PHY411 Lecture notes Part 5

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1 Introduction

In previous lectures we have seen *chaos* exhibited in different ways. We saw a change in dimension of orbits in Hamiltonian systems and in maps created from them. In iterative maps on the real line we saw examples of sensitivity to initial conditions. Here we will look at maps more abstractly, at maps on sets. In the realm of symbolic dynamics, we also find periodic orbits and maps that are strongly mixing. We will also discuss an abstract definition for a chaotic function on a metric space. We are building here on the properties of maps to cause strong mixing. Even though these maps are deterministic we find behavior typical of ergodic or stochastic systems.

Definition An *ergodic* system is one that, given sufficient time, includes or impinges on all points in a given space and can be represented statistically by a reasonably large selection of points.

Definition A *stochastic* system is one that is randomly determined; having a random probability distribution or pattern that may be analyzed statistically but may not be predicted precisely.

2 Symbolic Dynamics

Symbolic dynamics is concerned with maps on sets. We focus here on an example using the set Σ_2.

We define Σ_2 as the set of infinite sequences of 0's and 1's. An element or point in Σ_0 is something like

0000000 ... or 010101010101 ... repeating.

We can refer to the point s as

$$s: \qquad s_0 s_1 s_2 s_3 s_4.....$$

and the point t as

$$t: \qquad t_0 t_1 t_2 t_3...$$

We can think of s and t as close to one another if their sequences are similar at the beginning of the sequence.

This space has a nice metric that we can use to tell if two points in Σ_2 are close to each other.

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Since $|s_i - t_i|$ is always either 0 or 1

$$0 \le d[s,t] \le \sum_{i=0}^{\infty} 2^{-i} = 2$$

Definition Recall that a metric on a set X has the properties

- $d[x, y] \geq 0$
- $d[x, y] = d[y, x]$
- $d[x, y] \leq d[x, z] + d[z, y]$ triangle inequality

for all x, y, z in the set X.

A metric space is a set along with a metric allowing one to measure distances between points in the set.

A nice thing about this metric is that we can describe the distance between two points in terms of the number of digits in the beginning of the sequence that agree. For example, if s, t agree in their first 2 digits (first 2 numbers of the sequence) then $d[s, t] \leq 2^{-1}$. If s, t agree in their first n digits then $d[s, t] \leq 2^{-n+1}$. The size of a neighborhood can be described in terms of the numbers of matching digits of points in the neighborhood.

2.1 The Shift map

The shift map on Σ_2 is defined as

$$\sigma(s_0 s_1 s_2 ...) = s_1 s_2 s_3$$

where $s \in \Sigma_2$. The shift map forgets the first digit of the sequence. For example $\sigma(10010101...) = 0010101....$

If two points are close together with respect to our metric d then they remain close together after they are both shifted by the shift map. But if the shift map is reapplied many times then we eventually expect the orbits to differ.

There are periodic points of the shift map. For example the repeating sequence

$$s = 1010101010....$$

is a fixed point of σ^2.

The shift map has 2^n periodic points of period n. This follows as there are 2^n ways to chose 1,0 for n positions in a repeating sequence.

We can call a point *eventually* periodic if it starts with a sequence and then is periodic.

Definition It is useful to use the word *dense*. A set A is *dense* in another set X if for every point in $x \in X$, every neighborhood of x contains at least a point in A.

In other words for each point $x \in X$ and each $\epsilon > 0$ there exists a $y \in A$ such that $|x - y| < \epsilon$. Another equivalent statement is that for every point $x \in X$ there is a sequence of points in A that converge to x. A dense set is one that is present everywhere in X.

The set of periodic points of the shift map is *dense* in Σ_2. Suppose we choose an $x \in \Sigma_2$ that is not periodic. We will show that there are periodic points near it no matter what x is. We can define distance in terms of the number of digits that agree. Suppose we choose a neighborhood with all points that are the same as x up to 2^n digits. We can find a periodic point in this neighborhood by choosing a point that repeats with the same 2^n digits as x. Since for every size neighborhood (every possible n) and every x we can find a periodic orbit in the neighborhood, we have shown that the set of periodic points is dense in Σ_2.

There is a point in Σ_2 whose *orbit* is dense in Σ_2. To show that this is true, all we have to do is construct such an orbit and show that it is dense. The Morse sequence first has a block containing all single bit sequences, then a block containing all two bit sequences, then a block containing all three-bit sequences, to infinity;

$$M = 0\ 1\ 00\ 01\ 10\ 11...$$

As the shift map is applied this orbit will eventually cover all possible sized neighborhoods near any point in Σ_2.

Definition A function $f : D \to D$ is *topologically transitive* if for all open sets U, V in D there is an x in U and a natural number n such that $f^n(x)$ is in V.

If a function is topologically transitive, then we can arbitrarily chose two neighborhoods and we can always *find an orbit that contains points in both neighborhoods.* A function that is topologically transitive *mixes* the domain. We saw mixing in the chaotic regions of the area preserving maps and for certain parameters in the logistic map, though we didn't determine whether the mixing was present everywhere. We did not show that there was an orbit connecting *any* two neighborhoods.

2.2 Σ_2 is well mixed by the shift map

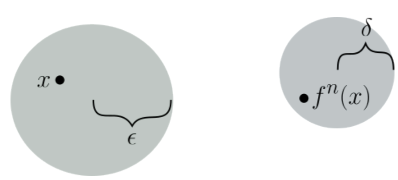

Figure 1: If a map is topologically transitive (and mixes) then an orbit exists between any two neighborhoods.

We can show that the shift map is *topologically transitive* and so well mixes Σ_2. Chose any two points, s, t, in Σ_2 and any two sized neighborhoods around them. We can specify the size the neighborhood around s with a number of digits n for points in Σ_2 at the beginning of the sequence that are the same as those of s. A point z in this neighborhood of s has distance $d[s, z] \leq 2^{-n}$. We can similarly specify the size of the neighborhood around t with m digits. We can construct a point x that has first n numbers in its sequence that are the same as s but the following m numbers in its sequence that are the same as t. This x has orbit such that it stars in the neighborhood of s and σ^n puts it in the neighborhood of t.

2.3 The shift map is sensitive to initial conditions

Definition For D a metric space with metric d. The function $f : D \to D$ exhibits *sensitive dependence* on initial conditions if for all $\forall x \in D, \forall \epsilon > 0$, and any $\delta > 0$, there is a natural number n and $y \in D$ such that

$$d[x, y] < \epsilon \qquad \text{and} \qquad d[f^n(x), f^n(y)] > \delta$$

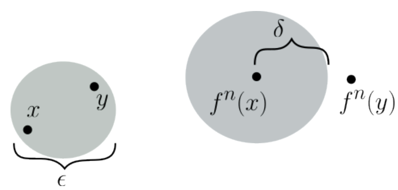

Figure 2: If a map is sensitive to initial conditions then two orbits that are initially close together get further apart.

No matter how small a neighborhood we chose around any point x, we can always find a nearby point y (in the neighborhood) that has an orbit that eventually sends it some distance δ away from the orbit of x. That means it is hard to predict orbits after many iterations if we have uncertainty in initial conditions.

The shift map is sensitive to initial conditions. Choosing an ϵ gives an m the number of digits of y that must be in common with x to satisfy $d[x, y] < \epsilon$. We set δ to be 1/4. We can chose y to have m digits in common with x and then differ in the next digit. After m applications of the shift map the orbits are 1/2 apart. This means for any ϵ we can find a y and an $n = m$ that has distance between the orbits greater than $\delta = 1/4$. Thus the shift map is *sensitive to initial conditions*, as defined above.

2.4 Symbolic Definition of Chaos (Devaney)

Definition due to Devaney 1989 in his book on chaotic dynamical systems.

Definition Let D be a metric space. The function $f : D \to D$ is *chaotic* if

a) The periodic orbits of F are dense in D.

b) The function f is topologically transitive. In other words f mixes the set really well.

c) The function f exhibits extreme sensitivity to initial conditions.

It turns out that a,b imply c if D is infinite (proved by Banks, Brooks Cairns, David and Stacey in 1992).

2.5 The shift map on the unit interval

A point in Σ_2 can be turned into a number on the unit interval by inserting a decimal point before the sequence and assuming that the number is in base 2. For example $s = 01011....$ is equivalent to the number 0.01011.. Recall our metric in Σ_2. This metric is equivalent to distance between two points in the unit interval.

What is the equivalent of the shift map but on the unit interval? If the number is less than 1/2 then the shift map simply multiplies by 2. If the number is greater than 1/2 then the shift map multiplies the number by 2 and then subtracts it by 1.

$$\sigma(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2x - 1 & 1/2 \leq x \leq 1 \end{cases}$$

Another way to write this map is

$$\sigma(x) = 2x \mod 1$$

It is easy for us to compute the Lyapunov exponent for this map from its derivative which is always 2.

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln|f'(x_i)| = \ln 2$$

How is this related to the Lyapunov exponent of the shift map? Consider two points s, t that are the same for their first 100 digits but differ afterwards. At the beginning the distance between the two points is $d(s,t) \leq 2^{-100}$ but after a single application of the shift map $d(\sigma(s), \sigma(t)) \leq 2^{99}$ and so on. The ratio of these distances is 2^n where the map is applied n times. The natural log of the ratio of these distances is $n \ln 2$.

Recall we can define a Lyapunov exponent for a map as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \frac{|\delta_n|}{|\delta_0|}$$

Instead of using Euclidian distances if we use the metric computing

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{d(\sigma^n(s), \sigma^n(t))}{d(s,t)} \right| = \ln 2$$

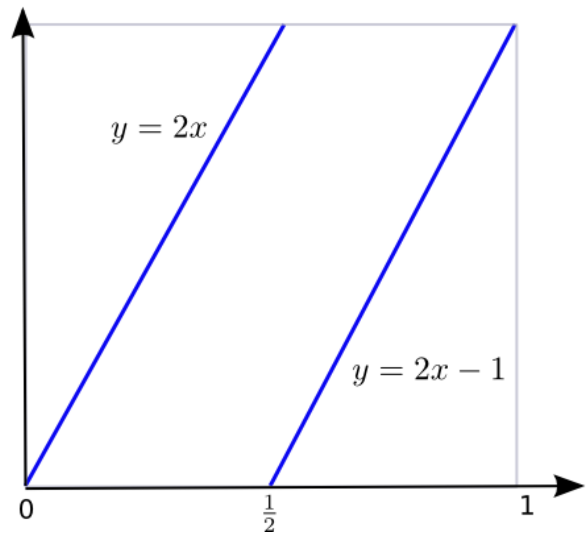

Figure 3: The shift map on Σ_2 is equivalent to this map on the unit interval. The map stretches the interval.

Question Do we have to be careful how we chose two initial points for computing trajectories? Periodic points are dense. What happens if we chose two periodic orbits?

Remark This map stretches the unit interval. It stretches both the region 0 to 1/2 and the region 1/2 to 1 back into the region 0 to 1. Stretching and folding is typical of chaotic maps and how they mix.

3 Scrambled sets and the Li-Yorke definition of Chaos

Definition A continuous map $f : X \to X$ on a compact metric space (X,d) is called *chaotic* (in the sense of Li and Yorke) if X contains an *uncountable* set called a *scrambled set* S with the following properties:

- $\lim_{n\to\infty} \sup d(f^n(x), f^n(y)) > 0$ for all $x, y \in S$, $x \neq y$

  orbits remain separate

- $\lim_{n\to\infty} \inf d(f^n(x), f^n(y)) = 0$ for all $x, y \in S$, $x \neq y$

  orbits get arbitrarily close

- $\lim_{n\to\infty} \sup d(f^n(x), f^n(p)) > 0$ for all $x \in S$, $p \in X$ with p periodic.

orbits don't intersect a periodic orbit

An alternative definition for a scrambled set replaces the first condition with

$$\lim_{n\to\infty} \sup d(f^n(x), f^n(y)) = 1$$

if the space has a maximum distance between two points of 1, and drops the last condition.

The first condition requires that orbits remain apart.

The second condition requires that all orbits to eventually be very close to each other.

The last condition requires all orbits to be somewhat distant from all periodic orbits. Periodic orbits cannot be in the set. (No more than 1 of them can be in it).

An alternative definition of a scrambled set can remove the last condition, and in that case periodic orbits can be contained in the scrambled set.

3.1 Recurrence

The Poincaré recurrence theorem for bounded volume preserving dynamical systems states that if an orbit starts within an open set, exits it and then returns to the open set, it must return to the same open set an infinite number of times.

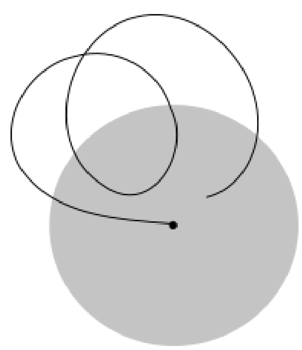

Figure 4: The setting for the Poincaré recurrence theorem. If an orbit leaves and then re-enters a neighborhood, it will return an infinite number of times.

4 Topological conjugacy and entropy

Suppose we have two metric spaces X, Y and two maps $f : X \to X$ and $g : Y \to Y$.

Definition Two pairs (f,X), (g,Y) are **topologically conjugate** if there is a homeomorphism $h : X \to Y$ such that $f = h^{-1}gh$.

This is an equivalence relation. There is a one-to-one correspondence between the orbits of f and those of g.

In particular, if f has a fixed point, so does g. For example, suppose $f(x_*) = x_*$ is a fixed point of f.

$$\begin{aligned} h(f(x_*)) &= g(h(x_*)) \\ h(x_*) &= g(h(x_*)) \qquad \rightarrow \quad h(x_*) \text{ is a fixed point of } g(). \end{aligned}$$

For every fixed point of f we find a fixed point of g.

Consider $f^2(x) = f(f(x))$.

$$\begin{aligned} h(f(x)) &= g(h(x)) \\ h(f(f(x)) &= g(h(f(x))) = g(g(h(x))) \\ h(f^2(x)) &= g^2(h(x)). \end{aligned}$$

Thus if f and g are topologically conjugate then so are f^2 and g^2. This implies that periodic orbits of f give periodic orbits of g.

As the orbits of f and g are similar, we can now think about invariants.

4.1 Topological entropy

Let's look at the orbits of f after $n-1$ iterations the sets of n points $x, f(x), f^2(x), ...f^n(x)$. Now consider the set

$$x, f(x), f^2(x), ...f^{n-1}(x) \qquad \text{for all} \quad x \in X$$

If n is large, then this is likely to be a pretty big set. However there are fixed points then the set is not as big a set. If there is a fixed point of f^{n-1} (a periodic point) then the above set is also not as big a set.

Define the number $S(n, \epsilon, f)$ to be the minimal natural number n such that there exist points $y_1, ..., y_n$ such that for every $x \in X$, there is a point y_j in $y_1, \ldots, y_n$ such that

$$d(f^i(x), f^i(y_j)) < \epsilon, \quad 0 \leq i \leq n-1$$

This is the minimal number of points we need to keep close to all orbits iterated $n-1$ times. In other words n is the number of points spanning the space we need to make sure that every orbit is near to at least one of them.

The orbits contain separated periodic points. So $S(n, \epsilon, f)$ for small ϵ is **at least the number of individual fixed points** of f^{n-1}. This implies that $S(n, \epsilon, f)$ is sensitive to the **number of periodic orbits with period** n. Note the number of fixed points counted includes fixed points for $f(x)$ or cycles that are integer dividers of $n-1$. In other words if n is 4 then the number S must include period 2 points and fixed points as well as period 4 points.

The number $S(n, \epsilon, f)$ is the number of words of length n needed to encode all points in terms of the behavior of n consecutive points in their orbit.

We can define a topological entropy as

$$h(\epsilon) = \sup_{n \to \infty} \frac{\log S(n, \epsilon, f)}{n}$$

for small ϵ or

$$h \equiv \lim_{\epsilon \to 0} \sup_{n \to \infty} \frac{\log S(n, \epsilon, f)}{n}$$

Topological entropy can also be understand as the number of separated regions for n iterations of the map. This can be described in terms of coverings of the space. We could restrict our study of the coverings and orbit sets to contain only periodic points to better understand the idea of complexity or topological entropy.

Topological entropy can be estimated from N_m, the number of periodic orbits of period m

$$h = \lim_{m} \frac{1}{m} \log N_m. \tag{1}$$

Topological entropy is consistent with the idea that information is lost as the map is iterated. Or equivalently how much information about a point is required to determine the location of previous iterations of the map. Or equivalently how precisely you need to define the neighborhood of a point to determine where it and its neighbors goes a number of iterations later.

Topological conjugacy is a useful equivalence relation. Two maps that are topologically conjugate have the same number and type of periodic points. If a map can be constructed establishing topological conjugacy for an unclassified map to a known system, then the behavior of the unclassified map can be learned. It make be possible to iteratively, using perturbation expansions construct the map. For example a circle map can in some cases be topologically conjugate to a rotation and in this case a perturbative approach to finding the map between the original system and the rotation should converge. This another setting that is considered part of KAM theory.

An alternate definition of topological entropy is that of Adler, Konheim, and McAndrew which is based on limits of minimum numbers of elements needed to cover intersections of the space and maps of the space.

5 The Baker map

The Baker map is a prototype for 2D chaotic dynamical systems. See Figure 5. On $x \in [0, 1]$ and $y \in [0, 1]$

$$(x_{n+1}, y_{n+1}) = \begin{cases} (cx_n, 2y_n) & \text{for} \quad y_n \leq 1/2 \\ (1 + c(x_n - 1), 1 + 2(y_n - 1)) & \text{for} \quad y_n > 1/2 \end{cases} \tag{2}$$

Or

$$x_{n+1} = \begin{cases} cx_n & \text{for} \quad y_n \le 1/2 \\ 1 + c(x_n - 1) & \text{for} \quad y_n > 1/2 \end{cases} \qquad y_{n+1} = \begin{cases} 2y_n & \text{for} \quad y_n \le 1/2 \\ 1 + 2(y_n - 1) & \text{for} \quad y_n > 1/2 \end{cases}$$

The form of the map depends on the critical line $y = 1/2$. We assume that $0 < c \le 1/2$. Looking at the y part of the map alone. The map is a sawtooth and so reminds us of the shift map on the unit interval. We already know that this map is 'chaotic'. The x part of the map is more complicated as it depends on the y value.

Remark The Baker map, and the Smale horseshoe map are topologically conjugate. The area preserving Baker map is topologically conjugate to the Arnold cat map. The 1D tent map and the shift map are not topologically conjugate. All these statements should be checked!!!!

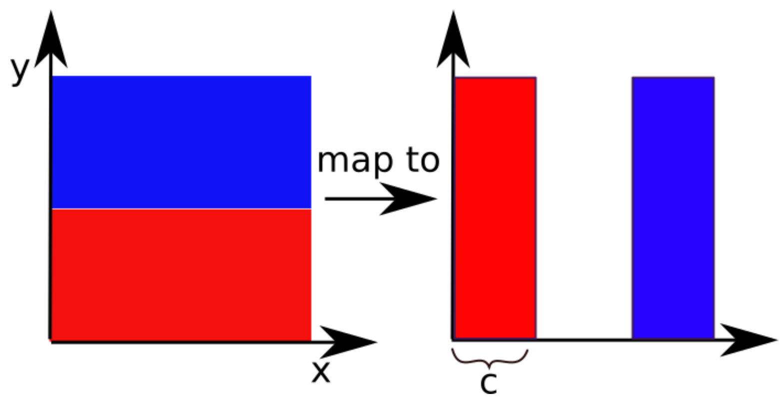

Figure 5: Illustration of the Baker map. The y direction is stretched while the x direction is compressed. The Lyapunov exponent is determined from the stretching in the y direction. The lower left and right most corners are hyperbolic fixed points of the map.

The origin $(0, 0)$ and $(1, 1)$ are fixed points. We will show that these two points are *hyperbolic* fixed points.

Linearizing the map about these two points Near the origin

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Near $(1, 1)$

$$\begin{pmatrix} x_{n+1} - 1 \\ y_{n+1} - 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_n - 1 \\ y_n - 1 \end{pmatrix}$$

The Jacobian matrix for transformations near both fixed points

$$J = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix}$$

The eigenvalues of this matrix are $2, c$. The determinant of the matrix is $J = 2c$ which is 1 if $c = 1/2$. The map is area preserving if the determinant of the matrix is 1 (and when $c = 1/2$). For $c < 1$ one direction shrinks volume and the other stretches volume. With $c < 1/2$ there is continual shrinking of volume.

5.1 Classification of fixed points in a linear two-dimensional map

Writing the map in this form

$$\begin{aligned} x_{n+1} &= f_x(x_n, y_n) \\ y_{n+1} &= f_y(x_n, y_n) \end{aligned}$$

Suppose we have a fixed point $x*, y*$ of a two-dimensional map Compute the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}$$

and evaluate it at the fixed point $x*, y*$. Near the fixed point the map looks like

$$\begin{pmatrix} x_{n+1} - x_* \\ y_{n+1} - y_* \end{pmatrix} = \mathbf{J} \begin{pmatrix} x_n - x_* \\ y_n - y_* \end{pmatrix}$$

Using a characteristic equation we can solve for eigenvalues of J, λ_1, λ_2. The size of the eigenvalues is important. If the eigenvalue is negative then Δx will oscillate back and forth. The distance away from the fixed point will grow if the absolute value of an eigenvalue is greater than 1.

1. If the eigenvalues are both real and $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then the map is expanding. Orbits are exponentially diverging away from the fixed point with distance along directions determined by the eigenvectors. This type of fixed point is also called a *node repeller*.

2. If both eigenvalues are real and absolute value less than 1. The fixed point is called a *node attractor*. Orbits are exponentially decaying toward the fixed point.

3. If both eigenvalues are real but one has absolute value greater than 1 and the other has absolute value less than 1, then the point is *hyperbolic*. There is one eigenvector along which the orbit decays to the fixed point. Along the other direction the trajectory moves outwards.

4. If the determinant of J is 1 then the map is area (or volume) preserving.
5. If both eigenvalues are complex then the fixed point is said to be *elliptic*. Orbits circulate about the fixed point.
6. If the fixed points have complex parts but their real part has absolute value less than 1 then the node is a *spiral attractor*.
7. if the fixed points have complex parts but their real part has absolute value greater than 1 then the node is a *spiral repeller*.
8. If the matrix is degenerate and their is only eigenvector then all trajectories are parallel to a single eigendirection.

Definition A **hyperbolic** fixed point has Jacobian matrix with two real eigenvalues and one of them has absolute value greater than 1.

Remark The classification of fixed points in a two-dimensional map is similar but not identical to that of a two-dimensional linear dynamical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. In that case the sign of the eigenvalue is important and determines whether trajectories exponential diverge from or converge to the fixed point.

5.2 Hyperbolic points in the Baker map

Recall for the Baker map, the eigenvalues for the fixed points at $0, 0$ and $1, 1$ were $2, c$. Because there is an eigenvalue greater than 1, there is a trajectory moving away from the fixed point exponentially fast. The fixed points at (0,0) and (1,1) are both hyperbolic.

5.3 Lyapunov exponent of the Baker map

The Baker map looks exactly like the shift map in the y direction. In fact the map in the y direction is independent of the x value. The Lyapunov exponent is equivalent to that of the shift map and is $\lambda = \ln 2$.

5.4 Periodic orbits in the Baker map

How many points, N_m are fixed points of the map B^m? There are two points that are fixed in the map (those at the corners). So $N_1 = 2$.

We can write the Baker map in terms of two linear transformations. From equation 2

$$\begin{aligned} B_-(\mathbf{x}) &= \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \\ B_+(\mathbf{x}) &= \begin{pmatrix} 1-c \\ -1 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \end{aligned}$$

The transformation B_- is used for $y \leq 1/2$ and that for B_+ for $y > 1/2$. A period two point is a fixed point of either B_-^2 or B_+^2 or B_+B_-. If we evaluate B_+^2 we find that the only fixed point is at (1,1). Likewise the only fixed pint of B_-^2 is the origin.

$$B_+B_-(\mathbf{x}) = \begin{pmatrix} 1-c \\ -1 \end{pmatrix} + \begin{pmatrix} c^2 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x}$$

This has fixed point $1/(c+1), 1/3$.

There is one 2-cycle giving two more fixed points for B^2. Altogether including the two fixed points of B there are four points that are fixed for B^2 so $N_2 = 4$.

We can similarly count the number of 3-cycles. There are two 3-cycles each giving 3 fixed points of B^3. Altogether giving $N_3 = 6 + 2 = 8$ fixed points of B^3 (we don't need the 2-cycles for B^3). And so on for other cycles. The number of fixed points of the map iterated m times or B^m is

$$N_m = 2^m.$$

5.5 Topological entropy from the number of periodic points

The quantity called *topological entropy* can be computed from the number of periodic points of the map. The number of fixed points of the map iterated m times is N_m. The topological entropy is

$$h = \lim_{m\to\infty} \frac{1}{m} \ln N_m$$

And this implies that

$$N_m \sim e^{hm}$$

The topological entropy is a measure of complexity of the attractor. Thus the topological entropy is the average (per iteration) amount of information needed to describe long iterations of the map.

We compute $\log N_m = \log 2^m = m \ln 2$ for the Baker map. The topological entropy defined with a natural log gives $h = \ln 2$.

5.6 KS entropy of the Baker map

Suppose we partition the space into two pieces a, b and consider the regions $B(a)$, $B(b)$, where B is the Baker map. We also look at the regions $B^2(a), B^2(b)$ and $B^3(a), B^3(b)$.

After $j-1$ iterations of the map, we partition the space into regions. Each region is labelled by a string that is j letters long. Each letter is either a or b. The i-th letter is a if the region was in a at the ith iteration of the map. The first letter gives the initial location. For example the regions are shown in the bottom row of Figure 6 for 3 iterations of the map. The strings look like *aaab* or *abba* or *bbaa*. There are 16 possible strings to label 16 different regions. Each of the areas is $1/2^n$ of the total area after $n-1$ iterations.

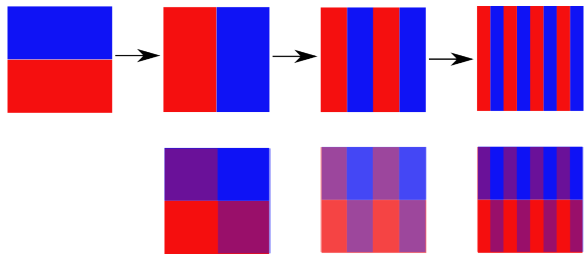

Figure 6: In the top row we show 3 iterations of the Baker map. Region a is in red and region b is in blue. In the bottom row we count partitions given by intersections of the different regions. After a single iteration there are 4 partitions. After 3 iterations there are 16 partitions. After $n-1$ iterations there are 2^n separate regions, each which have area $1/2^n$.

The KS entropy is

$$h_{KS} = \sup \lim_{n\to\infty} \frac{h_n}{n} \tag{3}$$

The function h_n depends on the areas of the pieces in the n-th partition. The function $h_n = -\sum_j p_j \ln p_j$ where p_j is the area (like a probability) in reach region in the n-th partition. For the Baker map

$$h_n = \sum_{j=1}^{2^n} (-) \left(\frac{1}{2}\right)^n \ln \left(\frac{1}{2}\right)^n = n \ln 2.$$

This gives KS entropy equal to the topological entropy we previously calculated.

5.7 Chaotic attractor for the area contracting Baker map

When $c < 1/2$ the map is not area preserving. Each application of the map shrinks the volume. As each application of the map shrinks the volume where points can go, every orbit must converge eventually onto something that has no area (has measure zero).

Reapplication of the map many times converges to a set. The set has measure zero when $c < 1/2$. The set is called a *chaotic attractor*. Points at the boundaries are fixed or periodic points. The chaotic attractor looks like a Cantor set. There are cycles of all integers and these are all part of the chaotic attractor (see Figure 7).

I found that to compute more than a hundred iterations of the Baker map I had to increase the precision of the computations above that of float.

The attractor contains all periodic fixed points. Like the Cantor set it has measure zero. Most members of the Cantor set are not endpoints of deleted intervals. Since each

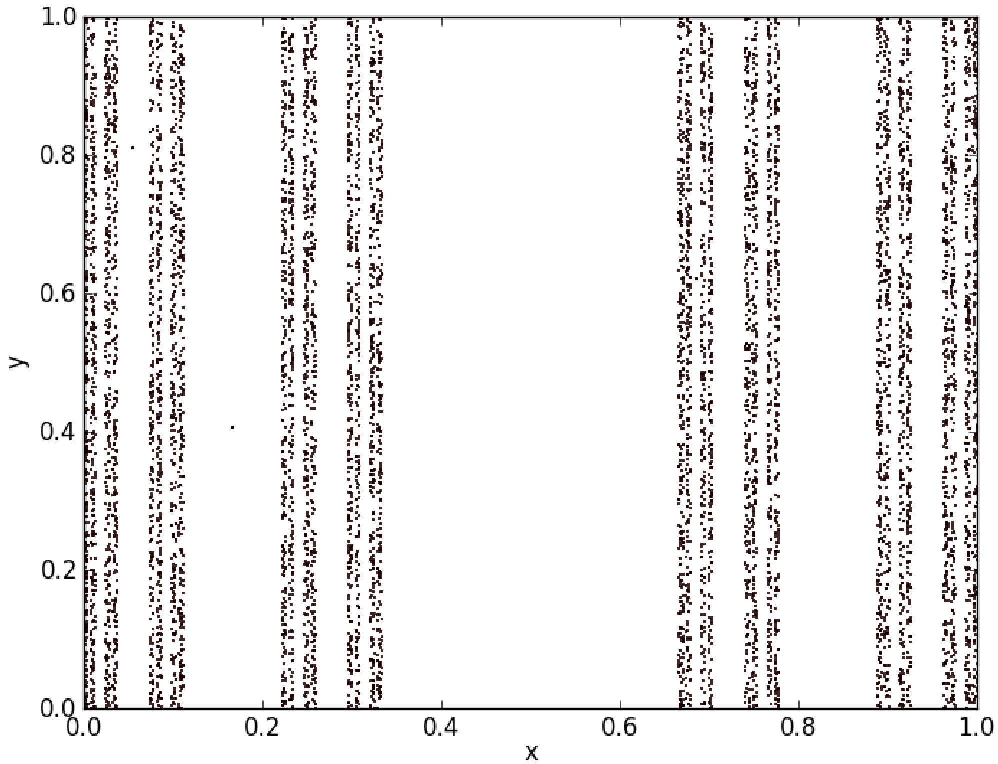

Figure 7: For $c = 1/3$ and initial conditions $x_0 = 0.5$, $y_0 = 2/\pi^2$, 8000 iterations of the Baker map are plotted. I found I had to use a high level of precision. With regular floats in python, the orbit was incorrectly converging on the origin. The self-similar structure of the chaotic attractor can be imagined from this illustration.

step removes a finite number of intervals and the number of steps is countable, the set of endpoints is countable while the whole Cantor set is uncountable. Are there members of the attractor for the Baker map that are not periodic points? (Yes as they can start from irrational initial conditions). And if so does that mean that it is uncountable? (Yes). The number of periodic points is countable.

An *attractor* attracts all orbits starting within a sufficiently small open setting containing the attractor. If the map $c = 1/2$ is area preserving then all orbits shuffle things around and orbits never decay to anything. An attractor is incompatible with area preservation and can only be possible in physical systems that are dissipative. An attractor cannot contain any repelling fixed points.

The Baker map has provided a setting for calculating various types of dimensions, Hausdorff, information, pointwise, fractal and Lyapunov dimensions for the attractor. For example, the Hausdorff dimension is the exponent used to describe how the number of balls needed to cover the set scales with the size of the balls.

What is the Hausdorff dimension of the Baker map's attractor?

6 Notes

Following 'A First Course in Discrete Dynamical Systems' by Richard A. Holmgren (1994). The Baker map discussion is from "Chaotic Dynamics" by Tel and Gruiz and currently has mistakes in the discussion of period three orbits. The Li-Yorke definition of chaos is very unclearly written. This whole set needs clarification, more examples, particularly in ergodic two-d systems and some problems.