

PHY411. PROBLEM SET 1.

September 14, 2023

There are sample python notebooks on the class website that show how to plot streamlines of a function and numerically integrate a bunch of orbits to illustrate phase wrapping. <https://astro.pas.rochester.edu/~aquillen/phy411/lectures.html>

1. The harmonic oscillator. The period is independent of amplitude

Consider the harmonic oscillator with Hamiltonian

$$H(p, q) = \frac{1}{2} (p^2 + q^2)$$

Show that the area $S(E)$ on the phase plane, (p, q) , of an orbit with energy E , is proportional to E and so the period of oscillations for the harmonic oscillator is independent of energy or oscillation amplitude.

2. Dimensional analysis on the pendulum.

Consider a Hamiltonian for the pendulum with coefficients a, ϵ

$$H(p, \phi) = a \frac{p^2}{2} - \epsilon \cos \phi \tag{1}$$

Let us take H in units of energy per unit mass and ϕ as an angle. With this convention H has unit of velocity² and p has units of velocity times distance.

- (a) Find all the fixed points of the Hamiltonian system
- (b) Consider the level curves of the system. Which fixed points are stable and which ones lie on a separatrix?
- (c) How do your previous answers change if ϵ is negative instead of positive?
- (d) How do your previous answers change if a is negative instead of positive?
- (e) In a physical system what are the units of a and ϵ ?
- (f) Construct a frequency using dimensional analysis and a, ϵ .

- (g) What is the frequency of libration about a stable fixed point?
- (h) Consider an initial condition that is very close to the separatrix. The orbit will move away from the separatrix exponentially fast. Compute the exponential timescale for the motion.
- (i) What is the value of energy for an orbit in the separatrix? What is the value of energy for an orbit near a stable fixed point?
- (j) Using a, ϵ construct a quantity with units of momentum, p .
- (k) What is the maximum p value in the separatrix?
- (l) What is the frequency of oscillation ($\dot{\phi}$) for $p \gg \sqrt{|\epsilon/a|}$?
- (m) Construct a map rescaling both time and momentum. For τ a unit less time variable and P a unit less momentum that are related to t, p by two coefficients t_r and p_r :

$$\begin{aligned}\tau &= t/t_r \\ P &= p/p_r\end{aligned}$$

find t_r, p_r such that the Hamiltonian in equation 1 becomes

$$H(P, \phi) = \frac{P^2}{2} + \cos \phi$$

3. Rescaling time and other transformations

- (a) Consider a Hamiltonian $H(p, q)$. How does the Hamiltonian change if t is rescaled by b to $\tau = bt$? This is relevant to a numerical technique known as *regularization* where the Hamiltonian is multiplied by a function (usually dependent on coordinates) that rescales time.
- (b) Show that $t \rightarrow -t$, $p \rightarrow q$ and $q \rightarrow p$ preserves Hamilton's equations.
- (c) Show that $t \rightarrow -t$ and $p \rightarrow -p$ preserves Hamilton's equations. This is time reversal.

4. Plotting level curves for an Andoyer Hamiltonian

A Hamiltonian that describes first order mean motion resonances in Celestial mechanics and Lindblad resonances in galactic dynamics is an Andoyer Hamiltonian

$$H(J, \phi) = J^2 + \delta J + J^{1/2} \cos \phi \tag{2}$$

Here J is an action momentum variable and ϕ an angle. It is customary to plot level curves for this Hamiltonian in a coordinate system (x, y)

$$x = \sqrt{2J} \cos \phi \quad y = \sqrt{2J} \sin \phi$$

so that radius on the plot gives larger J values and angle on the plot corresponds to ϕ . The coordinate transformation is canonical so Hamilton's equations describe the equation of motion in the new coordinate system.

- (a) Transfer the Hamiltonian into coordinates (x, y) showing that the Hamiltonian looks like

$$H(x, y) = \frac{1}{4}(x^2 + y^2)^2 + \frac{\delta}{2}(x^2 + y^2) + \frac{1}{\sqrt{2}}x \quad (3)$$

- (b) Plot the level curves as a function of x, y for different values of δ including positive and negative ones. Illustrate that there are either 1 or 3 fixed points.
 (c) Classify the fixed points as stable or unstable based on the phase curves.
 (d) How do the Hamiltonian level curves change if you flip the sign of the cosine term in the Hamiltonian?
 (e) Using Hamilton's equations for x, y find a cubic equation with roots giving the x values for the fixed points.
 (f) Explain why this system either has 1 or 3 fixed points. Using the cubic equation, find what δ values give three fixed points rather than just one.
 (g) Draw a bifurcation diagram for the fixed points.

5. Double well potential

Consider the following Hamiltonian

$$H(q, p) = \frac{p^2}{2} + V(x)$$

with quadratic potential function

$$V(x) = -2Bx^2 + x^4$$

With $B > 0$ the potential has two potential wells.

a) Orbits can remain in one potential well with $x > 0$ or $x < 0$ at all later times. What condition on the energy determines whether a particle remains confined to one of the potential wells?

b) Find the fixed points for the system and determine whether they are stable or hyperbolic.

- c) Plot level curves of the Hamiltonian in phase space (x, p) . What does the separatrix look like?
- d) What is the period of orbits that librate about one of the stable fixed points?
- e) What are orbits near the origin like?

6. Pendulum on a rotating axis

Consider a system with force $f(x) = -\sin(x) + M$;

$$\ddot{x} = -\sin x + M$$

- (a) What is a Hamiltonian description for this system?
- (b) Why is it called a pendulum on a rotating axis?
- (c) Plot level curves
- (d) Plot streamlines
- (e) How do the streamlines, level curves and number of fixed points depend on the value of M ?

Now add dissipation

$$\ddot{x} = \sin x + M - \alpha \dot{x}$$

$\alpha > 0$.

- (f) Again, find the fixed points
- (g) Taking into account energy loss with time, draw streamlines.
- (h) Discuss the appearance of a periodic orbit in the case $M > 1$
(this system is much discussed and illustrated by Strogatz in his book on Non-linear Dynamics and Chaos).

7. On the Legendre Transform

Find the Legendre Transform of

$$f(x) = x^3$$

Where is this function convex and allows a Legendre transform?

8. Not Time Crystals

Consider a separable Lagrangian

$$\mathcal{L}(q, \dot{q}) = T(\dot{q}) + V(q) \tag{4}$$

that does not depend on time. Because it does not depend on time, the associated Hamiltonian (energy) should be conserved.

A function $f(x)$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex if for $\forall x_1, x_2 \in \mathbb{R}$ and $\forall t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

A twice differentiable function of one variable is convex on an interval if and only if its second derivative is non-negative in the interval.

Taking a Legendre transform ($\dot{q} \rightarrow p$) The associated Hamiltonian

$$H(p, q) = p\dot{q}_*(p) - \mathcal{L}(q, \dot{q}_*(p))$$

where $\dot{q}_*(p)$ is a solution to this condition

$$p = \frac{\partial \mathcal{L}(\dot{q}_*)}{\partial \dot{q}}. \quad (5)$$

We assume that this function can be inverted and that we can find $\dot{q}_*(p)$.

(a) Equations 4 and 5 imply that $p = T'(\dot{q}_*)$. Show that

$$\frac{d\dot{q}_*(p)}{dp} = \frac{1}{T''(\dot{q}_*(p))}$$

(b) An extremum energy must occur at fixed points where

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial q} = 0.$$

Show that $\dot{q}_*(p) = 0$ at a fixed point.

(c) The second derivatives at a minimum must be positive. Show that the requirement

$$\frac{\partial^2 H}{\partial^2 p} > 0$$

implies that $T''(\dot{q}_*(p)) > 0$ at the fixed point. This means that the kinetic energy function must be convex at the minimum energy.

Most of these results are violated if the function for $\dot{q}_*$ cannot be inverted.

See *Classical Time Crystals*, Shapere, A. Wilczek F. 2012, Phys. Rev. Lett. 109, 160402

9. Invariance of equations of motion

- (a) Consider related Lagrangians $L(\dot{q}, q, t)$ and $\tilde{L}(\dot{q}, q, t)$ with

$$\begin{aligned}\tilde{L}(\dot{q}, q, t) &= L(\dot{q}, q, t) + \frac{d}{dt} [G(q, t)] \\ &= L(\dot{q}, q, t) + \dot{q} \frac{\partial}{\partial q} G(q, t) + \frac{\partial G}{\partial t},\end{aligned}\tag{6}$$

where $G(q, t)$ is a smooth function. The actions $S, \tilde{S}$ on a curve $q(t)$

$$\begin{aligned}S(q(t)) &= \int_{t_a}^{t_b} dt L(\dot{q}, q, t) \\ \tilde{S}(q(t)) &= \int_{t_a}^{t_b} dt \tilde{L}(\dot{q}, q, t).\end{aligned}\tag{7}$$

Show that the difference action $S - \tilde{S}$ over a path starting at q_a at $t = t_a$ and going to q_b at $t = t_b$ is only a function of q_a, q_b, t_a, t_b .

This means that the variations of the path integral are identical, $\partial S = \partial \tilde{S}$ and so that the equations of motion for L and $\tilde{L}$ are identical.

10. On dissipation

Consider a dissipation function $\mathcal{F}(\dot{\mathbf{q}})$ that will help us represent a velocity dependent force. Modify Lagrange's equation so that they become

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = - \frac{\partial \mathcal{F}}{\partial \dot{\mathbf{q}}}.\tag{8}$$

- (a) For a Lagrangian in the form $L = \frac{\dot{q}^2}{2} + V(q)$, show that Lagrange's equations in the form of equation 8 (with a dissipation function) are consistent with a damping force per unit mass $-\frac{\partial \mathcal{F}}{\partial \dot{\mathbf{q}}}$.
- (b) Defining energy as

$$E = \dot{\mathbf{q}} \frac{\partial L}{\partial \dot{\mathbf{q}}} - L$$

show that for a time independent Lagrangian (giving equation of motion consistent with equation 8)

$$\frac{dE}{dt} = - \dot{\mathbf{q}} \frac{\partial \mathcal{F}}{\partial \dot{\mathbf{q}}}\tag{9}$$

This gives work due to dissipation.

- (c) A damped harmonic oscillator is described by

$$m\ddot{x} + b\dot{q} + kq = 0\tag{10}$$

This has a damping force and energy is not conserved.

What dissipation function $\mathcal{F}$ and Lagrangian gives the equations of motion?

- (d) We can derive the equations of motion without a dissipation function and with a time dependent Lagrangian or Hamiltonian.

Show that the Euler-Lagrange equations yield the equation of motion (equation 10) with a time dependent Lagrangian

$$L(q, \dot{q}, t) = e^{bt/m} \left(\frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \right). \quad (11)$$

- (e) Do a Legendre transformation and show that the Hamiltonian corresponding to this Lagrangian is

$$H(p, q, t) = e^{-\frac{bt}{m}} \frac{p^2}{2m} + e^{\frac{bt}{m}} \frac{k}{2} q^2 \quad (12)$$

- (f) Show that Hamilton's equations are consistent with equation 10.
 (g) Hamilton's equations implies that $\frac{\partial H}{\partial t} = \frac{dH}{dt}$. What is $\frac{dH}{dt}$ for the Hamiltonian of equation 12? Is it equal to the dissipation rate with energy dissipation rate $\dot{E} = -b\dot{q}^2$ consistent with the equations of motion?

The second half of this problem is by K. S. Lam.

11. On phase wrapping

Consider the Hamiltonian

$$H(I, \theta) = \omega I + bI^2$$

in action angle variables where ω and b are constants. The action I is a conserved quantity because the Hamiltonian is independent of θ . One of Hamiltonian's equations is

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega + 2bI. \quad (13)$$

Suppose two particles are initially both at angle $\theta = 0$ and initially have action variables I_1 and I_2 .

How long does it take before the first particle is delayed or advanced in its orbit by $\Delta\theta = 2\pi$ compared to the second particle?

12. A numerical illustration of phase wrapping

For the pendulum, the libration period depends upon amplitude. The pendulum Hamiltonian

$$H(p, \phi) = \frac{p^2}{2} - \epsilon \cos \phi \quad (14)$$

Assign a bunch of particles initial conditions with $\phi \sim 0$, and momentum $p \sim p_0 \pm \Delta p$. This distribution is initially confined in a small region in phase space.

Show that after some integration time, the distribution shears out in phase space.

Example code showing phase wrapping for an anharmonic oscillator is available in a python notebook on the class website.

13. The phase space distribution function

The number of stars in the Milky Way Galaxy per unit volume in phase space can be described with a distribution function

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x}^3 d\mathbf{v}^3$$

- (a) Ignoring the birth and death rate of stars and encounters between them, explain how conservation of volume in phase space is consistent with

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (15)$$

where $\Phi(\mathbf{x})$ is the gravitational potential. This equation is known as the collisionless Boltzmann equation.

- (b) Suppose there is a frame moving with velocity $\mathbf{V}$ in which the velocity distribution is constant (independent of time). The distribution function can be written as

$$f(\mathbf{x} - \mathbf{V}t, \mathbf{v}).$$

The velocity $\mathbf{V}$ can be called a pattern speed. Compute the collisionless Boltzmann equation for this distribution function. What constraint can be inferred on Φ and f ?

- (c) With million of stars with precise distances and space motions from the GAIA DR3 release, it is now possible to compute spatial gradients of the velocity distribution function. Consider a velocity distribution in a small region (like the solar neighborhood) of the Galaxy. Explain why $\frac{\partial f}{\partial \mathbf{v}} = 0$ at a peak in the velocity distribution.

- (d) Assuming a constant pattern speed, and that we do not know the exact form of the gravitational potential Φ , explain how the pattern speed $\mathbf{V}$ might be measured from spatial gradients of the phase space distribution function.

Note: This is related to the Weinberg-Tremaine method for measuring a pattern speed of a bar or spiral arm. I attempted to do this with the GAIA DR2 data and it failed miserably. Apparently the assumption of a constant pattern speed is a bad one for the velocity distribution in the solar neighborhood.

14. Propose and work on your own problem!