase space, if we can find 3 independent conserved quantities, then the system is said to be *integrable* and the Hamiltonian can be transformed to $H(\mathbf{p})$. This gives 3 conserved components of p. As H is determined by these three conserved quantities it is not an additional conserved quantity. The three associated angles (or coordinates) are not conserved but advance in time with

$$\dot{\mathbf{q}} = \boldsymbol{\omega} = \boldsymbol{\nabla} H(\mathbf{p})$$

For the Kepler problem, we have 5 conserved quantities which is two extra than needed to make the system integrable. This means that we can find a coordinate system such that $H(\Lambda)$ only depends on a single momentum. In this coordinate system two associated coordinates or angles are conserved, as well as three momenta. The system is said to be 'super-integrable'.

The conservation of $\mathbf{L}$ implies that the Lie group of symmetries contains SO(3). Including the Runge-Lenz vector, the Lie algebra is equivalent to that of SO(4). The additional symmetry is related to a continuous transformation that maps an orbit to one with a different angular momentum while preserving energy. The Lie algebra is the Poisson algebra obeyed by the Poisson brackets and is comprised of infinitesimal transformations. The Lie group is the entire continuous symmetry group. It is possible to have different Lie groups that have the same algebra for their infinitesimal transformations.

5. Periodically forced pendulum

We will integrate the equations of motion for the time dependent system,

$$H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos\phi - \mu \cos(\phi - \nu t) \tag{5}$$

to make a map g^T from phase space ϕ, p to itself at a time T later

$$\phi(t), p(t) \xrightarrow{g^T} \phi(t+T), p(t+T)$$

We will use

$$T = 2\pi/\nu$$

the period of the time dependent perturbation in the Hamiltonian, to create the map.

Starting with an initial condition $\mathbf{x}_0 = (p_0, \phi_0)$ we can iteratively apply this map. The orbit of an initial condition is the collection of points

$$g^T(\mathbf{x_0}), g^{2T}(\mathbf{x_0}), g^{3T}(\mathbf{x_0})....$$

The orbits of the map are curves, points or area filling regions.

Our map is a function of parameters ν, ϵ, μ. The Hamiltonian has two resonances, one at $p = 0$ and the other at $p = \nu$. The two resonances are separated by a frequency ν. The peaks of the separatrices for one resonance are at $p = \pm 2\sqrt{|\epsilon|}$ and the other at $p = \pm 2\sqrt{|\mu|}$. Thus the resonances are *overlapped* if

$$2\sqrt{|\epsilon|} + 2\sqrt{|\mu|} > \nu \tag{6}$$

There is a sample python code linked here `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/perio.ipynb` or `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/perio.html` illustrating how to to use the python routine `odeint` to integrate the Hamiltonian, make the map and apply it iteratively.

By using different initial conditions (shown in different colors) a picture of the types of different orbits can be constructed; see Figure 1

a) Chose μ, ϵ, ν so that the resonances do not overlap (check condition 18). Integrate the map and plot some orbits. When the resonances do not overlap, show that most initial conditions give orbits that are curves.

b) Chose μ, ϵ, ν so that the resonances do overlap. Integrate the map and plot some orbits. When the resonances overlap, show that some initial conditions

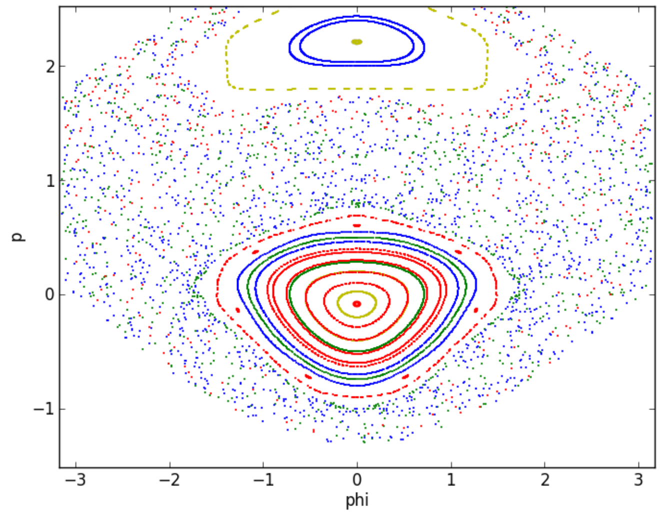

FIGURE 1. Mapping at a period for the Hamiltonian in equation 17 with $\nu = 2, \epsilon = 0.4, \mu = 0.15$.

give orbits that are area filling. These orbits are likely to be in regions near the separatrix of one of the resonances.

c) Explore different parameters to see if you can find periodic orbits (three resonant islands corresponding to period three orbits). If you look closely at Figure 1 there are some period 7 orbits at the edge of the lower stable resonant island.

d) We have three free parameters describing the map. Show that one of them can be removed by rescaling time and p.

Hint: multiplying the Hamiltonian by a constant is equivalent to shifting to a new time that is equal to the old time multiplied by a constant.

e) Create a map with only a single perturbation (set $\mu = 0$) and show that you see something like a pendulum.

f) What do you expect for the map with $\epsilon = 0$? Verify numerically that the map is what you expect.

g) Explore how the width (in p) of the area filling region depends on the values of $\nu/\sqrt{\epsilon}$ and μ/ϵ.

h) You can shift the phase of the second cosine term with a phase ϕ_0

$$H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos\phi - \mu \cos(\phi - \nu t + \phi_0) \tag{7}$$

but this is equivalent to starting your series of integrations with a different value for t_0 and still plotting every $T = 2\pi/\nu$. How does your map change and is this significant? (Think about slices in the 3D space).

6. On the restricted 3-body problem

The dynamics of a massless particle in orbit near two massive bodies is given by Hamiltonian

$$H(x, y; p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2}} - \frac{\mu}{\sqrt{(x + (1-\mu))^2 + y^2}} - (xp_y - yp_x)$$

where we have restricted the dynamics to the plane containing the two massive bodies and we have assumed that the two massive bodies are circular orbits about the center of mass. The above Hamiltonian is in the center of mass coordinate system that is rotating with the two massive bodies, so they remain in fixed positions. Here $\mu \equiv \frac{m_2}{m_1+m_2}$ where m_1, m_2 are the masses of the two bodies. We are working in units of distance between m_1 and m_2 and in units of time and mass so that $G(m_1 + m_2) = 1$. The above Hamiltonian is independent of time so it is conserved. The conserved quantity is known as the Jacobi integral of motion which we call E_J.

In polar coordinates the system is equivalent to

$$H(r, \theta; p_r, L) = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - L - \frac{1-\mu}{\sqrt{\mu^2 + r^2 - 2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2 + r^2 + 2(1-\mu)r\cos\theta}}$$

a) Suppose that we take into account the z degree of freedom. What is the Hamiltonian $H(x, y, z; p_x, p_y, p_z)$?

b) Consider an orbit with $z = 0$ that is nearly circular with radius r_c. What is its value of L? What is its value of the Jacobi integral E_J((the value of H)? In other words what is $L(r_c)$ and $E_J(r_c)$? You can assume that μ is very small to do this estimate so that the second potential term can be neglected and the first can be approximated as $-1/r$.

Hint: To solve for $L(r_c)$ use Hamilton's equation $\dot{p}_r = \frac{\partial H}{\partial r} = 0$. To check your answer, consider how angular momentum depends on radius for circular orbits in a Keplerian system.

If you are making a surface of section, it useful to know where to expect nearly circular orbits. You can choose E_J so that you will see nearly circular orbits near a desired radius.

When μ is not small, then one can manipulate the average value of the potential using a function $g(\alpha)$

$$g(\alpha) = \int_0^{2\pi} \frac{d\theta}{\sqrt{1-\alpha^2\cos\theta}} = 4K\left(\frac{2\alpha^2}{\alpha^2+1}\right)(1+\alpha^2)^{-1/2} \tag{8}$$

where the complete elliptic integral of the first kind $K(k)$ is defined as

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

For the nearly circular orbit case we can take r to be constant and replace

$$V(r,\theta) = -\frac{1-\mu}{\sqrt{\mu^2+r^2-2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2+r^2+2(1-\mu)r\cos\theta}}$$

with

$$\begin{aligned} V_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[-\frac{1-\mu}{\sqrt{\mu^2+r^2-2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2+r^2+2(1-\mu)r\cos\theta}}\right] \\ &= -\frac{(1-\mu)}{\sqrt{\mu^2+r^2}}g\left(\frac{2r\mu}{\mu^2+r^2}\right) - \frac{\mu}{\sqrt{(1-\mu)^2+r^2}}g\left(\frac{2r(\mu-1)}{(1-\mu)^2+r^2}\right) \end{aligned}$$

7. On Periodic Orbits

We have a 3-dimensional system in cylindrical coordinates θ, r, z.

Consider orbits that are nearly circular with mean radius r_c and are described by three angles: The azimuthal angle θ, radial oscillation angle θ_r and vertical oscillation angle θ_z. A setting would be a disk galaxy. Starts orbit the galaxy but also oscillate radially about the mean radius and vertically about the galactic midplane.

The radius and vertical height are related to action variables J_r and J_z with

$$r = r_c + \sqrt{2J_r/\kappa}\cos\theta_r$$
$$z = \sqrt{2J_z/\nu}\cos\theta_z$$

where the epicyclic frequency is $\kappa = \dot\theta_r$ and the vertical oscillation frequency is $\nu = \dot\theta_z$.

The quantity $\sqrt{2J_r/\kappa}$ is sometimes called the epicyclic amplitude. The quantity $\sqrt{2J_z/nu}$ is sometimes called the vertical epicylic amplitude.

A Hamiltonian for this system is $H(r, z, \theta; J_r, J_z, L) = L\Omega + \kappa J_r + \nu J_z$ where $\Omega = \dot{\theta}$ is the angular rotation rate. The frequencies Ω, κ, ν depend upon the mean radius of the orbit r_c.

a) Draw a closed or periodic orbit with $\phi = \theta_r - 2\theta = 0$. Assume that the epicyclic amplitude is not zero. Put the orbit in the xy plane (corresponding to $J_z = 0$). What does look orbit look like in the xy plane?

b) Draw a periodic orbit with $\phi' = \theta_z - 2\theta = 0$. Assume that the vertical amplitude is small but not zero but J_r can be zero. What does look orbit look like in the xy plane, xz plane and in the yz plane?

c) Draw a periodic orbit with $\phi = \theta_r - 2\theta = \pi$ and a small epicyclic amplitude.

d) Draw a periodic orbit with a small vertical amplitude and with $\phi' = \theta_z - 2\theta = \pi$.

e) Draw a periodic orbit with $\phi = \theta_r + 4\theta = 0$ and small epicyclic amplitude.

Which of these types of orbits is called *banana* shaped?

8. On Creating a Map for a separable Hamiltonian with the Dirac Comb

Suppose we have a separable Hamiltonian

$$H(p, q) = P(p) + Q(q)$$

The equations of motion

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{\partial Q}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{\partial P}{\partial p} \end{aligned}$$

Over a small time τ

$$\Delta p = \dot{p}\tau = -\frac{\partial Q}{\partial q}\tau$$

We can approximate the equations of motion using the Hamiltonian

$$H(p, q, t) = P(p) + Q(q)\tau D_\tau(t) \tag{9}$$

where D_τ is the Dirac comb or

$$D_\tau = \sum_{n=-\infty}^{\infty} \delta(t + n\tau)$$

Denote p_n, q_n as p, q at times $t = n\tau$.

a) Create a map for p_{n+1} and q_{n+1} as a function of p_n, q_n that is consistent with the equations of motion for the approximate Hamiltonian in equation 21.

b) Show that your map is area preserving. (Hint: compute the Jacobian and show that its determinant is 1.)

Hint: Because H is independent of q between delta functions, p is conserved and only changes at times $t = n\tau$. The Hamiltonian implies that $\dot{q}$ is constant at times other than $t = n\tau$.

9. Location of mean motion resonances

Jupiter has a semi-major axis of $a_J = 5.2$ AU. Jupiter's orbital period $P_J = (a_J)^{\frac{3}{2}}$ and is in years if computed with a_J in AU.

Consider an asteroid with orbital period P and semi-major axis a. The asteroid is in the 3:2 mean motion resonance with Jupiter; $3P \approx 2P_J$. What is its semi-major axis in AU?

Another object is also in a 3:2 resonance but $2P \approx 3P_J$. What is its semi-major axis in AU?

10. On Area Preserving maps

Consider the area preserving map

$$\begin{aligned} x_{n+1} &= x_n + aI_n^\gamma \\ I_{n+1} &= I_n + \sin x_{n+1} \end{aligned} \tag{10}$$

with $x \in [0, 2\pi]$.

See `http://astro.pas.rochester.edu/~aquillen/phy411/pylab/Keplermap.ipynb`

or

`http://astro.pas.rochester.edu/~aquillen/phy411/pylab/Keplermap.html`

for sample python code.

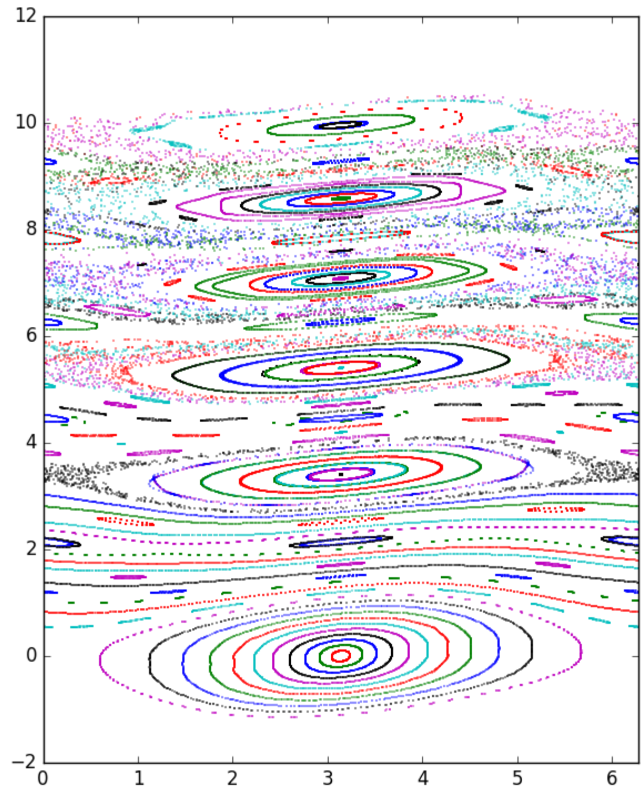

FIGURE 2. Orbits for the Kepler map (equations 26) with $K = 0.3$, $\gamma = 1.5$.

a) Show that the map is area preserving. (Hint: compute the Jacobian and show that its determinant is 1.)

b) Show that with $W_n = a^{\frac{1}{\gamma}} I_n$ that the map can be written in the form

$$\begin{aligned} x_{n+1} &= x_n + W_n^{\gamma} \\ W_{n+1} &= W_n + K \sin x_{n+1} \end{aligned} \tag{11}$$

and find K.

c) For what value of γ is this map equivalent to the standard map (Chirikov's map), written like this

$$\begin{aligned} \theta_{n+1} &= \theta_n + p_n \\ p_{n+1} &= p_n + K \sin \theta_{n+1} \end{aligned} \tag{12}$$

for $p, \theta \in [0, 2\pi)$?

d) The standard map is often written as this

$$\begin{aligned} p_{n+1} &= p_n + K \sin \theta_n \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned} \tag{13}$$

for $p, \theta \in [0, 2\pi)$. Is this essentially the same map as the map in equations 24?

e) Choose a value for γ and compute some orbits numerically for the map in equation 23. If $\gamma \neq 1$ then the above map might be ill defined for negative W. One approach is to use the map in equation 26.

$$\begin{aligned} x_{n+1} &= x_n + \text{sign}(y_n)|y_n|^\gamma \\ y_{n+1} &= y_n + K \sin x_{n+1} \end{aligned} \tag{14}$$

and with $x \in [0, 2\pi)$ but y not constrained.

See if you can identify different types of orbits.

f) The Kepler map with $\gamma = 1.5$ has lots of resonances (see Figure 2). Explain why the resonant islands get smaller with larger p. For a given K value at what y do the resonances overlap?

11. An area preserving map for the kicked harmonic oscillator

Consider the Hamiltonian

$$H(p, q, t) = \frac{p^2}{2} + \frac{q^2}{2} + f(q) \sum_n \delta(t - nT)$$

Find an area preserving map that maps p, q to p, q at a period T.

Hint: First find p_{n+1} and q_{n+1} in terms of p_n, q_n for the harmonic oscillator alone. This will involve cosines and sines. Then apply the change to p caused by the kick. Equivalently you can reverse the order of these two area preserving transformations.

12. Exploring drifting Hamiltonian systems

Choose a time dependent Hamiltonian system. Numerically integrate it to understand how it behaves as it drifts. Plot level curves at different values of the time dependent parameters to try and understand how phase space evolves.

A possible system to explore might be the drifting double well potential

$$H(q,p) = p^2/2 + aq^4 - bq^2$$

with $a(t)$ or/and $b(t)$ slowly varying functions of time.

Another possible system to explore would be the Andoyer Hamiltonian that is a model for second order mean motion resonances.

$$H(p,\phi) = a(t)p^2/2 + b(t)p + \epsilon(t)p\cos(2\phi),$$

with $a(t), b(t)$ and $\epsilon(t)$ slowly varying.

An example of a drifting Hamiltonian system is illustrated in `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/drift_integrate.ipynb` or `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/drift_integrate.html`

Another possible system to explore would be time dependent version of the periodically forced pendulum defined in equation 17.

$$H(p,\phi,t) = \frac{p^2}{2} + bp - \epsilon\cos\phi - \mu\cos(\phi - \nu t) \tag{15}$$

You can allow $\epsilon(t), b(t)$ and $\mu(t)$ to be time dependent. With ϵ and μ fixed but b drifting, you could measure how long it takes a particle to jump across the resonance. In the chaotic system, the particle must spend time in the chaotic region which can delay it from jumping across resonance.

Alternatively you could explore the behavior of a slowly varying version of an area preserving map such as the Kepler map.

13. Create and work on your own problems

Some possible ideas: Explore the Fermi map. Explore diffusive-like behavior in the standard map when K is large. Explore what happens to different types of orbits for drifting 2D are preserving maps. Add some dissipation to a 2D map and see how its properties change.