PHY411. PROBLEM SET 14.

December 25, 2023

1. The billiard problem with a circular boundary

Consider a 2D flat billiard problem, where the boundary is a closed loop. A particle bounces elastically off the boundary giving a map

$$T : (s, \mu) \to (s, \mu)$$

with $s \in [0, 1]$ along the boundary and with s periodic. Here $\mu \in [-1, 1]$ and $\mu = \cos\theta$ where θ is the angle between particle trajectory and tangent to the boundary at an impact point.

The angle of a trajectory has $\theta = 0$ if the particle is grazing the boundary and moving in the direction of increasing s. We assume that $T(s, 1) = T(s, 1)$ and $T(s, -1) = T(s, -1)$ are fixed points for all s.

One definition of **integrability** is that there exists a function f (piecewise continuous) $f : (s, \mu) \to \mathbb{R}$ such that each set $S_c : \{s, \mu\}$ defined by $f(s, \mu) = c$ is either a union of points or lines.

Consider a billiard problem with a **circular** boundary.

a) Find some periodic orbits.

Hint: look at the triangle that connects two boundary contact points and the center of the circle.

A *periodic* orbit is the set $\{T^i(x)\}$ for $i = 0, ..., n - 1$ where $T^n(x) = x$. Here n is a positive integer (usually $n > 1$) and x is a point that is pair of numbers (s, μ). Here we take *quasiperiodic* to mean an orbit that never repeats, though we could also include the concept of needing two incommensurable frequencies to describe the orbit.

b) Find some quasiperiodic orbits.

c) Show that the billiard problem with circular boundary is integrable.

2. On the classical D-Billiard

The D-Billiard is a 2d billiard system with rink that is a truncated circle, as shown in Figure 1.

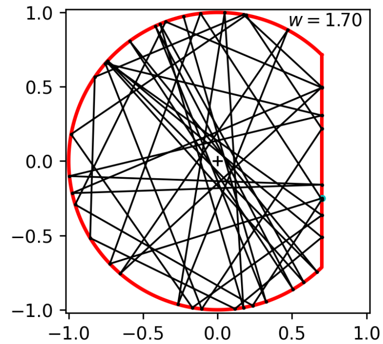

FIGURE 1. A D-Billiard system. The rink consists of a segment connected to an arc. Shown is an example of a chaotic orbit. The horizontal width of the rink is w times the radius of the circle.

Discuss whether the following statements are likely to be true or false for the D-billiard.

a) An orbit that never bounces off the segment must be periodic.

b) An orbit that bounces off the segment is either periodic or chaotic.

c) A chaotic orbit eventually crosses all points within the rink.

d) A chaotic orbit eventually bounces off all positions on the rink boundary.

e) A chaotic orbit eventually bounces off all positions on the rink boundary at all possible trajectory angles.

If you would like to gain intuition by integrating some orbits, example code is in the python notebook Ripple_and_D_Billiard.ipynb which is available on this page https://astro.pas.rochester.edu/~aquillen/phy411/lectures.html

3. On the Wigner function of a cat coherent state

We denote $|\alpha\rangle$ as a coherent state; $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ where $\hat{a}$ is the ladder operator and α is a complex number.

Consider a superposition state

$$|v\rangle = c(|\alpha\rangle + |-\alpha\rangle)$$

This is sometimes called a 'cat' state.

a) Find the normalization factor c so that $|\psi\rangle = c\,|v\rangle$ gives $\langle\psi|\psi\rangle = 1$. Note, coherent states are not orthogonal to one another.

For coherent states $|\alpha\rangle\,, |\beta\rangle$,

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta} \tag{1}$$

b) What is $\langle\hat{q}\rangle$ of the cat state?

Hint: use the fact that $\hat{q} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$ and that $\langle\alpha|\,\hat{q}\,|\alpha\rangle = \sqrt{2}\,\mathrm{Re}\{\alpha\}$.

c) What is $\langle\hat{p}\rangle$ of the cat state?

Hint: use the fact that $\hat{p} = \frac{1}{\sqrt{2}}\frac{\hbar}{i}\left(\hat{a} - \hat{a}^\dagger\right)$ and $\langle\alpha|\,\hat{p}\,|\alpha\rangle = \hbar\sqrt{2}\,\mathrm{Im}\{\alpha\}$.

Consider a mixed state described with density matrix

$$\hat{\rho}_{mix} = \frac{1}{2}\left(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|\right)$$

d) Compute the Wigner function of this mixed state, $W_{\hat{\rho}_{mix}}(q,p)$.

Hint: the Wigner function for a coherent state is

$$W_{|\alpha\rangle\langle\alpha|}(q,p) = \frac{1}{\pi\hbar} e^{-(q-\langle\hat{q}\rangle)^2} e^{-(p-\langle\hat{p}\rangle)^2/\hbar^2} \tag{2}$$

where $\langle\hat{q}\rangle = \sqrt{2}\,\mathrm{Re}\{\alpha\}$ and $\langle\hat{p}\rangle = \sqrt{2}\,\mathrm{Im}\{\alpha\}\hbar$. The resulting Wigner function should resemble a sum of Gaussians.

Consider the density matrix $\hat{\rho}_{cat} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{2}\left(|\alpha\rangle + |-\alpha\rangle\right)$.

e) Is the Wigner function $W_{\hat{\rho}_{cat}}$, near zero near the origin in phase space?

To answer this question you could compute the Wigner function for $|\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|$, which might take a while. Alternatively $\langle\hat{q}^2\rangle$ computed for the two different density operators might suggest that one of these two cases has a higher probability of being near the origin in phase space. Note that $\langle\alpha|\,\hat{q}^2\,|\alpha\rangle = (\sqrt{2}\,\mathrm{Re}\{\alpha\})^2 + \frac{1}{2}$.

4. Point operator and Weyl transformation

The Wigner function takes an operator and produces a function in phase space. The Weyl transformation takes a function of phase space and produces an operator.

The Wigner function of a state $|\psi\rangle$ can be written in terms of a point operator

$$\hat{A}(q,p) = \frac{1}{2\pi}\int dx e^{-ixp/\hbar} \left|q + x/2\right\rangle \left\langle q - x/2\right| \tag{3}$$

via

$$W_{|\psi\rangle\langle\psi|}(q,p) = \langle\psi| \hat{A}(q,p) |\psi\rangle . \tag{4}$$

(Here I am working in 1 dimensional coordinate space).

The Weyl transformation of a function in phase space $f(q,p)$ is the operator

$$\hat{\Phi}_f = \int\int da\ db\ dq\ dp\ e^{-i(aq+bp)} e^{ia\hat{Q}+ib\hat{P}} f(q,p) \tag{5}$$

where $\hat{Q}, \hat{P}$ are position and momentum operators, respectively.

Show that the Weyl transformation of function $f(q,p)$ can be written in terms of the point operator

$$\hat{\Phi}_f = \int dq\ dp\ f(q,p)\hat{A}(q,p) \tag{6}$$

It is handy to know that, if operators $\hat{A}$, $\hat{B}$ commute with their commutator, $[A,B]$, then

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

This is the Baker-Campbell-Hausdorff formula.

5. The displacement operator for coherent states

The displacement operator for coherent states $D_\alpha = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ where α is a complex number.

Show that $D_\alpha = e^{\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}$

Hint: Use the Baker-Campbell-Hausdorf formula.

6. Point operator for a discrete system

Consider a finite dimensional Hilbert space with basis $|n\rangle$ and $n \in \mathbb{Z}_N$. We create a basis using a discrete Fourier transform via $|k\rangle_F = \frac{1}{\sqrt{N}} \sum_n \omega^{kn} |n\rangle$ where $\omega = e^{2\pi i/N}$ is a complex root of unity.

We construct operators that look like Pauli operators

$$\hat{X} = \sum_{n=0}^{N-1} |n+1\rangle \langle n| \tag{7}$$

$$\hat{Z} = \sum_{n=0}^{N-1} \omega^n |n\rangle \langle n| \tag{8}$$

In the above expression for $\hat{X}$, addition for $|n+1\rangle$ is modulo N.

a) Find expressions for $\hat{X}, \hat{Z}$ in the $|k\rangle_F$ basis.

Hints: use identity $\mathbf{I} = \sum_k |k\rangle_F \langle k|_F$, the relations $\langle n|k\rangle_F = \frac{\omega^{nk}}{\sqrt{N}}$, ${}_F\langle k|n\rangle = \frac{\omega^{-nk}}{\sqrt{N}}$ and $\sum_j \omega^{jk} = N\delta_{k0}$.

We create a point operator

$$\hat{A}_{nk} = \frac{1}{2\sqrt{N}} \omega^{nk/2} \sum_{x \in \mathbb{Z}_N} \omega^{-xk} |n-x\rangle \langle n| \tag{9}$$

where n is a position index, and k is a momentum index.

b) Transfer the point operator into the Fourier basis. It should look similar to the point operator in the $|n\rangle$ basis!

c) Write the point operator in terms of $\hat{Z}, \hat{X}$ operators.

Suppose you have a state vector $|\tilde{\eta}\rangle$ that is an eigenfunction of the discrete Fourier transform. It satisfies $\hat{Q}_{FT} |\tilde{\eta}\rangle = |\tilde{\eta}\rangle$.

d) Show that $|\tilde{\eta}\rangle$ is also an eigenfunction of $Q_{FT}^\dagger$.

$\hat{X}$ is not Hermitian but $\hat{X} + \hat{X}^\dagger$ is Hermitian and so is $i(\hat{X} - \hat{X}^\dagger)$.

Let a variance $\sigma(\hat{f}) = \left\langle \hat{f}^2 \right\rangle - \left\langle \hat{f} \right\rangle^2$.

e) Show that $\sigma(\hat{X}+\hat{X}^\dagger) = \sigma(\hat{Z}+\hat{Z}^\dagger)$ and that $\sigma(\hat{X}-\hat{X}^\dagger) = \sigma(\hat{Z}-\hat{Z}^\dagger)$ for the state $|\tilde{\eta}\rangle$.

7. On the quantized kicked rotator

Sample example code is available with the python notebook quantum_kicked_rotor.ipynb which is available on this page https://astro.pas.rochester.edu/~aquillen/phy411/lectures.html

a) Numerically construct a propagator for the quantized kicked rotator which is directly related to the standard map.

I recommend you use approximate with a discrete Hilbert space. With dimension N, the angle operator and momentum operators are

$$\hat{\theta} = \sum_j \frac{2\pi j}{N} |j\rangle \langle j|$$
$$\hat{p} = \sum_m \hbar m \, |m\rangle_F \langle m|_F$$

where $|m\rangle_F$ is the Fourier basis.

The Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2I} + k \cos\hat{\theta} \sum_{j=-\infty}^{\infty} \delta(t - jT) \tag{10}$$

The propagator across time T is

$$\hat{U} = e^{-i\hat{H}T/\hbar} = Q_{FT}^\dagger \Lambda_A Q_{FT} \Lambda_B \tag{11}$$

where the diagonal matrices

$$\Lambda_A = \sum_j e^{-\frac{ik}{\hbar}\cos(2\pi j/N)} |j\rangle \langle j|$$
$$\Lambda_B = \sum_m e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} |m\rangle)F \langle m|_F . \tag{12}$$

and the discrete Fourier transform $\hat{Q}_{FT} = \sum_{jk} \omega^{jk} |j\rangle \langle k|$ and $\omega = e^{2\pi i/N}$.

Here we define two dimensionless parameters

$$\alpha \equiv \frac{T\hbar}{I} \qquad K \equiv \frac{kT}{I}$$

Note $K/\alpha = k/\hbar$ where K is the equivalent parameter in the standard map.

b) Compute the eigenvalues for the propagator. Compute the distribution of differences between phases. Adjust K and α and see if you can see a Wigner distribution for high K (corresponding to chaotic) and a Poisson distribution for low K (corresponding to nearly integrable).

c) Are there any obvious differences between the eigenstates for the two cases?

d) Display some eigenstates in phase space.

In part d there is more than one possible choice on how to do this. For the quantized kicked rotator, I computed the Wigner function with this

$$W_{\hat{\rho}}(n,k) = \frac{1}{2N} \sum_x \langle n-x| \, \hat{\rho} \, |n+x\rangle \, \omega^{2xk} \tag{13}$$

8. Propose and work on your own problem!