PHY411 Lecture notes Part 6

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Contents

0.1 Introduction

The material here is taken from a variety of sources. Newton's map section following Holmgren's book. Circle map stuff from some reviews (H. Bruin, Physica D, 1992, and a review by Predrag Cvitanovic). The list of rational/irrational number discussion is currently like a laundry list and not yet nicely related to the dynamical problems. It would help to add something on the circle map done with perturbation theory (as by Rajeev)

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to illustrate some KAM-like ideas and connect the rational/irrational number discussion to the dynamics. I had not yet added any complex number dynamics but Milnor's book on the topic is nicely accessible. The set of topics here is nicely set up so that I can in future add a section on KAM theory for maps of the circle (following original paper by Arnold) that includes a holonomic type of equation (that will look familiar in the context of near-integrable Hamiltonian KAM theory). I another set of notes I looked a some nice invertible maps of the circle in an attempt to try to converge to a topological conjugacy without doing Fourier expansion.

The exploration of the circle map is interesting because in some regimes, the map is either like a rotation or has periodic orbits depending upon whether its *winding number* is rational or irrational. The circle map provides an analogy to weakly perturbed nearly integrable Hamiltonian systems. A perturbed Hamiltonian system can be transformed via a series of canonical transformations to resemble an integrable system if the unperturbed system is sufficiently far away from resonances. This is the focus of Kolmogorov's theorem and part of a body of work known as KAM theory. The resonances in the perturbed Hamiltonian setting are played by the role of rational winding numbers for the circle map. Instead of using canonical transformations to transform the Hamiltonian system, perturbation techniques on the circle map can be used to find a topological conjugacy to a rotation map. A procedure giving the map (giving the topological conjugacy) will only converge if the winding number is sufficiently irrational. Likewise the procedure giving the canonical transformation in the Hamiltonian setting will only converge if the frequencies are distant from resonances.

1 Circle maps and Arnold Tongues

A dissipative oscillator can have asymptotic motion that is periodic. The attractor in this case is called a *limit cycle*. When two oscillators are coupled together, the motion is in a 2-d space equivalent to a torus. When the oscillators mode-lock, the attractor of the combined system is a periodic orbit on the torus, and when they unlock the trajectory is quasi periodic, densely covering the torus.

Taking a Poincaré section reduces the torus to a circle. The flow on the torus in this surface of section becomes an iterated mapping of the circle with a single iteration of the resulting circle map corresponding to a complete revolution about one axis of the torus.

A map from the unit interval back onto itself can be regarded as a map of the circle onto itself. The map

$$\theta_{n+1} = [\theta_n + \Omega] \mod 1 \tag{1}$$

corresponds to a shift or rotation by Ω.

This is in the class of maps

$$\theta_{n+1} = f(\theta_n)$$

where $f(\theta)$ is a periodic function on the unit interval $f(\theta + 1) = f(\theta)$.

We can define a winding number W (sometimes called the rotation number) as

$$W \equiv \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \Delta\theta_n \qquad (2)$$

with

$$\Delta\theta_n = [\theta_{n+1} - \theta_n] \mod 1$$

For the rotation map in equation 1 the winding number should be equal to Ω. Sometimes the winding number is written as

$$W = \lim_{n \to \infty} \frac{1}{n}\theta_n \qquad (3)$$

The two definitions are *should* be equivalent, but care should be taken with the modulo operation. For example for W computed with Equation 2, if the differences $\Delta\theta_n$ vary alternate between ϵ and $1 - \epsilon$ the winding number computed would converge to 0.5 but it should be 0 or 1.

The easiest way to compute the winding number sensibly is to use a *lift* to the real line. In other words let

$$\phi_{n+1} = f(\phi_n)$$

with no modulo operator. We no longer require f to be periodic. This sends numbers on the real line to numbers on the real line. The circle map is recovered with the modulo operation

$$\theta_n = \phi_n \mod 1$$

sending intervals on the real line into the unit interval. For a winding number W, ϕ_n would on average keep increasing by W with each application of the map. So we can compute the winding number with

$$W = \lim_{N \to \infty} \frac{1}{n}\phi_n$$

and by *not taking the modulo during the mapping.* So if you see the winding number written as given by equation 3 it is actually computed without the modulo operation.

If the winding number is rational and equal to $W = p/q$ then there are periodic fixed points with period q. The winding number depends on a limit so if the winding number is rational then orbit must converge to a periodic orbit.

If the winding number is irrational, then there are no periodic orbits. The behavior is called *quasi-periodic*. These orbits are *dense*. This means that given any point and any size neighborhood containing that point, the orbit will eventually come into that neighborhood.

1.1 Sine-circle map

We now look at maps that are more complicated than a rotation or a shift

$$\theta_{n+1} = [\theta_n + \Omega + g(\theta_n)] \mod 1$$

where $g(0) = g(1)$ so g is a periodic function $g(\theta + 1) = g(\theta)$. We can also assert that $\int_0^1 g(\theta)d\theta = 0$ as the constant or average value for g can be incorporated in the rotation Ω. It is useful to consider the lifted map on the real line

$$\phi_{n+1} = f(\phi_n)$$

with

$$f(\phi) = \phi + \Omega + g(\phi)$$

As the integral of g is zero, repeated operations of the lifted map with $\Omega > 0$ should cause ϕ to increase forever, giving a non-zero winding number, computed as

$$W = \lim_{n \to \infty} \frac{1}{n} f(\phi_n)$$

An example of a circle map, is the much studied sine-map giving rise to a phenomena called Arnold tongues

$$\theta_{n+1} = [\theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n)] \mod 1$$

This map we can write in terms of the function

$$f_{\Omega,K}(\theta) = \theta + \Omega - \frac{K}{2\pi} \sin(2\pi\theta)$$

If $K \le 1$ the map is invertible. This limit for being invertible can be understood by looking at the fixed point with $f'(\theta) = 0$.

When K is near zero we recover our shift or rotation map. When K is not zero (but not greater than 1) for each rational number there is a region in θ where the winding number is equal to that rational number. The region is *mode locked*.

The recurrence time N for a particular ϵ is the smallest integer N such that

$$\|\theta_N - \theta_0\| < \epsilon$$

and is a good way to numerically identify the rational winding numbers $W = p/q$ with low q (see Figure 2). Here $\|\theta_N - \theta_0\|$ should be computed so that it is a minimum of $|\theta_N - \theta_0|, |\theta_N - \theta_0 + 1|, |\theta_N - \theta_0 - 1|$ or a minimum of $(\theta_N - \theta_0) \mod 1$.
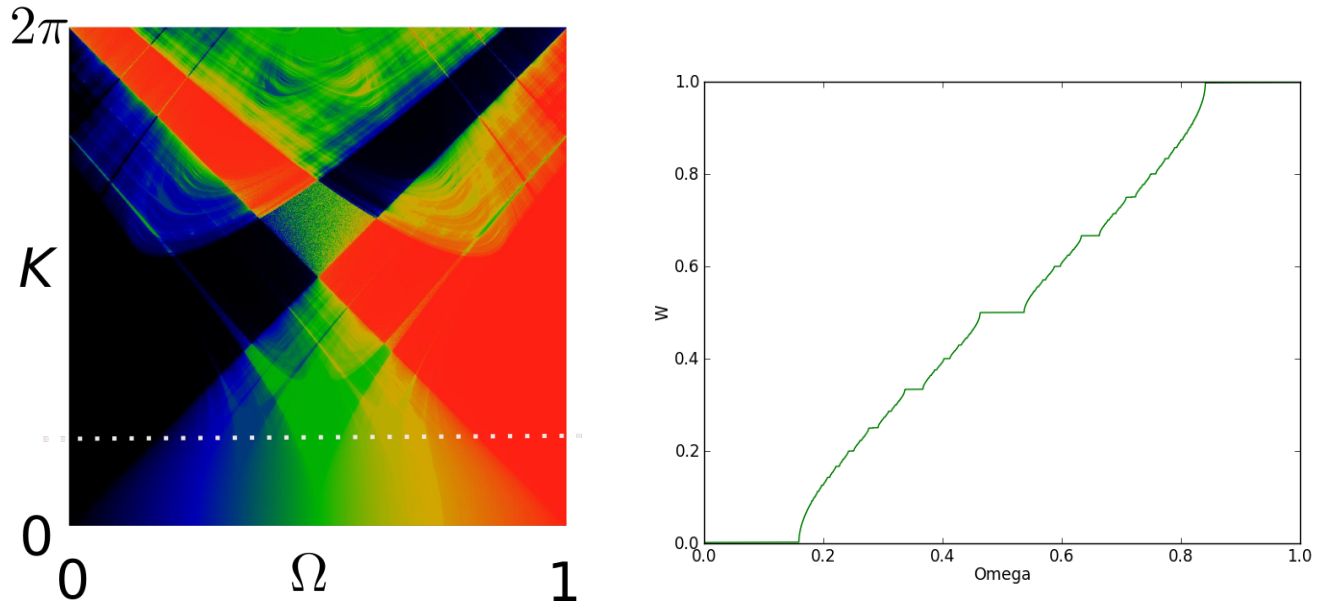
Figure 1: a) The winding number with black corresponding to $W = 0$ and red $W = 1$ with green $W = 1/2$. x axis is Ω and ranges from 0-1. y-axis is K and ranges from 0 to 4π. Original artwork created by Linas Vepstas (from wiki). The white dotted line is also shown on the right panel. b) Winding or rotation number as a function of Ω for $K = 1$, also known as the devil's staircase. I computed this with $K = 1 - 10^{-3}$ and using 300 iterations of the map.

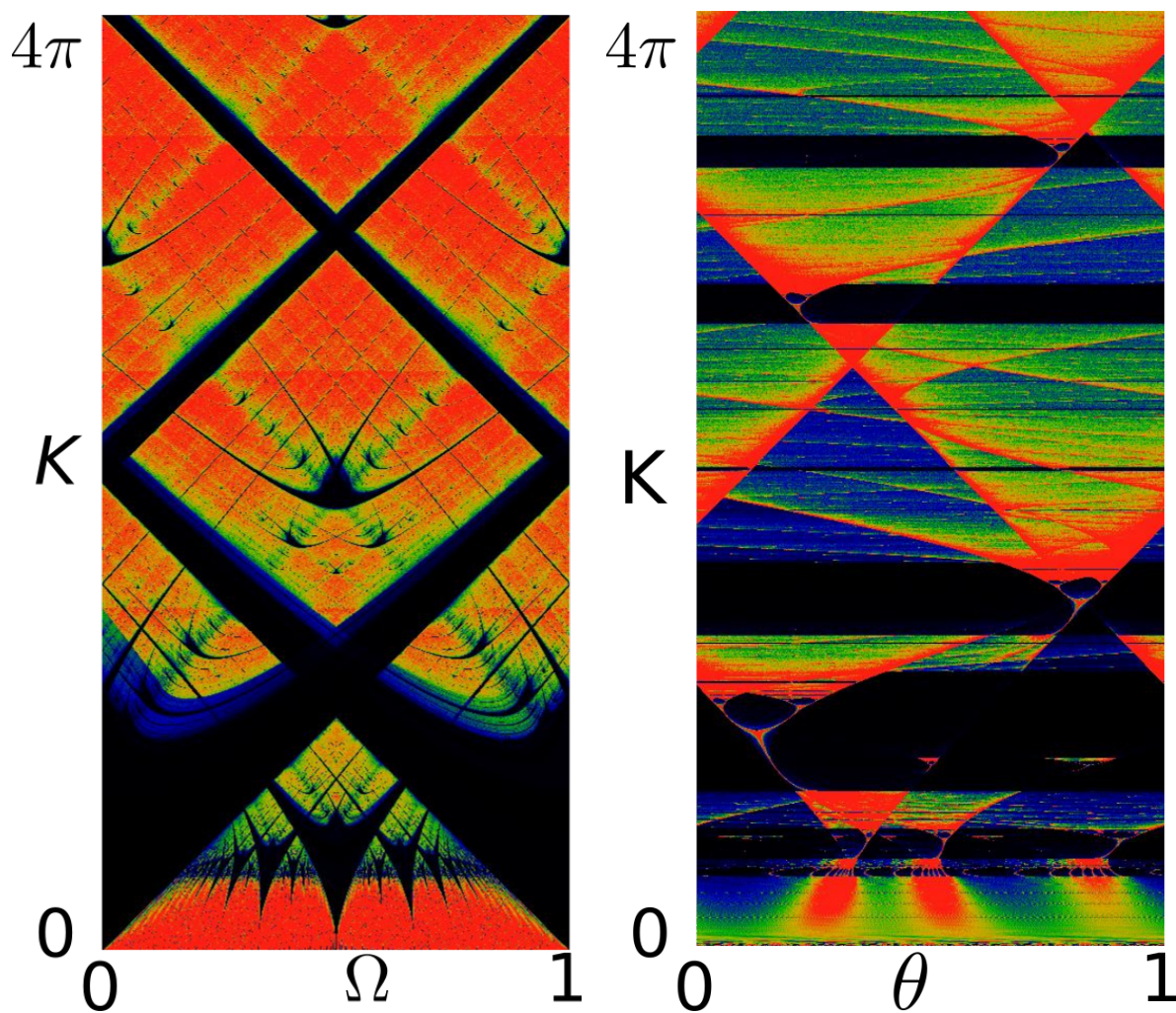Figure 2: a) The recurrence time N is plotted for $\epsilon = 0.003$ with black $N < 10$, blue values of about 50, green shows values of about 140 and red for $N > 250$. x axis is Ω and ranges from 0-1. y-axis is K and ranges from 0 to 4π. b) Bifurcation map for $\Omega = 1/3$. Here the x axis is θ_n plotted for many iterations. Again the y axis is $K \in (0, 4\pi)$. Original artwork created by Linas Vepstas (from wiki).
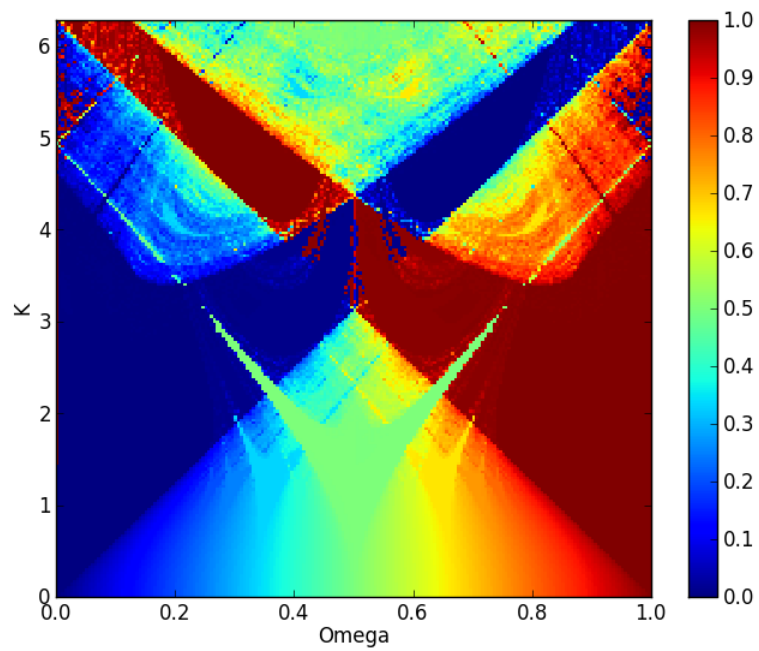
Figure 3: My attempt to compute the winding number using equation 3. After doing this I was much more impressed with Linas Vepstas's artwork!

1.2 $K < 1$ quasi-periodic and mode-locking behavior

We can worry about convergence of the winding number. Let

$$W_N \equiv \sum_{n=0}^{N-1} \frac{\Delta\theta_n}{n}$$

If the series

$$\lim_{N\to\infty} W_N$$

converges then the winding number exists and is either rational or irrational and behavior is either quasi-periodic or mode-locked, respectively.

For $K < 1$ there are only two types of behavior, quasi-periodic and mode-locked and the winding number series always converges.

If the winding number is rational, $W = p/q$, then the map converges to a periodic orbit with

$$f^q(\theta_i) \sim \theta_i + p$$

with θ_i having $q - 1$ values.

It is not true that if Ω is rational and $K < 1$ then the winding number is always rational. Only if K is small enough is the winding number is equal to Ω. For $K < 1$ there is a mode-locked region for every rational winding number.

The largest tongues, ordered by size, occur at *Farey* fractions. A Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to n. For two consecutive rational numbers in a Farey sequence of the same order, p_1/q_1 and p_2/q_2, it is true that $p_1 q_2 - p_2 q_1 = 1$. The largest Arnold tongue between the tongues with winding numbers p_1/q_1 and p_2/q_2 has winding number $(p_1 + p_2)/(q_1 + q_2)$, known as a Farey sum.

If the winding number is irrational then the map is *conjugate* to a shift or rotation. In other words there is an invertible function h on the unit interval and an irrational number λ, such that

$$h^{-1} \circ f_{\Omega,K} \circ h = g$$

and with $g(\theta) = \theta + \lambda$ a rotation where λ is irrational and equal to the winding number.

It is possible to develop a convergent perturbative method (usually based on Fourier expansion) to construct h and λ given Ω, K, providing that the winding number is irrational. For example, see recent work by Llave.

If the winding number is rational, then the map is *conjugate* to one containing rational periodic orbits. One could attempt to estimate the width of the Arnold tongues as a function of rational winding number $W = p/q$ and K. In this case I think it should be possible to find an invertible function h such that

$$h^{-1} \circ f_{\Omega,K} \circ h = g$$

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where g is something simple like

$$g(\theta) = \theta - \epsilon \sin(2\pi q \theta)$$

or

$$g(\theta) = \theta + \prod_{i=0}^{2q} (\theta - \pi i / q)$$

1.3 $K = 1$ Critical value giving a Cantor set

If $K = 1$ the situation is considered critical. All the mode-locked regions just barely touch. There are no irrational winding numbers, but the winding number as a function of Ω is fractal and called the *devil's staircase*.

It is sometimes said that the circle sine-map maps the rationals, a set of measure zero at $K = 0$, to a set of non-zero measure for $K \neq 0$, and a set of measure 1 for $K = 1$. The total integrated (integrated here in the sense of Lebesque integration) length of the frequency-locked intervals goes from $w = 0$ when $K = 0$ to $w = 1$ when $K = 1$.

Rational numbers are a set of measure zero. For $K \neq 0$ each rational winding number gives rise to a tongue of a finite width. But the rational numbers are an infinite set. How can this set of intervals with rational winding numbers not be of measure 1 and cover the entire interval? Consider a tongue with winding number p/q, with width Δw. If $\Delta w \sim K^q$ or K^{q-1}, decreasing sufficiently fast as both q increases and K decreases. then the total width w would be of measure less than 1.

There is a flat region in the devil's staircase for every rational number.

Question Is it possible to construct a devil's staircase in different ways? There must be a flat region for every rational number and the entire set must be of measure 1 and so cover the entire unit interval.

Question For the sine map, the set of rational winding numbers is of measure 1 at $K = 1$ and this is the same value of K where the tongues start to overlap. For other circle maps can tongues start to overlap in one region of Ω before another region?

1.4 $K > 1$ Period doubling and chaos

In this case tongues overlap and we can have chaotic behavior. See Figure 2 showing a bifurcation diagram for different values of K at $\Omega = 1/3$. Period doubling and chaotic trajectories are both seen as we saw previously in the logistic map.

The circle map exhibits a type of universality similar to the logistic map (and in fact behavior of periodic orbits of the logistic map can be seen in the circle map). Renormalization techniques have also been applied giving a set of universal exponents that describe convergence of W_n and renormalization of the function.

1.5 Perturbations

Consider
$$f(x) = x + \Omega + g(x)$$

$$
\begin{aligned}
f^2(x) &= x + 2\Omega + g(x) + g(x + \Omega + g(x)) \\
&= x + 2\Omega + g(x) + g(x + \Omega) + g'(x + \Omega)g(x) + g''(x + \Omega)\frac{(g(x))^2}{2} + ...
\end{aligned}
$$

$$
\begin{aligned}
f^3(x) &= x + 3\Omega + g(x) + g(x + \Omega) + g'(x + \Omega)g(x)... + g(x + 2\Omega + g(x + 2\Omega)) \\
&= x + 3\Omega + g(x) + g(x + \Omega) + + g(x + 2\Omega) + g'(x + \Omega)g(x) + g''(x + \Omega)\frac{(g(x))^2}{2} + ...
\end{aligned}
$$

To first order in g

$$f^q(x) = x + q\Omega + \sum_{i=0}^{q-1} g(x + \Omega i)$$

If $\Omega \sim 1/q$ then this first order term averages to zero. Likewise for second order terms as long as Ω differs from $1/2$. The q-th order terms do not average to zero if $\Omega \sim 1/q$,

$$f^q(x) \sim x + q\Omega + \prod_{i=0}^{q} g'(x + i\Omega)$$

For $g(x) = K/(2\pi)\sin(2\pi x)$ the product gives something like

$$f^q(x) \sim x + q\Omega + K^q \sin(2\pi q x)$$

This implies that the width of the tongues scales with K^q. The terms that do not average to zero of lower order would contribution to an offset of the tongue giving a winding number $W = p/q = \Omega + \epsilon$ where ϵ is an offset.

Remark It would be nice to use a perturbation technique similar to that used for Diophantine numbers and using a conjugacy map, that could be used to estimate width of Arnold tongues.

1.6 Density maps

$f(\theta)$ an invertible map of the unit interval

$$\theta_{i+1} = f(\theta_i)$$

It is convenient to compute a derivative

$$\frac{d\theta_{i+1}}{d\theta_i} = f'(\theta_i) = f'(f^{-1}(\theta_{i+1}))$$

$$\frac{d\theta_i}{d\theta_{i+1}} = Df^{-1}(\theta_{i+1})$$

Density distribution (initially smooth) $\rho(\theta_i)$

$$\rho(\theta_{i+1})d\theta_{i+1} = \rho(\theta_i)d\theta_i$$

The map gives a new density distribution

$$
\begin{aligned}
\rho(\theta_{i+1}) &= \rho(\theta_i)\frac{d\theta_i}{d\theta_{i+1}} \\
&= \rho(f^{-1}(\theta_{i+1}))\frac{1}{f'(f^{-1}(\theta_{i+1}))} \\
&= \rho(f^{-1}(\theta_{i+1}))Df^{-1}(\theta_{i+1})
\end{aligned}
$$

Let $g(\theta) = f^{-1}(\theta)$

$$\rho(\theta_{i+1}) = \rho(g(\theta_i))Dg(\theta_i)$$

Starting with a constant initial density distribution $\rho_0(\theta) = 1$, we generate a map between density distributions at each iteration of the map,

$$\rho_i(\theta) = \rho(\theta_i)$$

Attracting fixed points are points θ_* where

$$\lim_{i \to \infty} \rho_i(\theta_*) \to \infty$$

The distribution approaches a delta function with peak at θ_*.

2 Ways to describe rational and irrational numbers

2.1 Farey sequence

A Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to n. An example

$$F_5 = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

The sequence has the nice property that

$$P_i Q_{i-1} - P_{i-1} Q_i = 1$$

The number of terms in a given Farey sequence $\phi(Q)$ turns out to be an irregular function of Q.

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2.2 Continued Fractions

$$\frac{P_n}{Q_n} = [a_1, a_2, a_3....a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + ...\cfrac{1}{a_n}}}$$

To compute the expansion take the integer part. Then invert the remainder and take its integer part. Repeat. The expansion terminates for any rational number.

Rational approximations get better if we have large numbers in the continued fraction of the value we are approximating. By that definition the most irrational number is the one that has a continued fraction expansion with all 1s. This is the golden mean.

For any irrational number we given an estimate of how irrational it is by summing the digits in the continued fraction expansion.

A *best rational approximation* to a real number x is a rational number p/q, $q > 0$, that is closer to x than any approximation with a smaller or equal denominator. The simple continued fraction for x generates all of the best rational approximations for x. To obtain the best rational approximation from a truncated continued fraction expansion it is sometimes necessary to increment the last digit in the sequence.

It is possible to estimate the best rational within an interval. Given two numbers x, y defining an interval, they have the same continued fraction expansion $[a_1, a_2...a_{k-1}..]$ up to a particular sequence number a_{k-1} at which point they start to differ with x having next integer a_k and y have next integer in the sequence b_k. The best rational in the interval has continued fraction expansion $[a_1, a_2...a_{k-1}, \min(a_k, b_k) + 1]$

There is a pretty close relation between the Farey sequence and best rational approximations.

2.3   Irrational Numbers and Diophantine approximation

Rational winding numbers exhibit mode-locking in circle maps. When the winding number is rational, the orbit converges to a periodic orbit. The orbit is then not dense. There is no way to make a transformation that transfers the map to a rotation.

In Hamiltonian systems, when the frequency vector of the unperturbed system contains a resonance $\boldsymbol{\omega} \cdot \mathbf{k} = 0$ for some integer vector $\mathbf{k} \in Z^n$, then it is not possible to remove perturbations using a perturbative expansion.

In both settings the desired transformation can be done if the perturbation is small enough and either the frequency or winding number is sufficiently *irrational* or distant from rational numbers or resonances, depending on the setting.

A vector $\boldsymbol{\omega}$ satisfies a *diophantine condition* if

$$|\mathbf{k} \cdot \boldsymbol{\omega}| > \frac{\gamma}{|\mathbf{k}|^\tau}$$

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for every $\mathbf{k} \in \mathbf{Z}^n$, $\mathbf{k} \neq 0$ for some positive numbers γ, τ. Only numbers that are poorly approximated by rational series satisfy diophantine conditions. Another way of saying this is that the number is so irrational that it is always pretty far away from any rational numbers or resonances.

What if I keep looking for higher values of integers? With larger integers you can get closer to any irrational number. The diophantine condition takes this into account in how the distance scales with $|\mathbf{k}|$.

Roots of polynomials can be irrational and can be written in terms of a continued fraction expansion. For example, the gold mean $\alpha_g = (\sqrt{5} - 1)/2$ is the solution to $x^2 + x - 1 = 0$ and can be written as

$$\alpha_g = (1 + \alpha_g)^{-1} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

Taking the vector

$$\mathbf{v}_g = (1, \alpha_g)$$

it is possible to show using the continued fraction expansion that

$$|\mathbf{k} \cdot \mathbf{v}_g| \geq \frac{1}{\sqrt{5}\|\mathbf{k}\|}$$

where $\|k\| = \sum_i |k_i|$. Thus the vector v_g is Diophantine with $\tau = 1$ and coefficient $\gamma = 1/\sqrt{5}$.

2.4 Pigeonholes

The pigeonhole principle: If one puts $n + 1$ pigeons into n pigeonholes, then at least one pigeonhole will contain at least two pigeons. More generally, if one puts $kn + 1$ pigeons into n pigeonholes, then at least one pigeonhole will contain at least $k + 1$ pigeons.

Dirichlet used this principle to prove a result about approximating irrational numbers by rationals.

Theorem 2.1 *Theorem (Dirichlet) Let α be an irrational number, and let N be a positive integer. Then there is a rational p/q such that the denominator q is between 1 and N and such that*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{qN}.$$

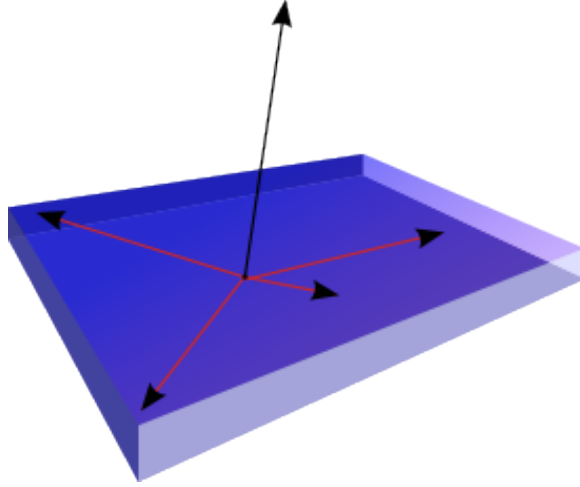Another way to state this is $q\alpha$ is within $1/N$ of an integer p.

Figure 4: A frequency vector $\boldsymbol{\omega}$ is shown as the black arrow. Integer vectors $\mathbf{k}$ near the plane perpendicular to $\boldsymbol{\omega}$ have low values of $|\boldsymbol{\omega} \cdot \mathbf{k}|$ and so are near resonance. Larger integer vectors $\mathbf{k}$ are likely to be closer to this plane.

Proof Take the interval $[0, 1)$ and divide it up into N intervals,

$$[0, 1/N), [1/N, 2/N), [2/N, 3/n)...$$

Now look at the fractional parts of

$$0, \alpha, 2\alpha, 3\alpha...N\alpha$$

There are $N + 1$ of these fractional parts and we need to fit them into our N intervals. So one interval has to contain two of these fractional parts. Let us call them $\text{int}(r\alpha)$ and $\text{int}(s\alpha)$. This means that

$$\text{int}(r\alpha) - \text{int}(s\alpha) < 1/N$$

Set $q = (r - s)$ and we have $q\alpha$ within $1/N$ of an integer. If $q\alpha$ is within $1/N$ of an integer then there exists an integer p such that $|q\alpha - p| < 1/N$.

This proof can be generalized for higher dimensions just keeping track of more pigeon-holes. For example, consider a frequency vector $\boldsymbol{\omega}$ of 2 rational numbers (ω_x, ω_y). We can think of this as a direction and ask when is the vector near in integer point on the lattice (see Figure 5). Then $\alpha = \omega_x/\omega_y$ and there exists p, q with $q < N$ such that

$$|\alpha q - p| < \frac{1}{N}$$

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We can also write this as

$$\omega_x q - p\omega_y < \frac{\omega_y}{N}$$

giving a Diophantine like condition for distance to resonance.

In three dimensions we can normalize the vector by the first frequency

$$(\boldsymbol{\omega} = 1, \alpha_1, \alpha_2)$$

and using the pigeonhole principle there exists p_1, p_2 and $q, q_2 < N$ such that

$$|\alpha_1 q_1 - p_1| < \frac{1}{N^2} \qquad |\alpha_2 q_2 - p_2| < \frac{1}{N^2}$$

These integers give *nearly periodic orbits* rather than *near resonances*. Nearly periodic orbits are specified by an array of integers $\mathbf{q}$ such that

$$\mathbf{q} \cdot \frac{\boldsymbol{\omega}}{\omega_0}$$

is very near an array of integers $\mathbf{p}$ rather than an array of integers $\mathbf{k}$ such that $|\mathbf{k} \cdot \boldsymbol{\omega}|$ is small.

Remark Iteration techniques for finding a conjugacy map for a circle map to a pure rotation often rely on specifying that the winding number is *Diophantine*. This is specified so that at each level of the perturbation expansion, the winding number would remain distant from rational numbers. Estimates of the width of Arnold tongues on the other hand find that the width depends on which Farey sequence in which it appears. Kolmogorov's theorem for a perturbed Hamiltonian system, uses a perturbation expansion of canonical transformation to transform the Hamiltonian into action angle variables. A diophantine condition on the frequencies of the unperturbed system is used to ensure that the perturbation expansion will converge. This will ensure that at all levels of the perturbation expansion that no resonances (giving small divisors) will occur.
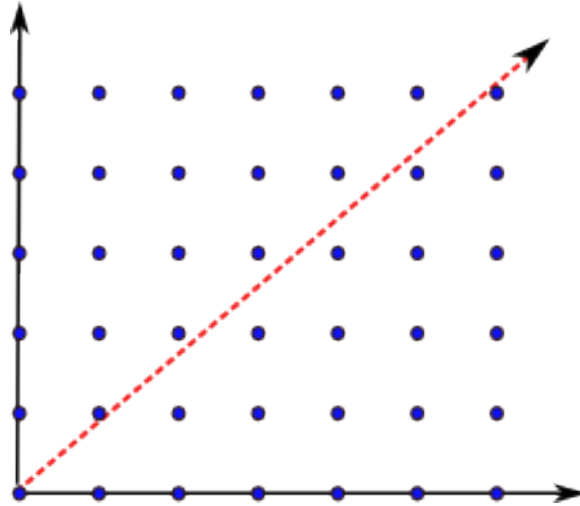
Figure 5: A two dimensional frequency vector with irrational slope is likely to be nearer integer lattice points further from the origin. If we describe the lattice points as integers $k_x, k_y = p, q$ then the pigeonhole principle implies that there is a integer $q < N$ and am integer p such that $|\omega_x/\omega_y q - p| < 1/N$. In two dimensional this also implies that there is an integer vector $\mathbf{k} = q, p$ such that $\mathbf{k} \cdot \boldsymbol{\omega} < C/N$ which is a diophantine condition.

3 Newton's method

Given a differentiable function $f(x)$ on real numbers how can we find its roots? Newton's method is a iteration procedure for converging on a root. The idea is to make a map, N_f using f so that if we keep applying the map to an initial value of x we will converge on a root. We want the map to have orbits that converge on to roots.
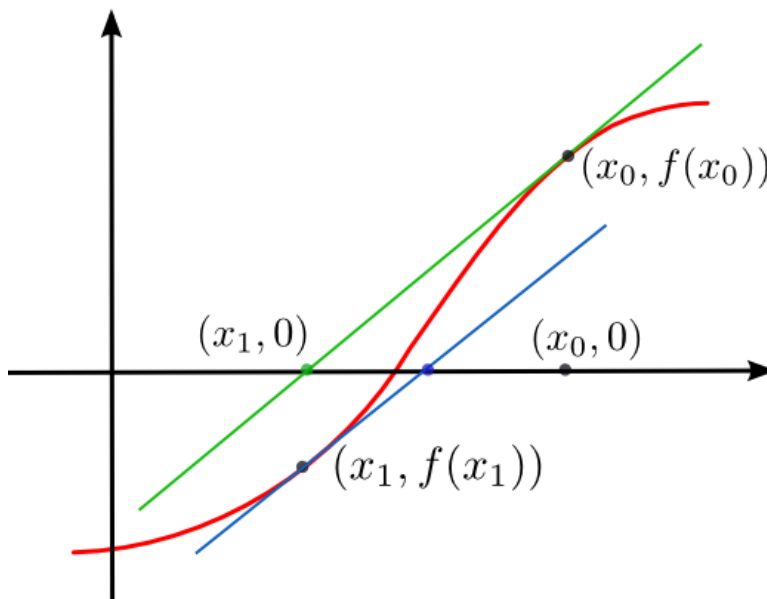

Figure 6: Illustration of Newton's method for converging to a root of a function.

The map can have interesting dynamics. For example, convergence is not assured. There may be no roots. In this case the orbits of the map can be chaotic or go to infinity. If there are more than one root, the basins of attraction for convergence to each root may be difficult to determine. The map does not always converge onto the nearest root. However, if there is a nearby root, then Newton's map converges very quickly, quadratically quickly in fact.

We start with a point $x_0, f(x_0)$ and compute the tangent of the function at this point. We look at the tangent line (with slope $f'(x_0)$) that goes through $x_0, f(x_0)$ and find where it crosses the y axis. The line has equation

$$y - y_0 = f'(x_0)(x - x_0) = y - f(x_0) = f'(x_0)(x - x_0)$$

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and it crosses the y axis at $y = 0$ and where

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We construct a map

$$x_{n+1} = N_f(x_n)$$

with

$$N_f(x) = x - \frac{f(x)}{f'(x)}$$

We can call N_f Newton's function for f.

Example Let $f(x) = x^3 - x$. The roots are $-1, 0, 1$. We compute

$$N_f(x) = x - \frac{x^3 - x}{3x^2 - 1} = \frac{2x^3}{3x^2 - 1}$$

The function is well behaved except at $x = 1/\sqrt{3}$. Near roots, the map converges very quickly. But there are initial conditions that give problems and those for which convergence is very slow (near $x = 1/\sqrt{3}$).

3.1 Quadratic functions

Let $f(x) = ax^2 + bx + c$ and $q(x) = x^2 - A$. The two functions N_f, and N_q are topologically conjugate using the function $h(x) = 2ax + b$ and with $A = b^2 - 4ac$. We can show that this is true by showing that $h \circ N_f = N_f \circ h$. We compute $N_f(x) = \frac{ax^2 - c}{2ax + b}$ and $N_q(x) = \frac{x^2 + A}{2x}$.

$$h \circ N_f = 2a \frac{ax^2 - c}{2ax + b} + b = \frac{2a^2 x^2 + 2abx + b^2 - 2ac}{2ax + b}$$

$$N_p \circ h = \frac{(2ax + b)^2 + A}{2(2ax + b)} = \frac{2a^2 x^2 + 2abx + b^2 - 2ac}{2ax + b}$$

As a consequence study of Newton's method on quadratic functions need only consider two cases $f(x) = x^2 - 1$ which has real roots and $f(x) = x^2 + 1$ which does not.

When $f(x) = x^2 + 1$ we find $N_f = (x^2 - 1)/2x$ which is pretty badly behaved near $x = 0$. Trajectories are chaotic. In fact there is a topological conjugacy with $h(x) = \cot(x/2)$ to the map on the unit circle $D(\theta) = 2\theta$ which is chaotic.
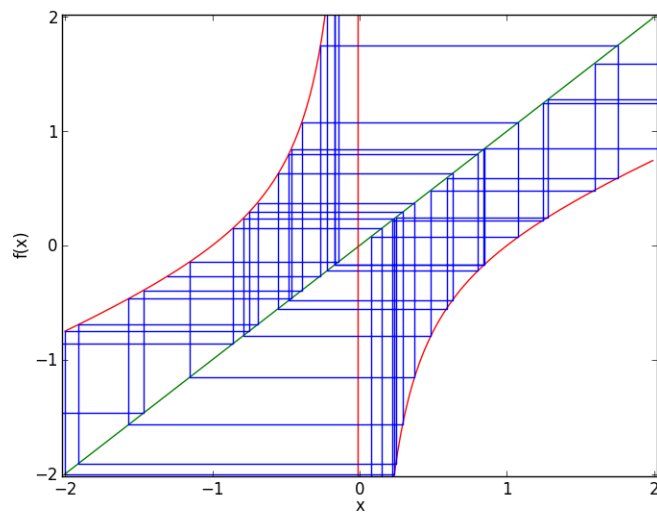
Figure 7: A cobweb plot showing trajectory for Newtons map $N_f(x) = x - (x^2 - 1)/(2x)$ for the function $f(x) = x^2 + 1$ which has no roots. The orbit is chaotic.

3.2 Convergence of Newton's map

Consider the difference

$$\delta_n \equiv |x_{n+1} - x_n|$$

that describes a distance away from convergence. If the map converges δ_n should decrease. As we get closer to a root this difference should decrease. We can write this difference

$$\delta_n = \left| x_n - \frac{f(x_n)}{f'(x_n)} - x_n \right| = \left| \frac{f(x_n)}{f'(x_n)} \right|$$

Let us define a function

$$r(x) = \frac{f(x)}{f'(x)}$$

so that

$$\delta_n = |r(x_n)|$$

Now we compute the next difference

$$
\begin{aligned}
\delta_{n+1} &= |r(x_n + \delta_n)| \\
&= |r(x_n + r(x_n))| \\
&= |r(x_n) + r'(x_n)r(x_n)| \\
&= |r(x_n)(1 + r'(x_n))|
\end{aligned}
$$

$$r'(x) = \frac{f'(x)}{f'(x)} - \frac{f(x)f''(x)}{f'(x)^2} = 1 - \frac{f(x)f''(x)}{f'(x)^2}$$

$$
\begin{aligned}
1 + r'(x) &= -\frac{f(x)f''(x)}{f'(x)^2} \\
&= -r(x)\frac{f''(x)}{f'(x)}
\end{aligned}
$$

Inserting this back into δ_{n+1}

$$
\begin{aligned}
\delta_{n+1} &= \left| r(x_n)r(x_n)\frac{f''(x_n)}{f'(x_n)} \right| \\
&= \delta_n^2 \left| \frac{f''(x_n)}{f'(x_n)} \right|
\end{aligned}
$$

This expressions shows that the distance to the root should decrease quadratically. This explains why when one is near a root, convergence is rapid and the technique is quite efficient.

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3.3   Newton's method as a first order approximation

We want x^* such that $f(x*) = 0$. But we are starting at different value x_0. We can search for δ such that

$$f(x_0 + \delta) = 0$$

Let us expand this to first order in δ.

$$f(x_0) + f'(x_0)\delta = 0$$

To first order

$$\delta = -\frac{f(x_0)}{f'(x_0)}$$

or

$$x_1 = x_0 + \delta = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Here x_1 is the solution to the first order approximation.

Remark Newton's method appears to be a simple root finding technique. However it can be applied to functions and estimates of its convergence rate are useful. Add examples of equations (perhaps cohological equation? or that for allowing one to find a topological conjugacy to a rotation for the circle map.)

3.4   Examples

Also there are nice examples of connections between maps on the complex plane with Newton's method along with some proofs involving topological conjugacy in Holmgren's book. It would be nice to add an example, proofing that Newton's method gives a chaotic map, using a topological conjugacy to a map on the complex plane?