PHY411 Lecture notes - Resonances

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1 What is a resonance?

Resonances can be important when perturbations are weak. In fact when perturbations are extremely weak, resonances are the only locations where perturbation can over long periods of time constructively add and so be important. In celestial mechanics, planets are much less massive than the Sun. Jupiter, the largest planet in our Solar system is only 1/1000-th the mass of the Sun. Nevertheless regions in the inner asteroid belt that are resonant with Jupiter (and seen in the semi-major axis distribution where they are called Kirkwood gaps) are almost devoid of asteroids. In contrast, most asteroids in the outer asteroid belt are resonant with Jupiter.

The angular rotation rate of an asteroid as it goes around the Sun is approximately the mean motion n. The mean motion for Jupiter we call n_J. A mean motion resonance is a condition

$$in \sim jn_J$$

for integers i, j. This condition is one where we expect the perturbations by Jupiter can strongly affect an asteroid. Otherwise over a long period of time the perturbations will be out of phase and will average to zero. Integrate the above equation to obtain an angle

$$\phi = i\lambda - j\lambda_J \sim \text{constant}$$

where λ, λ_J are angles known as mean anomalies and $\dot{\lambda} = n$ and $\dot{\lambda}_J = n_J$. A commensurability can be described in terms of a slowly moving angle. We might see two different types of motion depending upon wether the angle increases or decreases or stays near a particular value. We describe the motion as librating when the angle oscillates about a fixed value

or as circulating when the angle keeps increasing or decreasing. A low dimensional system that illustrates these two types of behavior is the pendulum model.

An integrable model

$$H(p,\phi) = H_0(p) + \epsilon f(p)\cos(\phi) \tag{1}$$

where ϕ is an angle. We can expand the unperturbed function H_0 about a particular momentum value, p_0. Let $J = p - p_0$. The transformation $p, \phi \to J, \phi$ is canonical.

To second order in J

$$H_0(J) = \frac{a}{2}J^2 + bJ + H_0(p_0)$$

with

$$b = \frac{\partial H_0}{\partial p}(p_0)$$

$$a = \frac{\partial^2 H_0}{\partial p^2}(p_0)$$

Neglecting the constant $H_0(p_0)$, the new Hamiltonian

$$H(J,\phi) = \frac{a}{2}J^2 + bJ + \epsilon g(J)\cos(\phi) \tag{2}$$

where $g(J) = f(J + p_0)$. In the case of $g(J) = 1$, this system is the pendulum except shifted in J by a frequency b. This can be seen with a canonical transformation shifting J or $J = P - b$ or by plotting the Hamiltonian level curves on a J, ϕ coordinate plot.

Why is b a frequency? This follows as ϕ is an angle, $\frac{\partial(bJ)}{\partial J} = b$ and Hamilton's equations give $\dot{\phi} = \frac{\partial H}{\partial J}$.

1.1 Dangers of low order approximations

Again consider our Hamiltonian model

$$H(p,\phi) = H_0(p) + \epsilon f(p)\cos(\phi)$$

Suppose we only expanded H_0 to first order in p about p_0, again with $J = p - p_0$. Then we would have neglected the quadratic term and we would find

$$H(J,\phi) = bJ + \epsilon g(J)\cos\phi$$

Let us look for fixed points.

$$\dot{J} = -\frac{\partial H}{\partial \phi} = \epsilon g(J)\sin\phi = 0$$

We find that $\phi = 0, \pi$ at a fixed point.

$$\dot{\phi} = \frac{\partial H}{\partial J} = b + \epsilon g'(J)\cos\phi = b \pm \epsilon g'(J)$$

where on the last step the plus sign corresponds to $\phi = 0$ and the minus sign for $\phi = \pi$. If $g(J) =$ constant and $b \neq 0$ then there are no fixed points! But we know that the system expanded to p^2 (equation 2 is like a pendulum. When a is neglected, we loose some important aspects of the dynamics.

If $g(J)$ is not constant then can consider inverting $g'(J)$. For example if $g(J) = J^{1/2}$ and

$$H(J, \phi) = bJ + \epsilon J^{1/2} \cos \phi$$

then the fixed point at π occurs at $b = \frac{\epsilon}{2J^{1/2}}$ or

$$J_{fix} = \left(\frac{\epsilon}{2b}\right)^2 \tag{3}$$

For small values of b (near resonance) we find only a single fixed point at large J.

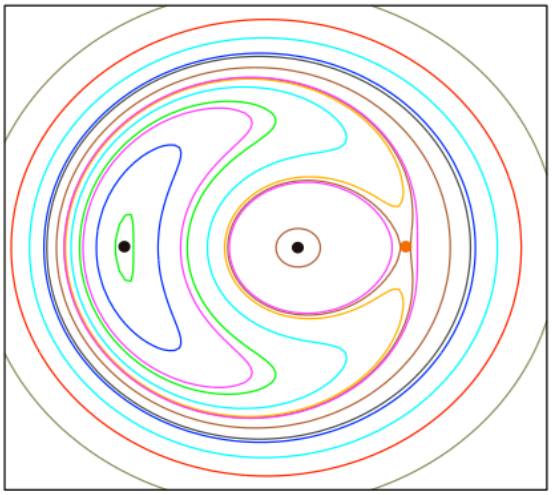

Figure 1: Level curves of Hamiltonian $H(J, \phi) = J^2/2 + \delta J + \epsilon J^{1/2} \cos \phi$ in the coordinate system with $x = \sqrt{2J} \cos \phi$, $y = \sqrt{2J} \sin \phi$. Here we have chosen $\delta < 0$ so that there are three fixed points, all on the x axis. The rightmost one (shown as an orange point) is unstable.

We can transform coordinates with the canonical transformation

$$x = \sqrt{2J} \cos \phi \qquad y = \sqrt{2J} \sin \phi$$

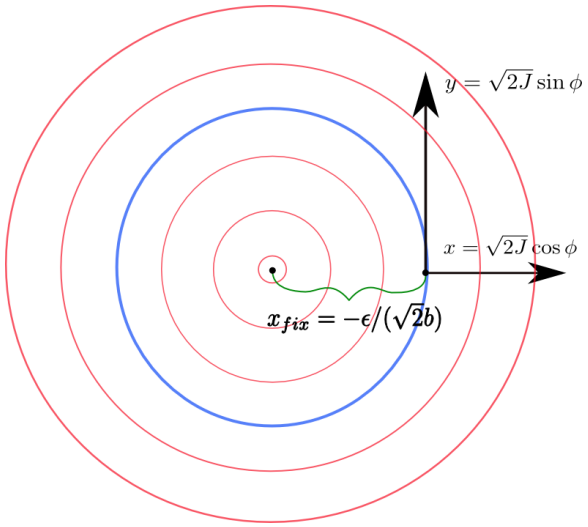

Figure 2: Level curves of Hamiltonian $H(J, \phi) = bJ + \epsilon J^{1/2}\cos\phi$ in the coordinate system with $x = \sqrt{2J}\cos\phi$, $y = \sqrt{2J}\sin\phi$. The system is equivalent to a harmonic oscillator but at position $x = -\epsilon/(\sqrt{2}b), y = 0$. The shift diverges as $b \to 0$. The divergence is an artifact of the low order of approximation.

giving

$$H(x,y) = \frac{b}{2}(x^2+y^2) + \frac{\epsilon x}{\sqrt{2}}$$

The fixed point at $\phi_{fix} = \pi, J_{fix} = \sqrt{\frac{\epsilon}{2b}}$ is at

$$x_{fix} = -\frac{\epsilon}{\sqrt{2}b} \qquad y_{fix} = 0$$

We can shift the coordinate system so that $x' = x - x_{fix}$. Inserting this into the new Hamiltonian we find

$$H(x',y) = \frac{b}{2}(x'^2+y^2) + \text{constant}$$

The new Hamiltonian is just like a harmonic oscillator! Again we have lost some important aspects of the dynamics as with the quadratic term we would have found stable and unstable fixed points.

If b is small then in our original coordinate system, the distance to the fixed point is large. With initial condition $x, y = 0$, the orbit is a large circle with radius $2x_{fix}$. An extremely large response in J occurs when b is small, or near resonance. And the limit $b \to 0$ gives an infinite response in J (see equation 3). This too is an artifact of the low order of approximation. When the quadratic term is added, initial conditions near the origin do not have orbits that diverge as $b \to 0$.

We can regard a resonance as a setting where there is a relatively large response. Here, the distance to resonance is b and the resonance described by the pendulum Hamiltonian.

1.2 A resonance is a commensurability

Now consider a multiple dimensional system with a perturbation in the form $\cos(\mathbf{k}\cdot\boldsymbol{\theta})$ where $\mathbf{k}$ is a vector of integers. We assume that the unperturbed Hamiltonian is *integrable*;

$$H(\mathbf{p},\boldsymbol{\theta}) = H_0(\mathbf{p})$$

and expand this about initial momenta $\mathbf{p}_0$.

$$H_0(\mathbf{p}) = H_0(\mathbf{p}_0) + \boldsymbol{\nabla}H_0(\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0) + \frac{1}{2}(\mathbf{p}-\mathbf{p}_0)^t\mathbf{M}(\mathbf{p}-\mathbf{p}_0)$$

where $\mathbf{M}$ is the Hessian matrix comprised of second derivatives evaluated at $\mathbf{p}_0$. With $\mathbf{J} = \mathbf{p} - \mathbf{p}_0$

$$H_0(\mathbf{J}) = H_0(\mathbf{p}_0) + \boldsymbol{\nabla}H_0(\mathbf{p}_0)\mathbf{J} + \frac{1}{2}\mathbf{J}^t\mathbf{M}\mathbf{J}$$

We identify

$$\boldsymbol{\omega} = \boldsymbol{\nabla}H_0(\mathbf{p}_0)$$

Because $\dot{\boldsymbol{\theta}} = \boldsymbol{\nabla}_J H(\mathbf{J}, \boldsymbol{\theta})$, the frequencies $\boldsymbol{\omega}$ are the angular rotation rates $\dot{\boldsymbol{\theta}}$ so we can write

$$H_0(\mathbf{J}) = H_0(\mathbf{p}_0) + \boldsymbol{\omega}\mathbf{J} + \frac{1}{2}\mathbf{J}^t\mathbf{M}\mathbf{J}$$

We can think of a *resonance* as a condition

$$\mathbf{k} \cdot \boldsymbol{\omega} \sim 0 \tag{4}$$

We can construct an angle

$$\phi = \mathbf{k} \cdot \boldsymbol{\theta} \tag{5}$$

The resonance condition (equation 4) implies that

$$\dot{\phi} = \mathbf{k} \cdot \dot{\boldsymbol{\theta}} = \mathbf{k} \cdot \boldsymbol{\omega} \sim 0$$

so ϕ is a *slow moving angle*.

We now consider a full Hamiltonian that is the sum of a part that only depends on momenta and one that is also a function of angles

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + H_1(\mathbf{J}, \boldsymbol{\theta}) \tag{6}$$

We often assume that H_1 is small. The perturbation term H_1 arises from planet-planet interactions in celestial mechanics or bar/spiral perturbations in the galactic disk setting. The perturbation term can be expanded in Fourier series.

For the moment we focus on a single cosine term for H_1. Taking H_0 expanded to second order, and H_1 comprised of a single cosine, the full Hamiltonian

$$H(\mathbf{J}, \boldsymbol{\theta}) = \boldsymbol{\omega}\mathbf{J} + \frac{1}{2}\mathbf{J}^t\mathbf{M}\mathbf{J} + \epsilon\cos(\mathbf{k} \cdot \boldsymbol{\theta}) \tag{7}$$

where we have dropped the constant.

Using a generating function of old coordinates $\boldsymbol{\theta}$ and new momenta I_1 and I_i for $i \in 2...N$

$$F_2 = (\mathbf{k} \cdot \boldsymbol{\theta})I_1 + \sum_{i=2..N} I_i\theta_i$$

we find new coordinates

$$\begin{aligned} \frac{\partial F_2}{\partial I_1} &= \mathbf{k} \cdot \boldsymbol{\theta} = \phi \\ \frac{\partial F_2}{\partial \theta_1} &= k_1 I_1 = J_1 \end{aligned}$$

and for $i > 1$

$$\begin{aligned} \frac{\partial F_2}{\partial I_i} &= \theta_i = \phi_i \\ \frac{\partial F_2}{\partial \theta_i} &= k_i I_1 + I_i = J_i \end{aligned} \tag{8}$$

Inserting these back into the Hamiltonian

$$H(\mathbf{I},\boldsymbol{\phi}) = I_1^2 \sum_{i,j} \frac{1}{2} k_i M_{ij} k_j + I_1 \left(\boldsymbol{\omega}\cdot\mathbf{k} + \sum_{i>1} k_i M_{ij} I_j \right) + \epsilon\cos\phi + \sum_{i>1} \omega_i I_i + \frac{1}{2}\sum_{i,j>1} I_i M_{ij} I_j \tag{9}$$

The Hamiltonian only depends on a single angle ϕ conjugate to I_1. This means all the other I_i are conserved. The Hamiltonian reduces to

$$H(I,\phi) = \frac{aI^2}{2} + bI + \epsilon\cos\phi + \text{constant}$$

where b and the constant depend on the conserved quantities and the coefficients a, b can be read off equation 9. If the I are small (and we started by expanding about some value of momentum so they should be) then

$$b \sim \mathbf{k}\cdot\boldsymbol{\omega}$$

is a frequency that describes the distance to resonance. Conserved quantities in equation 8 can be written in terms of the old mometa

$$I_i = J_i - k_i I_1 = J_i - \frac{k_i}{k_1} J_1. \tag{10}$$

The momentum J_1 varies due to the resonance. The other momenta J_i must vary along with J_1. Correlated changes between momenta can help identify a resonance.

- If there is a single resonant perturbation, then we can transform the Hamiltonian so that it resembles a pendulum, and the system is integrable.
- The above transformation is robust even if the resonant angle is slow or if $\mathbf{k}\cdot\boldsymbol{\omega} \sim 0$.
- The transformation allows you to calculate the coefficients a, b. These can be used to estimate a resonant width (in momentum $\sqrt{\epsilon/a}$), a libration frequency in resonance ($\sqrt{a\epsilon}$) and a distance to resonance (b).
- If there are more than one resonant perturbations then it is usually not possible to transform the system to a 1 dimensional system.
- A resonance can be identified as important by searching for a slowly varying resonant angle and correlated momenta variations arising from the conserved quantities.

1.3 Removal of perturbations with fast angles and the small divisor problem

Now let us consider the same Hamiltonian

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta}) \tag{11}$$

Now we consider the situation of small ϵ. Because ϵ is small we try a canonical transformation that is near the identity transformation. A canonical transformation that is near the identify is caused by the generating function

$$F_2(\boldsymbol{\theta}, \mathbf{I}) = \mathbf{I} \cdot \boldsymbol{\theta}$$

A near identity canonical transformation with generating function of old coordinates and new momenta

$$F_2(\boldsymbol{\theta}, \mathbf{I}) = \mathbf{I} \cdot \boldsymbol{\theta} - \frac{\epsilon}{\mathbf{k} \cdot \boldsymbol{\omega}} \sin(\mathbf{k} \cdot \boldsymbol{\theta})$$

giving relations between old coordinates $(\boldsymbol{\theta}, \mathbf{J})$ and new coordinates $(\boldsymbol{\theta}', \mathbf{I})$

$$\begin{aligned} \frac{\partial F_2}{\partial I} &= \boldsymbol{\theta} = \boldsymbol{\theta}' \\ \frac{\partial F_2}{\partial \theta} &= \mathbf{I} - \frac{\epsilon \mathbf{k}}{\mathbf{k} \cdot \boldsymbol{\omega}} \cos(\mathbf{k} \cdot \boldsymbol{\theta}) = \mathbf{J} \end{aligned} \tag{12}$$

Inserting these into the old Hamiltonian (equation 11) gives

$$\begin{aligned} H(\mathbf{I}, \boldsymbol{\theta}) &= H_0 \left(I - \frac{\epsilon \mathbf{k}}{\mathbf{k} \cdot \boldsymbol{\omega}} \cos(\mathbf{k} \cdot \boldsymbol{\theta}) \right) + \epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta}) \\ &= H_0(\mathbf{I}) - \frac{\epsilon \boldsymbol{\nabla} H_0(\mathbf{I}) \cdot \mathbf{k}}{\mathbf{k} \cdot \boldsymbol{\omega}} \cos(\mathbf{k} \cdot \boldsymbol{\theta}) + \epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta}) \end{aligned}$$

Where in the second step we have expanded H_0 to first order in ϵ. As long as we associate $\omega = \boldsymbol{\nabla} H_0(\mathbf{I})$ then we can arrange for the $\epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta})$ term to be cancelled.

- As $\boldsymbol{\omega}$ depends on $\mathbf{J}$ or $\mathbf{I}$ we should have taken into account its dependence in performing the canonical transformation. However to first order in ϵ the transformation is correct.
- The canonical transformation can be used to remove non-commensurate or non-resonant perturbation terms from a Hamiltonian. The behavior of the system in the new coordinate system is calculated easily, and the behavior in the old coordinates predicted using the transformation. We only considered a single perturbation term added to H_0 but any perturbation can be expanded in Fourier series with $\mathbf{k}$ a vector of integers. Because non-resonant terms (containing fast angles) can be removed from the Hamiltonian (via canonical transformation), they are often ignored altogether.

- The new momenta are close to the old ones (equation 12) as long as the frequency $\mathbf{k} \cdot \boldsymbol{\omega}$ is fast (not small). Fast angles can be ignored. Another way to say this is that fast perturbations on average do not affect the dynamics.

- If the frequency is not small then the new momenta are not close to the old ones and the expansion fails. This is the *small divisor problem*. Attempts to remove the perturbations with a near identity expansion fail if there are any small divisors (or slow frequencies or commensurability's) in the problem.

1.4 Examples of resonances: Lindblad and Mean-motion resonances

In the setting of nearly circular orbits in the mid plane of a disk galaxy we can describe orbits with

$$H_0(\theta, L; \theta_r, J_r) \approx g_0(L) + \kappa(L) J_r$$

where L its the angular momentum, θ is azimuthal galaxy in the mid plane of the galaxy, J_r is the action variable associated with epicyclic oscillations and θ_r the angle associated with these oscillations. Here $\dot{\theta} = \Omega(L) = g_0'(L)$ is the angular rotation rate of a star in a circular orbit and $\dot{\theta}_r = \kappa(L)$ is the epicyclic oscillation frequency. Here L sets the mean radius of the orbit.

A bar or spiral arm pattern can rotate through the galaxy inducing a perturbation in the gravitational potential that rotates with angular rotation rate (pattern speed), Ω_b. Lindlblad resonances are locations (radii) in the galaxy where

$$\kappa \sim \pm m(\Omega - \Omega_b)$$

with m an integer.

In the setting of an asteroid and a planet orbiting the Sun. The mean motion (approximately the angular rotation rate) of the planet we denote n_p and that of the asteroid n_a. A mean motion resonance is a radial location where

$$j n_p \sim k n_a$$

with j, k integers.

For both of these examples the resonance condition can be written as

$$\mathbf{k} \cdot \boldsymbol{\omega} \sim 0$$

with $\mathbf{k}$ a vector of integers and $\boldsymbol{\omega}$ a vector of frequencies.

While the pendulum Hamiltonian looks like this

$$H(p, \theta) = \frac{p^2}{2} + \epsilon \cos\theta$$

Lindblad and first order mean motion resonances are often modeled with the Andoyer Hamiltonian

$$H(p,\theta) = \frac{p^2}{2} + bp + \epsilon p^{\frac{1}{2}} \cos\theta$$

Here first order refers to an expansion in eccentricity and b gives a distance to resonance. Where does the $p^{1/2}$ come from?

If we consider a time dependent potential perturbation (from a nearby planet on an asteroid or a bar perturbation in a galaxy) a Fourier component of the potential $V_m \sim f(R)\cos(m(\theta - \Omega_p t)$ where R is a radius. Radial motions can be approximated as motion about a circle with an action variable J_r that is related to radius $\delta R \propto \sqrt{2J_r/\kappa}\cos\phi$ where ϕ is the epicyclic angle and κ the epicylic frequency. If we then take $R = R_0 + \delta R$ and expand the potential to first order term in δR that term depends on $J^{\frac{1}{2}}$. Likewise first order expansions in eccentricity scale with $\Gamma^{\frac{1}{2}}$ where Γ is a Poincaré or Delaunay action variable and conjugate to an angle sensitive to the angle of pericenter.

2 The periodically varying pendulum

Above we showed that a system perturbed by a single resonance is integrable. But a system with two resonant perturbations might not be integrable. The periodically varying pendulum can be written as if the Hamiltonian contains three resonant perturbations and so lets us explore the situation when two nearby resonances make the system non-integrable (aka chaotic). Another reason to study this system is that it is a dynamical system that gives a map from the plane onto itself, and introducing the connection between dynamical systems and Poincaré maps.

The Hamiltonian for a pendulum

$$H(p,\phi) = \frac{p^2}{2} - \epsilon\cos\phi$$

We know that fixed points occur at $p = 0$ and $\phi = 0, \pi$ For ϵ positive the stable fixed point is at $\phi = 0$. The energy on the separatrix $E = H(0,\pi) = \epsilon$. To estimate the width of the separatrix we now look for p at $\phi = 0$ with this energy $\frac{p^2}{2} - \epsilon = \epsilon$, giving peaks at

$$p_{sep} = \pm 2\sqrt{\epsilon}.$$

These are the high and low points of the separatrix.

A pendulum that has length that oscillates with period $P_\nu = 2\pi/\nu$

$$\begin{aligned} H(p,\phi,t) &= \frac{p^2}{2} - \epsilon\left(1 + \frac{2\mu}{\epsilon}\cos(\nu t)\right)\cos\phi \\ &= \frac{p^2}{2} - \epsilon\cos\phi - \mu\cos(\phi + \nu t) - \mu\cos(\phi - \nu t) \end{aligned} \tag{13}$$

Each of these terms can be thought as an individual resonance. To explain this we can consider each term separately. Suppose we have a Hamiltonian with only one cosine term but it is one of the time dependent terms

$$H(p, \phi, t) = \frac{p^2}{2} - \mu \cos(\phi - \nu t)$$

We perform a time dependent canonical transformation

$$F_2(\phi, p', t) = (\phi - \nu t)p'$$

giving new coordinate and momentum

$$\phi' = \phi - \nu t \qquad p' = p$$

with new Hamiltonian (gaining a term $\frac{F_2}{\partial t} = -\nu p$)

$$\begin{aligned} H(p, \phi') &= \frac{p^2}{2} - \nu p - \mu \cos \phi' \\ &= \frac{1}{2}(p - \nu)^2 - \nu^2 - \mu \cos \phi' \end{aligned}$$

We can perform another canonical transformation

$$F_4(P, \phi'') = (p - \nu)\phi''$$

giving

$$P = p - \nu \quad \phi'' = \phi'$$

and a new Hamiltonian

$$H(P, \phi'') = \frac{p^2}{2} - \mu \cos \phi''$$

(with a constant ignored). This looks just like a pendulum and has stable orbits near $P \sim 0$ which is equivalent to $p \sim \nu$. So the system essentially looks just like a pendulum but with stable fixed point shifted so that it is centered at a different p.

Going back to the Hamiltonian in equation 13 we find that there are three resonances, one at $p = 0$ (with width set by ϵ), and the others at $p = \pm\nu$ each with width set by μ.

2.1 A system with periodically varying parameters

Let us neglect one of the resonances from the forced pendulum model and consider a time dependent perturbation with just a single perturbation

$$H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos \phi - \mu \cos(\phi - \nu t) \tag{14}$$

Hamilton's equations give

$$\begin{aligned} \dot{\phi} &= p \\ \dot{p} &= -\epsilon \sin\phi - \mu(\sin\phi - \nu t) \end{aligned} \tag{15}$$

This system is in the class of systems with periodically varying parameters

$$\dot{\mathbf{x}} = f(\mathbf{x}, t)$$

with

$$f(\mathbf{x}, t+T) = f(\mathbf{x}, t)$$

and $\mathbf{x} = (\phi, p)$. For our system with Hamiltonian given by equation 14,

$$T = 2\pi/\nu.$$

We can look at the map g^T

$$(p(t), \phi(t)) \xrightarrow{g^T} (p(t+T), \phi(t+T))$$

that is induced by the time dependent Hamiltonian. This is called mapping at a period, and it sends a position on the plane p, ϕ to another position on the plane. This map can be iterated over and over again.

A fixed point in the map g^T is a position $\mathbf{x} = p, \phi$ such that $g^T(\mathbf{x}) = \mathbf{x}$. This point corresponds to a periodic orbit in the time dependent Hamiltonian system.

2.2 The system is equivalent to a 2 dimensional Hamiltonian system

The Hamiltonian in equation 14 is a function of three variables q, ϕ, t. Phase space is 2 dimensional. Because the Hamiltonian is a function of time, energy is not conserved. This system is equivalent to a Hamiltonian system that is time independent but on a 4 dimensional phase space.

$$K(p, \phi; J, \theta) = \frac{p^2}{2} - \epsilon\cos\phi - \mu\cos(\phi - \theta) + J\nu \tag{16}$$

Hamilton's equations give

$$\frac{\partial K}{\partial J} = \dot{\theta} = \nu$$

so that $\theta = \nu t +$ constant. Inserting $\theta = \nu t$ into the Hamiltonian we can see that Hamilton's equations for ϕ, p are the same as for the lower dimensional system (equations 15. Energy conservation is achieved in the higher dimensional system because $J\nu$ varies.

Let us set $\theta = \nu t$ so that $\theta = 0$ at time $t = 0$. The mapping at a period g^T with $T = 2\pi/\nu$, does not change θ because θ has period $T = 2\pi/\nu$. The map g^T sends p, ϕ

to new values while keeping $\theta = 0$. Because total energy is conserved we can compute J from the values of q, ϕ, θ at any later time as long as we know the initial conditions. The equations of motion imply that the new values of p, ϕ are independent of J at any time, though the equations of motion for J depend on the values of p, ϕ, θ. The mapping at a period of the 2D time dependent Hamiltonian is equivalent to a surface of section (choosing a fixed θ for each map) of the 4D time independent Hamiltonian.

In a previous lecture we showed that Hamiltonian flows preserve the symplectic two-form, so g^T preserves the two form

$$\omega = dJ \wedge d\theta + dp \wedge d\phi$$

The map preserves θ, and on the surface with $\theta = 0$ any integral involving $dJ \wedge d\theta$ would be zero. In other words, if we start with a distribution of particles initial that is infinitely thin in θ, it will remain infinitely thin in θ This implies that the two form $dp \wedge d\phi$ on the subspace p, ϕ is conserved. Thus the mapping g^T of p, ϕ is area preserving in the subspace defined by p, ϕ. Because θ advances in time at a steady rate, a mapping at any time g^t is area preserving in the p, ϕ subspace.

So what is special about mapping at g^T?

2.2.1 Why mapping at a period?

Starting with $H(p, q, t)$ with $H(p, q, t+T) = H(p, q, t)$ we can construct a new Hamiltonian in an *extended phase space* by adding a new angle θ and a a new action J conjugate to our new angle θ. We define $\theta = t\nu$ with $\nu = 2\pi/T$. We replace t in the old Hamiltonian with $t = \theta/\nu$. Our new and time independent Hamiltonian in the extended phase space is

$$K(p, q, \theta, J) = H(p, q, t = \theta/\nu) + J\nu$$

We construct surfaces of section using a plane of constant θ. We can do this for *any* time periodic Hamiltonian. We could not replace the time uniquely with an angle if H were not time-periodic. This gives us a map from t to an interval or phase.

The fact that I can construct a constant Hamiltonian in extended phase space and have it depend upon an angle is what makes it possible to think of the orbits mapped at a period as a surface of section. The surface of section gives slices through the orbits in the larger dimensional space. When mapping at a period, θ is the same for each map. If I mapped at a different period I would still get an area preserving map but I would be sampling a different part of the 4D orbit each time as θ would be different for each map.

2.3 Orbits of the mapping at a period

Given an initial conditions $\mathbf{x}_0 = (q_0, p_0)$, at time $t = 0$, the orbit of this point in the 2-dimensional plane (p, q) is the set of points

$$g^T(\mathbf{x}_0), g^{2T}(\mathbf{x}_0), g^{3T}(\mathbf{x}_0), g^{4T}(\mathbf{x}_0)....$$

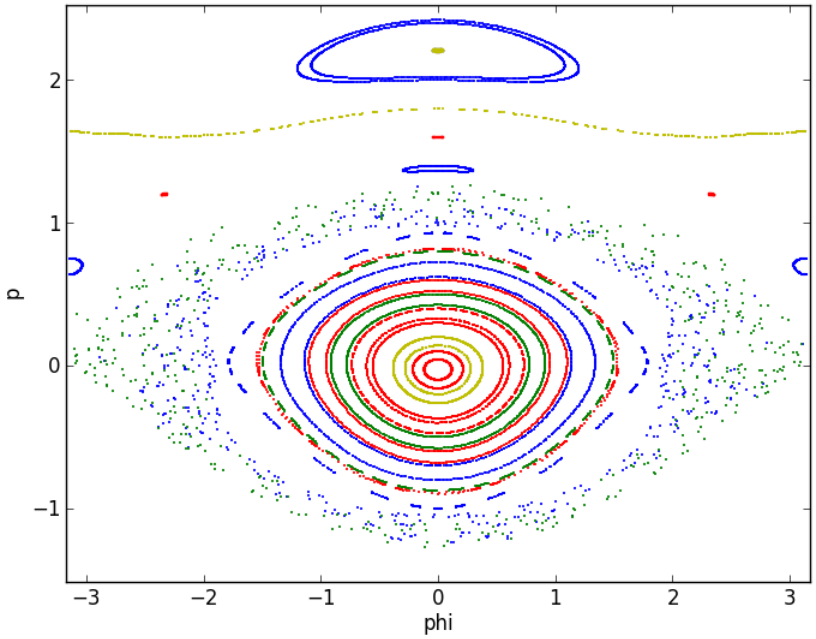

Figure 3: Mapping at a period $T = 2\pi/\nu$ with $\nu = 2$, $\epsilon = 0.4, \mu = 0.05$ and $H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos\phi - \mu \cos(\phi - \nu t)$. Orbits generated from different initial conditions are shown in different colors.

There are three types of orbits in the map at T.

1. Periodic points with $g^{iT}(\mathbf{x}) = \mathbf{x}$ with postive integer i. These correspond to periodic orbits in the 3-dimensional space of q, p, θ. These include fixed points $g^T(\mathbf{x}) = \mathbf{x}$. Periodic points can be classified as hyperbolic or stable depending upon the two eigenvalues of the linearized flow near them.

2. Quasiperiodic orbits, or tori. The orbit is a set of points on a oval. We can call them tori if we think about them in the 3-dimensional space of q, p, θ.

3. Area filling orbits.

The three classes of orbits are seen in the map generated in Figure 3.

If the orbits in the surface of section are lines, there is an additional conserved quantity in the orbit. If the orbits are area filling then there is no additional conserved quantity.

2.4 Lagrangian for the periodically varying pendulum

What is the Lagrangian formalism of this problem like? Starting with the time dependent Hamiltonian (equation 14), taking the Legendre transformation with $p \to \dot{\phi}$ is straightforward giving us a time dependent Lagrangian

$$\mathcal{L}(\phi, \dot{\phi}, t) = \frac{\dot{\phi}^2}{2} + \epsilon \cos\phi + \mu \cos(\phi - \nu t)$$

However if we started with the 2D time independent system(equation 16 with 4D phase space), we have a problem taking the Legendre transform of the second variable as $H \propto J$.

2.5 Poincaré Maps

Definition We start with an N dimensional space and a dynamical system with $\dot{\mathbf{x}} = f(\mathbf{x})$. Let S be an $N-1$ dimensional subspace called a *surface of section*. S must be transverse to the flow so that no trajectories flow in S, all trajectories flow through S. The *Poincaré map* is a mapping from S to itself obtained by following trajectories from one intersection with S to the next (see Figure 4).

Closed orbits in the dynamical system correspond to fixed points in the Poincaré map.

To make a Poincaré map orbits must cross the chosen surface S many times. Not all orbits need cross this surface. It is sometimes necessary to specify the direction that an orbit can cross the plane.

The mapping at a period for the periodically forced pendulum is an example of a Poincaré map!

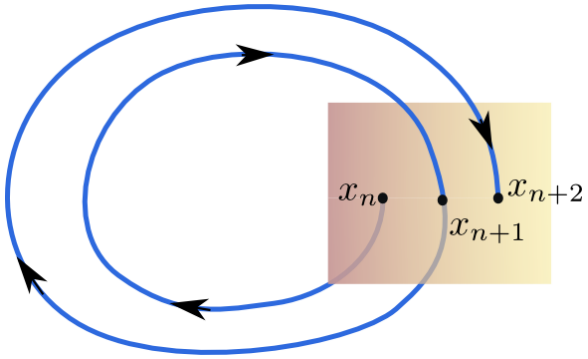

Figure 4: A Poincaré map turns a continuous dynamical system into a discrete map.

2.6 Melnikov's integral and Deriving Separatrix Maps

Consider a pendulum

$$H(p,\phi) = \frac{p^2}{2} - \epsilon\cos\phi. \tag{17}$$

With $\epsilon > 0$ the sign of the cosine term makes $p, \phi = 0$ a stable fixed point and about this point is pendulum libration. Near $\phi = \pi$ with $x = \phi - \pi$

$$H \approx \frac{p^2}{2} - \epsilon\frac{x^2}{2} + \epsilon$$

This is a hyperbolic fixed point. The separatrix orbit has energy $E = \epsilon$. Let us write frequency

$$\omega_0 \equiv \sqrt{\epsilon}.$$

At the energy of the separatrix (and using equation 17) the orbit

$$p_{sx}^2 = 2\epsilon(1+\cos\phi_{sx}) = 4\epsilon\cos^2(\phi_{sx}/2)$$
$$p_{sx} = \pm 2\omega_0\cos(\phi_{sx}/2) = \frac{d\phi_{sx}}{dt}$$

$$\frac{d\phi_{sx}}{2\omega_0\cos(\phi_{sx}/2)} = dt.$$

This can be integrated. The separatrix has orbit

$$\begin{aligned}\omega_0 t(\phi_{sx}) &= \ln\left(\tan\left(\frac{\phi_{sx}(t)}{4} + \frac{\pi}{4}\right)\right) \\ \phi_{sx}(t) &= 4\arctan(e^{\omega_0 t}) - \pi\end{aligned} \tag{18}$$

with $\phi_{sx}(-\infty) = -\pi$ and $\phi_{sx}(\infty) = \pi$.

Consider again the periodically forced pendulum

$$H(p, \phi, t) = \frac{p^2}{2} - \omega_0^2 \cos\phi - \mu \cos(\phi - \nu t - \tau_0)$$

where τ_0 let's us specify an initial phase offset for the two terms. We consider an orbit that starts at the hyperbolic fixed point of the unperturbed system. Because

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = \mu\nu \sin(\phi - \nu t - \tau_0)$$

we can estimate the energy change for a orbit that starts near this hyperbolic fixed point. The change in energy is approximated from the separatrix orbit

$$\Delta H \approx \int_{-\infty}^{\infty} dt\ \mu\nu \sin(\phi_{sx}(t) - \nu t - \tau_0)). \tag{19}$$

Here ϕ_{sx} is given in equation 18. The integral goes from negative infinity to positive infinity.

We expand

$$\begin{aligned}
\Delta H &\approx \int_{-\infty}^{\infty} dt\ \mu\nu \left[\sin(\phi_{sx}(t) - \nu t)\cos\tau_0 - \cos(\phi_{sx}(t) - \nu t')\sin\tau_0\right] \\
&= \int_{-\infty}^{\infty} dt'\ \mu\lambda\Big[\sin(4\tan^{-1}(e^{t'}) - \pi - \lambda t')\cos\tau_0 \\
&\qquad\qquad - \cos(4\tan^{-1}(e^{t'}) - \pi - \lambda t')\sin\tau_0\Big]
\end{aligned} \tag{20}$$

with $\lambda = \nu/\omega_0$.

The integral in equation 19 and equation 20 can be written in terms of the Melnikov-Arnold integral

$$A_m(\lambda) \equiv \int_{-\infty}^{\infty} \cos\left(\frac{m}{2}\phi_{MA}(t) - \lambda t\right)\ dt \tag{21}$$

with

$$\phi_{MA}(t) \equiv 4\tan^{-1} e^t - \pi. \tag{22}$$

The integral when evaluated is

$$A_m(\lambda) = \frac{4\pi(2\lambda)^{m-1}}{\Gamma(m)} e^{-\pi\lambda/2} \qquad \text{for} \qquad \lambda > 0 \tag{23}$$

$$= -\frac{4\ \Gamma(m+1)\sin(\pi m)}{|2\lambda|^{m+1}} e^{-\pi|\lambda|/2} \qquad \text{for} \qquad \lambda < 0. \tag{24}$$

Following Chirikov (1979), we approximate by taking only one of the terms in equation 20. Perhaps they are about the same size. Using Melnikov's integral with $m = 2$, $\lambda > 0$, equation 19 becomes

$$\Delta H \approx -4\pi\mu\lambda^2 e^{-\frac{\pi\nu}{2\omega_0}} \sin\tau_0. \tag{25}$$

See Chirikov, B. V. "A Universal Instability of Many-Dimensional Oscillator Systems." Phys. Rep. 52, 264-379, 1979.

In the unperturbed system, only the separatrix itself has infinite period. With an energy near the separatrix energy of $\epsilon = \omega_0^2$, what is the orbital period? Let

$$E = \epsilon(1+\delta) = \omega_0^2(1+\delta).$$

We again write $d\phi/dt$ in terms of energy

$$\omega_0 dt = \frac{d\phi}{\sqrt{2(E/\omega_0^2 + \cos\phi)}} \tag{26}$$

The orbital period $T(\delta)$ (as a function of distance from the separatrix)

$$\begin{aligned}
\omega_0 T(\delta) &= 2\int_0^{\pi} \frac{d\phi}{\sqrt{2(1+\cos\phi+\delta)}} \\
&\approx 2\int_{\pi-c}^{\pi} \frac{d\phi}{\sqrt{2(1+\cos\phi+\delta)}} \\
&\approx 2\int_{-\sqrt{c}}^{0} \frac{dy}{\sqrt{y^2+2\delta}} \\
&\approx 2\ln\sqrt{2\delta} \sim \ln(2\delta)
\end{aligned} \tag{27}$$

using Laplace's method to asymptotically estimate the integral for small δ. (I tried Laplace's method and had no success, however the expression can be written in terms of an elliptic function which has an asymptotic limit that you can look up. The result is a log).

A **separatrix map**, capturing the chaotic behavior near the separatrix can be constructed by considering the change in energy in equation 25 and coupling it to a change in the phase angle τ_0. Together

$$w_{n+1} = w_n + W\sin\tau_0 \tag{28}$$

$$\tau_{0,n+1} = \tau_{0,n} + \lambda\ln(32/w_{n+1}) \tag{29}$$

$$W = -4\pi\mu\lambda^2 e^{-\pi\lambda/2} \tag{30}$$

with $\lambda = \nu/\omega_0$ and w the distance in normalized energy (normalized so that it is divided by ω_0^2) from the separatrix.

The change in τ arises from the dependence of the period on energy near the separatrix where the period goes to infinity. I see where the factor of λ comes from as the period of the orbit w.r.t to the perturbation is needed to compute τ_0. The choice for the signs are weird, and I might be missing some factors of π and 2 when I did the period integral in equation 27.

By considering variations in energy and phase near an unstable region, other stochastic maps have been created. The Kepler map can be derived by looking at a comet's orbit. Each time it enters the inner solar system it is perturbed by a planet. Assumed is a constant pericenter distance. The change in energy depends on the phase difference between planet and comet's orbit. The system is approximated with the Kepler map, a map of energy and phase. The map is similar to the standard map except the phase advances with a 3/2 power of the momentum and the map is not doubly periodic.

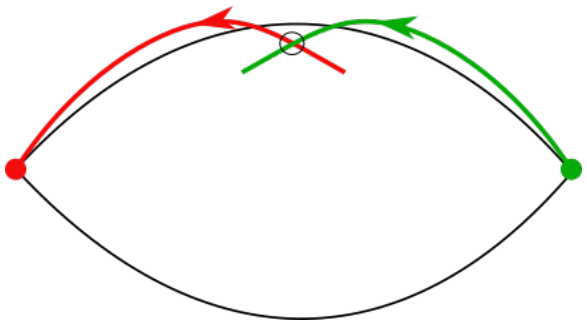

Figure 5: The stable (red) and unstable (green) manifolds of two fixed hyperbolic points must intersect. The unperturbed separatrix for the pendulum is shown in black.

2.7 Stable and unstable invariant manifolds

Consider a dynamical system $\dot{x} = f(x)$ with a hyperbolic fixed point x_*.

In the **stable manifold** is the set of points that asymptotically *approaches* the hyperbolic fixed point.

The **unstable manifold** is the set of points that originates from the hyperbolic fixed point (asymptotically backwards in time); see Figure 5 for an illustration of the pendulum. In a pendulum system with ϕ periodic, then the hyperbolic fixed points at $\phi = 0, 2\pi$ are the same point.

With s a point that we take to be an initial condition, we denote $x_s(t)$ as a solution to the dynamical system with $x_s(t = 0) = s$.

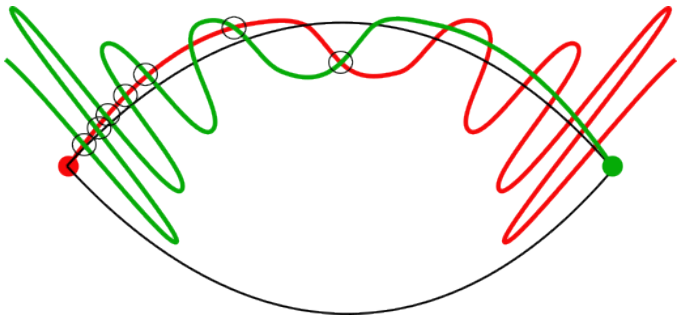

Figure 6: The stable (red) and unstable (green) manifolds of two fixed hyperbolic points must intersect an infinite number of times.

The **stable invariant manifold** W_s associated with a hyperbolic fixed point x_* is all points s such that

$$W_s : \text{ points } s \text{ such that } \lim_{t\to\infty} x_s(t) = x_*.$$

The **unstable invariant manifold** W_u associated with a hyperbolic fixed point x_* is all points s such that

$$W_u : \text{ points } s \text{ such that } \lim_{t\to-\infty} x_s(t) = x_*.$$

Consider again the periodically forced pendulum and set the phase of the perturbing function so that $\tau_0 = 0$.

$$H(p,\phi,t) = \frac{p^2}{2} - \epsilon\cos\phi - \mu\cos(\phi - \nu t). \tag{31}$$

We look at the Poincaré map of the plane (p,ϕ) to itself, made by mapping at a period or every 2π. This is an area preserving map.

For the unperturbed case of the perturbed pendulum, the stable and unstable manifolds are the same orbits, which are the separatrix of the conserved and unperturbed Hamiltonian. For a nonzero perturbation, the Hamiltonian is no longer conserved, as it is time dependent, and the stable manifolds and unstable manifolds no longer coincide. We assume that the perturbation is small and that there still exists a hyperbolic fixed point in the perturbed system.

2.8 The heteroclinic tangle

If the stable and unstable manifolds no longer coincide, where do they go? A stable manifold cannot cross another stable manifold, and an unstable manifold cannot cross another unstable manifold, because the crossing point would be asymptotic to two different

fixed points. A stable manifold or unstable manifold may not cross itself, otherwise that point would be inconsistent with the uniqueness of solutions to first order differential equations. However, stable and an unstable manifolds may cross one another.

The point where stable and unstable manifolds cross is called a **homoclinic** intersection if the stable and unstable manifolds arise from the same unstable fixed point. It is called a **heteroclinic** intersection if the stable and unstable manifolds belong to different fixed points, as show in figure 5. When the separatrix connects two different hyperbolic points, the orbit is called heteroclinic. If it forms a loop, as shown in Figure 7, then it is called homoclinic.

For the perturbed pendulum of equation 31, the first intersection point must occur at angle $\phi = \pi$ because of symmetry in ϕ.

The stable and unstable manifolds must cross at some point. If they don't cross then they must go to infinity or spiral inward. The manifolds cannot go to infinity because there are orbits that form a barrier and contain them. Area preservation excludes the existence of attractors. So the unstable and stable manifolds cannot spiral inward as the dynamics is Hamiltonian and area preserving.

If the stable and unstable manifolds cross once, then they must cross an infinite number of times. This is because we can apply the mapping at a period to the intersection point and we can also apply its inverse to the intersection point. The inverse procedure gives a new point on the stable manifold that has to be in both stable and unstable manifolds but it is is closer to a hyperbolic fixed point. Because the mapping at a period is area preserving the area bounded by the different regions must be the same.

The procedure for estimating the distance between stable and unstable manifolds using the unperturbed orbit is known as *Melnikov's method*. The Melnikov function measures the distance between stable and unstable manifolds. A zero in this function is a necessary condition for a hetero or homoclinic tangle and chaotic behavior. The tangle itself is sometimes called *Smale's horseshoe*.

2.9 The Melnikov Method

The Melnikov method establishes when a time dependent perturbation causes chaotic behavior in an integrable system.

We have an unperturbed Hamiltonian dynamical system for $x = (q, p)$ with $f(x) = (\partial_p H_0, -\partial_q H_0)$ for a Hamiltonian function $H_0(q, p)$. The equation of motion is

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^2, t \in \mathbb{R}. \tag{32}$$

The system has a particular solution $x_0(t)$ that forms a loop and contains a hyperbolic fixed point. We will call $x_0(t)$ the separatrix (it could also be called a homoclinic orbit). This orbit is at the fixed point at $t = \pm\infty$.

We perturb the system with a periodic function $g(x, t + T) = g(x)$,

$$\dot{x} = f(x) + \epsilon g(x, t) \tag{33}$$

The unperturbed system should also be Hamiltonian so $g(x,t) = (\partial_p H_1, -\partial_q H_1)$ for a periodic (in time with period T) function $H_1(q,p,t)$.

Lemma 2.1 *The stable and unstable manifolds of the Poincaré map of the dynamical system in equation 33 intersect at a point not on x_0 if and only if the Melnikov function $M(t_0)$ has simple zeros. The Melnikov function is*

$$M(t_0) = \int_{-\infty}^{\infty} dt\ f(x_0(t-t_0)) \wedge g(x_0(t-t_0), t) \tag{34}$$

We try to explain where equation 34 comes from.

Here $a \wedge b$ denotes the area spanned by the 2d vectors a and b. In other words if $a = (a_x, a_y)$ and $b = (b_x, b_y)$ then $a \wedge b = a_x b_y - a_y b_x$.

As long as the perturbation is periodic, perturbed orbits wiggle but have to return to the same energy ($H = H_0 + H_1$) value every period T. That places a limit on how far they can go from x_0 if they start on the separatrix orbit x_0. The perturbed invariant manifolds diverge from the separatrix in the p, q plane. We define a normal vector $\hat{\mathbf{n}}_z$ that is perpendicular to the separatrix at a position z. The tangent to z for the separatrix orbit is $\dot{x} = (\partial_p H_0, -\partial_q H_0)$ evaluated at z. This means that the normal

$$\hat{\mathbf{n}} \propto (\partial_q H_0, \partial_p H_0) = \nabla H_0 \qquad \text{evaluated at } z.$$

Consider points z_s, z_u on the stable and unstable manifolds of the perturbed system. Choose time t so that z_s, z_u, z are all nearby. As the perturbation causes the unstable and stable manifolds to diverge from the separatrix, we would like to know the component of $z_s - z_u$ in the $\hat{\mathbf{n}}$ direction which is perpendicular to the separatrix. This gives the distance between unstable and stable invariant manifolds at a particular time. The distance between the points on the stable and unstable manifolds

$$d(z_s, z_u, t) = \left| \frac{\nabla H_0(z) \cdot (z_s - z_u)}{|\nabla H_0(z)|} \right|.$$

Here t determines z_s, z_u, z positions. Now expand this to first order in ϵ. Because $z_s = z_u$ if $\epsilon = 0$ the first term in the expansion is first order in ϵ.

$$d(z_s, z_u, t) \sim \frac{\epsilon}{|\nabla H_0|} \left| \nabla H_0 \cdot (\partial_\epsilon z_s - \partial_\epsilon z_u)|_{\epsilon=0} \right|. \tag{35}$$

The derivatives w.r.t. ϵ only depend upon H_1 and they are evaluated at $\epsilon = 0$. Because the term on the right is evaluated at $\epsilon = 0$, you do not need to know the orbits on the perturbed invariant manifolds! The right hand side of equation 35 is evaluated on the separatrix. All you need to know is the separatrix orbit and how the orbits are perturbed via the functions H_1 or g.

The wedge product arises when we replace ∇H_0 with $f(z)$, as $\nabla H_0 = (\partial_q H_0, \partial_p H_0) = (-f_p, f_q)$ where $f = (f_q, f_p)$. Equation 35 becomes

$$d(z_s, z_u, t) \propto M(t) = f(z) \wedge (\partial_\epsilon z_s - \partial_\epsilon z_u)|_{\epsilon=0}. \tag{36}$$

We call this function $M(t)$. With cumbersome math (take time derivative, write itself in terms of integral of self) you find equation 34. The derivatives w.r.t ϵ become the function g evaluated on the separatrix.

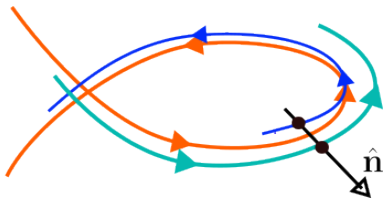

Figure 7: Distance between stable and unstable manifolds is computed using the normal to the separatrix orbit. For points z_s and z_u on the stable and unstable manifolds the distance $d = |(z_s - z_u) \cdot \hat{\mathbf{n}}|$.

If we can show that there is an isolated zero of the Melnikov function for a perturbed system, this lemma implies that the invariant manifolds intersect and this then implies an infinite number of intersections and chaotic behavior. Thus finding a zero of a Melnikov function for a dynamical system is equivalent to showing that it exhibits chaotic behavior.

Apparently, the intersection between unstable and stable manifods is *transverse* (though I am not sure how to show that or what it specifically means). This then makes it possible to say that the dynamical system in the vicinity of x_0 is conjugate to the Smale horseshoe map which exhibits a number of properties associated with chaotic systems.

We could add an example and associated problem on using Melnikov's method. It seems much easier to actually apply the lemma than to derive it. To apply the method, first identify a closed orbit with a hyperbolic orbit in an unperturbed Hamiltonian system. Then integrate equation 34 along it using the perturbation. If the function has an isolated zero, then the system must be chaotic. It seems fairly straightforward to follow this numerical recipe for perturbations on a Hamiltonian system where you can analytically write down a separatrix orbit. There is series of literature on the derivation of Melnikov's method in various settings and applying it to various dynamical systems.

3 The restricted 3-body problem

The restricted 3-body problem is restricted in a number of ways. There are two massive bodies and a third massless particle, interacting under the influence of gravity. Because

the third body is massless it does not affect the other two, but its dynamics can be complex. The orbit of the two massive bodies is circular and the third body restricted to the plane containing the other two; this makes the problem of the third body's dynamics a 2 dimensional problem with a four dimensional phase space. With respect to the center of mass we can describe the position of the first and second bodies in terms of their radius from the center of mass r_1, r_2 and angles in the orbital plane $\lambda_1 = n_p t$, and $\lambda_2 = n_p t + \pi$ where n_p is the angular rotation rate of the orbit. We denote the masses of the two bodies m_1, m_2 giving

$$n_p = \sqrt{\frac{G(m_1 + m_2)}{(r_1 + r_2)^3}}$$

Working in the center of mass frame implies that

$$m_1 r_1 = m_2 r_2$$

There is one important unit of time in the problem, n_p^{-1}, and one important unit of length $r_1 + r_2$. We can also set

$$G(m_1 + m_2) = 1.$$

This is equivalent to adjusting the units of mass, distance and time. However the problem is sensitive to the mass ratio

$$\mu \equiv \frac{m_2}{m_1 + m_2}.$$

Using these units and the condition that we are working in the center of mass frame

$$m_1 \to (1 - \mu) \qquad m_2 \to \mu$$

$$r_1 \to \mu \qquad r_2 \to 1 - \mu$$

$$n_p \to 1$$

The position of the two masses as a function of time is

$$\begin{aligned} \mathbf{x_1} &= \mu(\cos t, \sin t) \\ \mathbf{x_2} &= (\mu - 1)(\cos t, \sin t). \end{aligned}$$

Rotation is counter clockwise. In the frame with origin at the center mass and rotating with the two masses, the two masses are at positions

$$\begin{aligned} \mathbf{x_1} &= \mu\hat{\mathbf{x}} \\ \mathbf{x_2} &= -(1 - \mu)\hat{\mathbf{x}} \end{aligned}$$

where $\hat{\mathbf{x}}$ is a unit vector in the x direction. In this rotating frame, the two masses do not move.

The motion of the third body depends on the Hamiltonian

$$H(p_x, p_y, x, y, t) = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\mu}{|\mathbf{x}_1(t) - \mathbf{x}|} - \frac{\mu}{|\mathbf{x}_2(t) - \mathbf{x}|} \tag{37}$$

As shown in previous lectures, and more easily in polar coordinates, we can transfer into the rotating frame finding a time independent Hamiltonian

$$H(p_x, p_y, x, y) = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\mu}{|\mathbf{x}_1 - \mathbf{x}|} - \frac{\mu}{|\mathbf{x}_2 - \mathbf{x}|} - (xp_y - yp_x) \tag{38}$$

where the z component of the angular momentum $L_z = xp_y - yp_x$. Since the new Hamiltonian is independent of time, it is conserved. This integral of motion is known as the Jacobi constant.

Inserting the positions of the planets back into equation 38

$$H(x, y; p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2}} - \frac{\mu}{\sqrt{(x + (1-\mu))^2 + y^2}} - (xp_y - yp_x) \tag{39}$$

or in polar coordinates

$$H(r, \theta; p_r, L) = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - L - \frac{1-\mu}{\sqrt{\mu^2 + r^2 - 2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2 + r^2 + 2(1-\mu)r\cos\theta}}. \tag{40}$$

Equations of motion in the corotating Cartesian frame are

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x + y \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y - x \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = p_y - xf(x, y) \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -p_x - yf(x, y)
\end{aligned}$$

with

$$f(x, y) = \frac{1-\mu}{((x-\mu)^2 + y^2)^{3/2}} + \frac{\mu}{((x+1-\mu)^2 + y^2)^{3/2}}$$

The equations of motion in polar coordinates are

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = p_r \\
\dot{\theta} &= \frac{\partial H}{\partial L} = \frac{L}{r^2} - 1 \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{L^2}{r^3} - \frac{(1-\mu)(r - 2\mu\cos\theta)}{(\mu^2 + r^2 - 2r\mu\cos\theta)^{\frac{3}{2}}} - \frac{\mu(r + (1-\mu)\cos\theta)}{((1-\mu)^2 + r^2 + 2r(1-\mu)\cos\theta)^{\frac{3}{2}}} \\
\dot{L} &= -\frac{\partial H}{\partial \theta} = r\mu(1-\mu)\left[(\mu^2 + r^2 - 2r\mu\cos\theta)^{-\frac{3}{2}} - ((1-\mu)^2 + r^2 + 2r(1-\mu)\cos\theta)^{-\frac{3}{2}}\right]\sin\theta
\end{aligned} \tag{41}$$

3.1 In three dimensions and the effective potential

More generally in 3D the dynamics of a third massless particle orbiting two massive objects that bound in a circular orbit

$$H(x,y,z;p_x,p_y,p_z) = \frac{1}{2}(p_x^2+p_y^2) - \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2+z^2}} - \frac{\mu}{\sqrt{(x+(1-\mu))^2+y^2+z^2}} - (xp_y - yp_x) \tag{42}$$

with equations of motion

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = p_x + y \\
\dot{y} &= \frac{\partial H}{\partial p_y} = p_y - x \\
\dot{z} &= \frac{\partial H}{\partial p_z} = p_z \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = p_y + xf(x,y,z) \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -p_x + yf(x,y,z) \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = zf(x,y,z)
\end{aligned}$$

with

$$f(x,y,z) = \frac{1-\mu}{((x-\mu)^2+y^2+z^2)^{3/2}} + \frac{\mu}{((x+1-\mu)^2+y^2+z^2)^{3/2}}$$

We can compute the accelerations in the rotating frame

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x + xf(x,y,z) \\
\ddot{y} &= -2\dot{x} + y + yf(x,y,z) \\
\ddot{z} &= zf(x,y,z)
\end{aligned}$$

The velocity dependent term is recognized as the Coriolis force. The x, y term is centripetal. The other term is due to the forces from each point mass.

It is helpful to define r_1 the distance to the more massive body and r_2 the distance to the less massive body.

$$r_1 \equiv ((x-\mu)^2 + y^2 + z^2)^{1/2}$$
$$r_2 \equiv ((x+1-\mu)^2 + y^2 + z^2)^{1/2}$$

The equations of motion can be written in terms of an effective potential

$$V_{eff}(x,y,z) = -\frac{1}{2}(x^2+y^2+z^2) - \frac{1-\mu}{((x-\mu)^2+y^2+z^2)^{1/2}} - \frac{\mu}{((x+1-\mu)^2+y^2+z^2)^{1/2}} \tag{43}$$

$$= -\frac{1}{2}r^2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}. \tag{44}$$

The term on the left is the centrifugal term and the potential energy terms from the point masses are on the right. Using the effective potential the equations of motion are

$$\ddot{x} = 2\dot{y} - \frac{\partial V_{eff}}{\partial x}$$
$$\ddot{y} = -2\dot{x} - \frac{\partial V_{eff}}{\partial y}$$
$$\ddot{z} = -\frac{\partial V_{eff}}{\partial z}.$$

3.2 Lagrange Points

The Lagrange points are equilibrium points in the rotating frame. That means $\dot{x} = \dot{y} = \dot{z} = 0$, $\ddot{x} = \ddot{y} = \ddot{z} = 0$ and $\dot{p}_x = \dot{p}_y = \dot{p}_z = 0$. The Lagrange points are critical points of the effective potential. That means they satisfy

$$\boldsymbol{\nabla} V_{eff} = 0.$$

With the identity

$$(1-\mu)r_1^2 + \mu r_2^2 = x^2 + y^2 + z^2 + \mu(1-\mu) \tag{45}$$

we can write the effective potential as

$$-V_{eff} = (1-\mu)\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) + \mu\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right) - \frac{\mu(1-\mu)}{2} \tag{46}$$

The Lagrange points satisfy

$$\begin{aligned}
\frac{\partial V_{eff}}{\partial x} &= \frac{\partial V_{eff}}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial V_{eff}}{\partial r_2}\frac{\partial r_2}{\partial x} = 0 \\
&= (1-\mu)\left(r_1 - \frac{1}{r_1^2}\right)\frac{x+\mu}{r_1} + \mu\left(r_2 - \frac{1}{r_2^2}\right)\frac{x-1+\mu}{r_2} \\
\frac{\partial V_{eff}}{\partial y} &= \frac{\partial V_{eff}}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial V_{eff}}{\partial r_2}\frac{\partial r_2}{\partial y} = 0 \\
&= (1-\mu)\left(r_1 - \frac{1}{r_1^2}\right)\frac{y}{r_1} + \mu\left(r_2 - \frac{1}{r_2^2}\right)\frac{y}{r_2}.
\end{aligned} \tag{47}$$

The requirement that $\dot{p}_z = 0$ at an equilibrium point gives us $z = 0$. That means we can use the 2D system to classify the fixed points.

One solution of equation 47 satisfies

$$\begin{aligned}
r_1 - \frac{1}{r_1^2} &= 0 \\
r_2 - \frac{1}{r_2^2} &= 0
\end{aligned}$$

A solution is $r_1 = r_3 = 1$. These are the L3, L4 Lagrange points and they are at

$$L3, L4: \qquad x = \frac{1}{2} - \mu \qquad y = \pm\frac{\sqrt{3}}{2} \tag{48}$$

The $\dot{L} = 0$ condition (equation 41) would give Lagrange points at $\theta = 0, \pi$ or where $y = 0$. These are the L1, L2, and L3 Lagrange points. Using $\ddot{x} = 0$ we find that this condition is satisfied

$$(1-\mu)\left(1 - r_2\frac{1}{(1-r_2)^2}\right) - \mu\left(r_2 - \frac{1}{r_2^2}\right) = 0 \tag{49}$$

which can be written as

$$\left(\frac{\mu}{3(1-\mu)}\right) = r_2^3\frac{1 - r_2 + r_2^2/3}{(1 + r_2 + r_2^2)(1-r_2)^3} \tag{50}$$

It is convenient to define

$$\alpha = \left(\frac{\mu}{3(1-\mu)}\right)^{\frac{1}{3}}$$

and use α an expansion parameter. Notice this parameter is directly related to the Hill radius.

The distance from the lower mass body for L1 (between the two masses) and L2 (outside the lower mass body) are

$$\begin{aligned} L1: &\qquad r_2 = \alpha - \frac{\alpha^2}{3} - \frac{\alpha^3}{9}... \\ L2: &\qquad r_2 = \alpha + \frac{\alpha^2}{3} - \frac{\alpha^3}{9}... \end{aligned}$$

where these are both on the x axis and near the lower mass body. For small μ, the L1, L2 Lagrange points are about 1 Hill sphere radius away from the lower mass body. The distances to the L3 point (on the opposite side) and also on the x axix

$$\begin{aligned} L3: &\qquad r_1 = 1 - \frac{7}{12}\left(\frac{\mu}{1-\mu}\right) + \frac{7}{12}\left(\frac{\mu}{1-\mu}\right)^2 - \frac{13223}{20736}\left(\frac{\mu}{1-\mu}\right)^3 ... \\ L3: &\qquad r_2 = 1 + r_1 \end{aligned}$$

Here I have adopted the expansions given by M+D and Celletti which could be based on those by Brouwer and Clemence.

3.3 Surface of section

Nearly circular orbits would advance in θ and have radii that oscillate about a fixed value. Consider an initial condition in the plane with $\theta = 0$, and with $r_0, p_{r,0}$. And let us chose a value for the Jacobi constant E_J or energy. Given E_J and initial values for three of the the 4 coordinates we can determine the angular momentum, L. This is enough to specify initial conditions for all variables. So given an E_J, and in the plane $\theta = 0$, for every point in the planar subspace r, p_r we can determine all initial conditions for an orbit (see Figure 9). We can integrate the orbit until it crosses $\theta = 0$ again. This is a map from $r, p_r, L \to r, p_r, L$ that preserves E_J and keeps θ fixed. The symplectic two-form in polar coordinates

$$d\omega = d\theta \wedge dL + dr \wedge dp_r$$

is preserved because the map is determined by the orbits of the system. Within the subspace with $\theta = 0$ we can neglect the $d\theta \wedge dL$, finding that the map is area preserving in the subspace r, p_r. This gives us a map from the plane r, p_r to itself.

Compare this surface of section to the mapping at a period example that is equivalent to a 2 dimensional system with Hamiltonian in equation 16, but repeated here

$$H(p, \phi; J, \theta) = \frac{p^2}{2} - \epsilon \cos\phi - \mu \cos(\phi - \theta) + J\nu \tag{51}$$

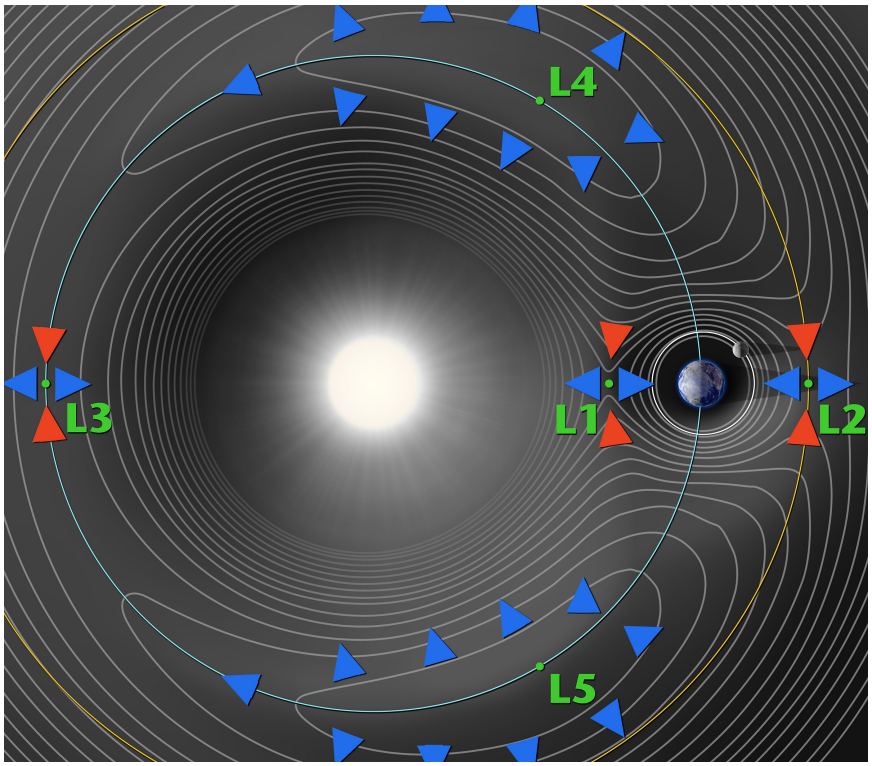

Figure 8: An illustration of the Lagrange points. The L1, L2 are unstable – this means that a particle placed near one of them will not stay in the vicinity. The L3, L4 points are stable. Trojan asteroids are found in the L3 or L4 region associated with Jupiter. The contours are those of the effective potential. The contours get higher at large radii due to the centripetal term in the effective potential. They get increasingly negative near each mass.

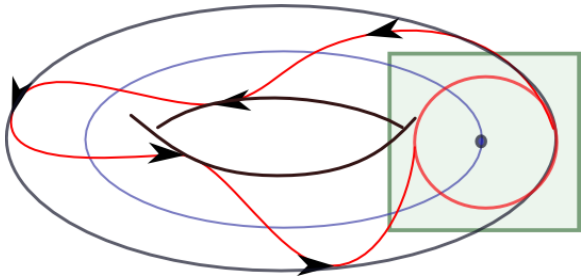

Figure 9: For the restricted 3-body problem, a map can be constructed from the position of the orbits of energy E_J each time they pass through a plane with a particular value of θ. This is known as a surface of section and is another way to create an area preserving map from the Hamiltonian flow. The flow lines are vortex lines and so are null vectors with respect to the symplectic two-form. Within the subspace for $\theta = 0$ the symplectic two-form $dr \wedge dp_r$ is preserved. The map from $r, p_r \to r, p_r$ is area preserving and it is a Poincaré map.

Oddly in that system, the map at $T = 2\pi/\nu$ is only dependent on the initial values of p, ϕ and not on the energy, whereas the surface of section for the restricted 3-body problem is dependent on the choice of energy. For the mapping at a period example, there is a symmetry of the equations of motion $J \to J+$ constant, making it irrelevant what value of energy is initially chosen. Given initial values of p, ϕ, the energy sets J (or vice versa) but J does not influence the equations of motion of the other variables. Consequently the choice of the energy does not influence the equations of motion.

Surfaces of section can also be constructed for other two-dimensional systems. For example, the dynamics of stars in the plane of a barred galaxy can be studied in a similar way – replacing the function $f(x, y)$ with one that is derived from the barred galaxy potential.

3.4 Nearly circular orbits

Recall that the restricted three-body problem in polar coordinates (equation 40) is

$$H(p_r, L; r, \theta) = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - L - \frac{1-\mu}{\sqrt{\mu^2 + r^2 - 2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2 + r^2 + 2(1-\mu)r\cos\theta}}$$

Let us look at the potential

$$V(r, \theta) = -\frac{1-\mu}{\sqrt{\mu^2 + r^2 - 2r\mu\cos\theta}} - \frac{\mu}{\sqrt{(1-\mu)^2 + r^2 + 2(1-\mu)r\cos\theta}}$$

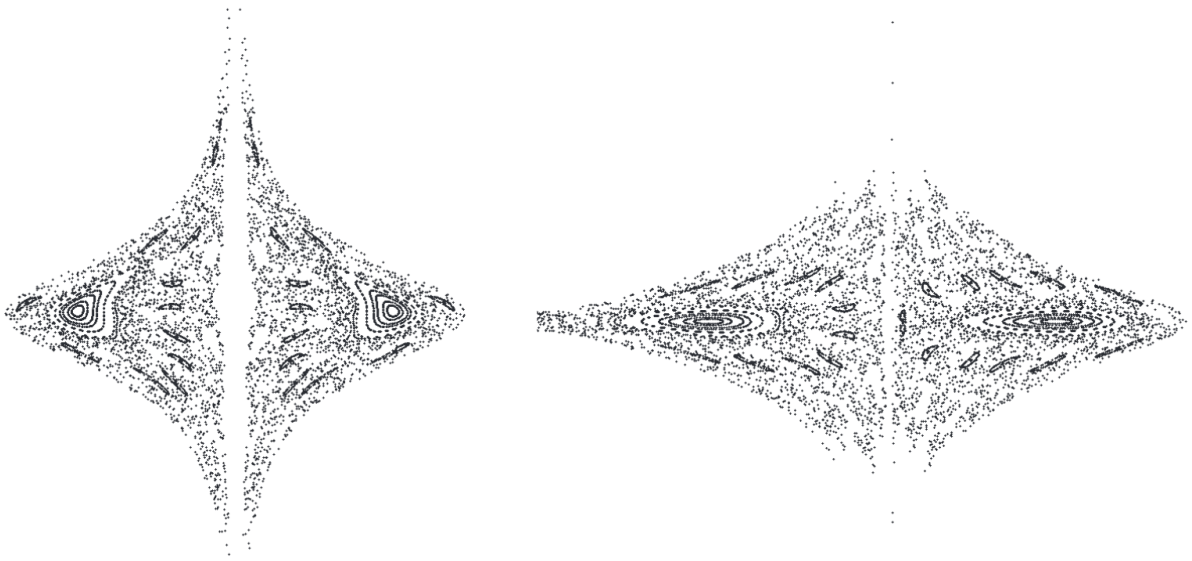

FIGURE 2. Poincaré sections in the planar circular restricted three-body problem corresponding to a Jacobi constant $C = 3.95$, for $\mu = 0.5$. Left – in $(y, \dot{y})$-coordinates, corresponding to $x = 0.5$; right – in $(x, \dot{x})$-coordinates, corresponding to $y = 0$. The small elliptic islands in the plot surround a periodic orbit of period 12 for the Poincaré map.

Figure 10: A surface of section for the restricted circular 3 body problem. Thess figures are from *Phase space reconstruction in the restricted three-body problem*, by Marian Gidea, Frederick Deppe, and Gregory Anderson, AIP Conference Proceedings 886, 139 (2007); doi: http://dx.doi.org/10.1063/1.2710051

We can expand the potential in a Fourier series in θ

$$V(r,\theta) = \sum_{m=-\infty}^{\infty} V_m(r)\cos(m\theta)$$

Our Hamiltonian we can write as a sum

$$H = H_0 + H_1.$$

Working in the rotating frame with angular rotation rate equal to one, the unperturbed system (that is independent of azimuthal angle θ)

$$H_0(p_r, L, r) = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - L + V_0(r)$$

and the perturbation (that is independent of p_r)

$$H_1(L, r, \theta) = \sum_{m\neq 0} V_m(r)\cos(m\theta)$$

In a previous lecture we showed how to transform a Hamiltonian in the form of H_0 into action angle variables, J_r, θ_r, within the context of the epicyclic approximation, using the assumption that the orbit is nearly circular. To first order in J_r and ignoring a constant, this gives

$$H_0(\theta_r, \theta; J_r, L) = g_0(L) + J_r\kappa(L) - L + ...$$

with $g_0'(L) = \Omega$ the angular rotation rate for a circular orbit. A canonical transformation relates mean radius to action angle variables J_r, θ_r

$$r = r_c(L) + \sqrt{\frac{2J_r}{\kappa(L)}}\cos\theta_r \tag{52}$$

Let us insert this into H_1 and expand to first order in $J_r^{1/2}$

$$\begin{aligned}
H_1 &= \sum_{m\neq 0} V_m\left(r_c + \sqrt{2J_r/\kappa(L)}\cos\theta_r\right)\cos(m\theta) \\
&\approx \sum_{m\neq 0}\left[V_m(r_c)\cos(m\theta) + \frac{dV_m}{dr}(r_c)\sqrt{\frac{2J_r}{\kappa(L)}}\cos\theta_r\cos(m\theta)\right] \\
&= \sum_{m\neq 0}\Bigg[V_m(r_c)\cos(m\theta) + \\
&\quad \frac{dV_m}{dr}(r_c)\sqrt{\frac{2J_r}{\kappa(L)}}\frac{1}{2}\left[\cos(\theta_r + m\theta) + \cos(\theta_r - m\theta)\right]\Bigg]
\end{aligned} \tag{53}$$

Locations where $\theta_r - m\theta$ are slowly varying angles are called mean motion resonances or Lindblad resonances. Remember we are in a corotating frame. In the inertial frame and recovering units these have slow moving angles

$$\phi = \theta_r - m(\theta - n_p t)$$

or slow frequencies

$$\kappa - m(\Omega - n_p) \sim 0$$

where $\Omega = \dot{\theta}$ is the angular rotation rate of the particle in its nearly circular orbit.

Note the appearance of the $J_r^{1/2}$ in the coefficient in equation 53! This is the source of the resonant Hamiltonian we introduced in a previous lecture that looked like

$$H(J, \phi) = J^2 + \delta J + \epsilon J^{1/2} \cos\phi \tag{54}$$

We can expand H_1 to higher orders in J_r to produce resonant terms with coefficients $J_r^{n/2}$ with integer n. We can also expand our Hamiltonian with other systems of action angle variables (like Poincaré coordinates) to find similar types of terms.

Consider the system near a resonance. Because non-resonant terms can contain fast angles, they can be neglected. They give only small variations to the action angle variables. Keeping only a single resonant term

$$H(\theta_r, \theta, L, J_r) = g_0(L) + \kappa(L) J_r + a(L) J_r^2 - L + \epsilon_m J_r^{1/2} \cos(\theta_r - m\theta) \tag{55}$$

where

$$\epsilon_m = \frac{dV_m}{dr} \sqrt{\frac{2}{\kappa}} \frac{1}{2}. \tag{56}$$

We have added a second order term $a(L)J_r^2$ here expanding the unperturbed Hamiltonian to second order. We discussed earlier the dangers of a low order approximation near resonance. The value of $a(L)$ depends on the rotation curve and can be derived with the canonical transformations introduced in the previous lecture. The Hamiltonian contains only a single angle $\phi = \theta_r - m\theta$. Following a canonical transformation the Hamiltonian can be put in a form that only depends on this single angle. In this new coordinate system, there is a conserved quantity (conjugate to the angle that does not appear in the Hamiltonian). The system can be reduced to the one-dimensional system in equation 54.

3.5 Fixed points and periodic orbits

In the previous example we examined a 2-dimensional Hamiltonian (4d phase space) in the rotating frame and so independent of time (equation 55) and reduced it to the one dimensional Hamiltonian equation 54. We know how to find fixed points in this system. But what are the *orbits* of these fixed points?

Recall the relation between radius and θ_r

$$r = r_c(L) + \sqrt{\frac{2J_r}{\kappa(L)}}\cos\theta_r$$

When $\theta_r = 0$ the orbit is at apocenter and when $\theta_r = \pi$, the orbit is at pericenter.

A fixed point has a fixed angle $\phi = \theta_r - m\theta$ where θ is the angle in the rotating frame. Supposing we have a fixed point with $\phi = 0$. We find that $\theta_r = m\theta$ (in the rotating frame). This lets us draw the shape of the orbit in the rotating frame.

For example if $m = 2$ the orbit looks like an oval. If $m = 4$ the orbit looks like a square. If $\phi = \pi$ and $m = 2$ the orbit is perpendicular to the $\phi = 0$ case. For $\phi = \pi$ and $m = 4$ the orbit is a diamond.

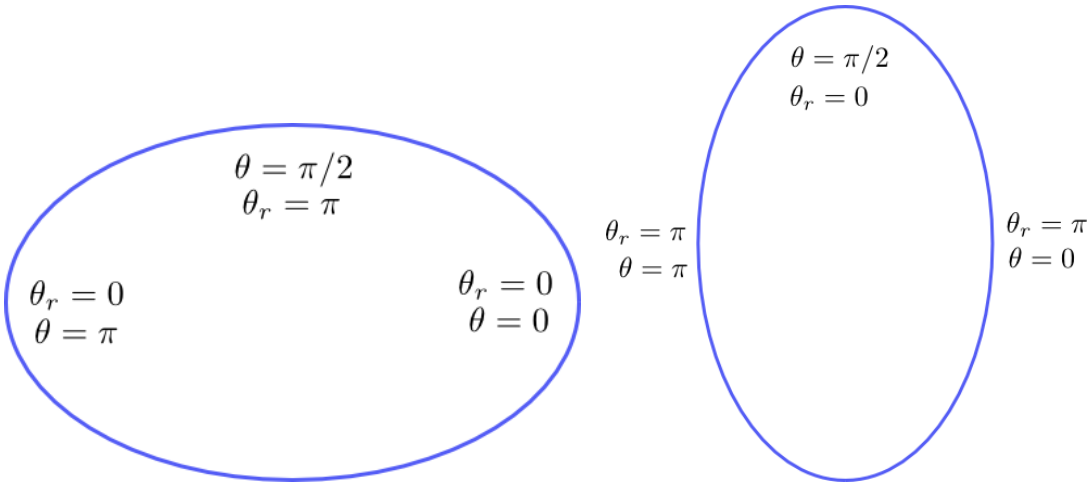

Figure 11: a) Orbits with $\phi = 0$ and $\phi = \theta_r - 2\theta$. b) Orbits with $\phi = \pi$ and $\phi = \theta_r - 2\theta$.

In the rotating frame these orbits are not fixed, but they are closed or *periodic*. How long does it take for an orbit to close? Consider $\phi = \theta_r - m\theta$ fixed. This gives us a relation between frequencies

$$\dot{\theta}_r = m\dot{\theta}$$

The oscillation period for θ_r is

$$P_r = \frac{2\pi}{\dot{\theta}_r}$$

and that for θ

$$P_\theta = \frac{2\pi}{\dot{\theta}}$$

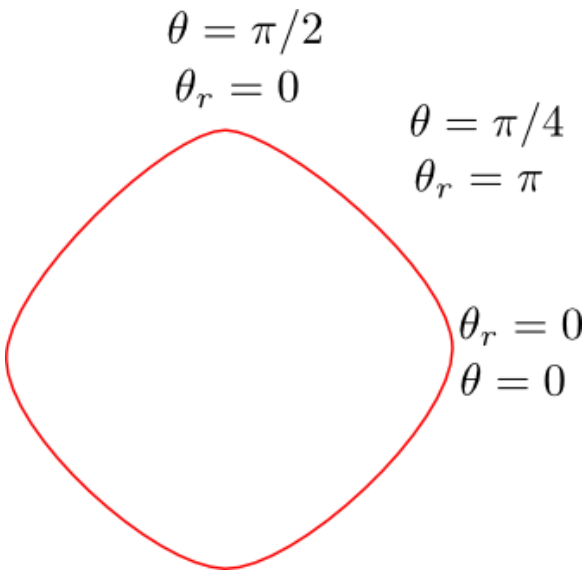

Figure 12: Orbits with $\phi = 0$ and $\phi = \theta_r - 4\theta$.

Using our relation between frequencies we find

$$P_\theta = mP_r$$

We can see that there are m radial oscillation periods per orbital period for these periodic orbits (in the rotating frame), that correspond to fixed points in our Hamiltonian model.

4 The kicked rotor and the standard map

The *Dirac comb* is a periodic function that is a sum of delta functions

$$D_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The spacing between the delta functions is period T.

Consider a Hamiltonian that is separable into two terms and one of the terms contains the Dirac comb

$$H(p, \theta, t) = H_0(p) + f(\theta) D_T(t)$$

Hamilton's equation for $\dot{\theta}$ gives

$$\begin{aligned} \dot{\theta} &= \frac{\partial H_0}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial \theta} = -f'(\theta) D_T(t) \end{aligned}$$

Because of the delta functions, $\dot{p} = 0$ except at the times $t_n = nT$. Let us integrate over a small time window

$$t_n - \mu < t_n < t_n + \mu$$

where $\mu > 0$ is small. At each time t_n the momentum p is kicked by

$$\Delta p = -f'(\theta(t_n))$$

In between these times p is constant and θ advances with frequency $\frac{\partial H_0}{\partial p}$.

Let us take $H_0(p) = \frac{p^2}{2}$ and $f(\theta) = -K\cos\theta$, giving

$$H(p,\theta,t) = \frac{p^2}{2} + K\cos\theta \sum_{n=-\infty}^{\infty} \delta(t-n)$$

(and we have set $T = 1$ which is equivalent to rescaling time so that we use $\tau = t/T$ and then rescaling momentum and K). Here

$$\begin{aligned} f(\theta) &= K\cos\theta \\ f'(\theta) &= -K\sin\theta \end{aligned}$$

This gives

$$\dot{\theta} = p \qquad \text{and} \qquad \Delta p = K\sin\theta$$

Note $\theta \in [0, 2\pi]$.

Let us choose to have p_n, θ_n the values just before the delta function. So p_n, θ_n are the values for p, θ at times $t = n$, but just before the delta function. We want to make a map giving p_{n+1}, θ_{n+1} from p_n, θ_n. To advance to p_{n+1}, θ_{n+1} we first apply the delta function and update p giving

$$p_{n+1} = p_n + K\sin\theta_n$$

Until the next delta function p is constant and now equal to p_{n+1} so $\dot{\theta} = p_{n+1}$ and

$$\theta_{n+1} = \theta_n + p_{n+1}$$

With the additional condition that p has periodic boundary conditions, this is the standard map (or Chirikov-Taylor map) which we restate

$$\begin{aligned} p_{n+1} &= p_n + K\sin\theta_n \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned}$$

and both p_n and θ_n are modulo 2π.

We can rewrite the map as

$$\begin{aligned} \theta_{n+1} &= \theta_n + p_n + K \sin\theta_n \\ p_{n+1} &= p_n + K \sin\theta_n \end{aligned}$$

so that terms on the left are for $t = n + 1$ and terms on the right for the previous time at $t = n$. In this form the standard map is related to another map called the circle map.

$$\theta_{n+1} = \theta_n + \Omega - K \sin\theta_n$$

that forces p to be constant Ω.

When K is not small the standard map is very pretty and displays fixed points, periodic orbits, orbits that are linear, and area filling orbits. Periodic orbits are surrounded by tori which can dissolve into objects called Cantori or Aubry-Mather sets. Sometimes linear orbits cross the entire space (from $\theta = 0$ to 2π) and these are called invariant circles. Some orbits for $K = 0.971635$ are shown in Figure 13. The standard map has been discussed in the context of *twist maps*.

We can derive an area preserving map from any Hamiltonian in the form

$$H(p, q, t) = P(p) + Q(q)D_T(t)$$

giving

$$\begin{aligned} \dot{q} &= P'(p) \\ \dot{p} &= -Q'(q)D_T \end{aligned}$$

and $\Delta q = P'(p)T$ and $\Delta p = -Q'(q)$.

Question: Why is it that we are choosing a separable Hamiltonian?

Answer: taking $H(p, q) = P(p) + f(p, q)D_T(t)$ we get equations of motion

$$\begin{aligned} \dot{q} &= P'(p) + \partial_p f(p, q)D_T(t) \\ \dot{p} &= -\partial_q f(p, q)D_T(t) \end{aligned}$$

with f depending upon both p, q with both changing across the delta function, making it hard to decide how we should use p, q to evaluate the function f at the transition times.

The most common generalization of the standard map is

$$\begin{aligned} p_{n+1} &= p_n + K \sin\theta_n \\ \theta_{n+1} &= \theta_n + \text{sign}(p_{n+1})|p_{n+1}|^\gamma \end{aligned} \tag{57}$$

with $\gamma = 1.5$ known as the Kepler map. For this map p extends to large values and is not restricted to a periodic interval. This map is sometimes use to model particle escape, comet trajectories or cosmic ray acceleration.

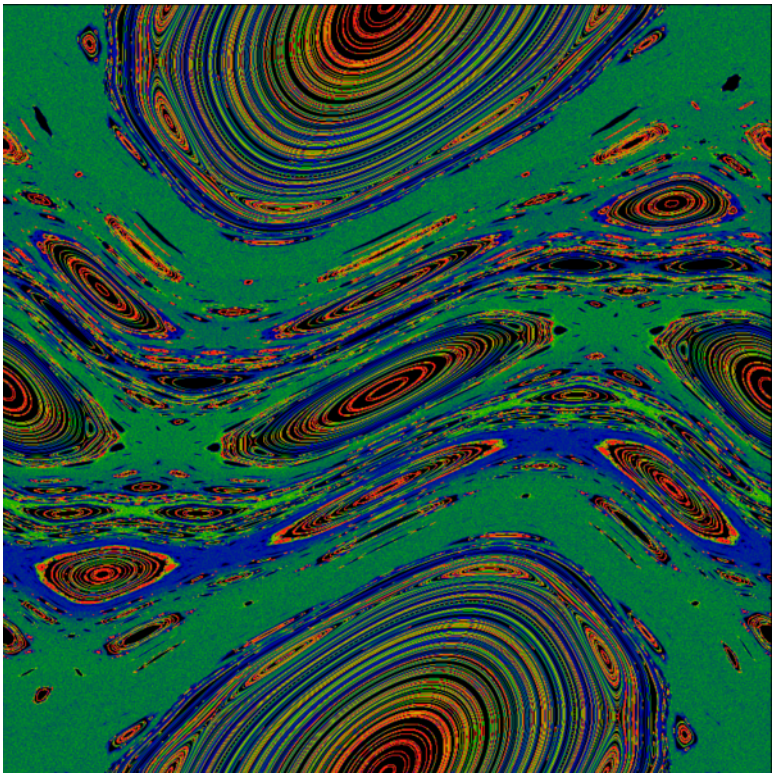

Figure 13: Standard map for $K = 0.971635$ from wikipedia. Here is the link: `https://commons.wikimedia.org/wiki/File:Std-map-0.971635.png`

4.1 Lagrangian and minimized discrete orbit path for the standard map

$$A(\boldsymbol{\theta}) = \sum_n \frac{1}{2}(\theta_{n+1} - \theta_n)^2 - K\cos\theta_n$$

where $\boldsymbol{\theta}$ is an orbit θ_0, θ_1..... Instead of integration over a continuous path we have a sum over the discrete positions in the orbit. Lagrange's equations give

$$\frac{\partial A}{\partial \theta_n} = -\theta_{n+1} - \theta_{n-1} + 2\theta_n + K\sin\theta_n = 0$$

With the identification that

$$p_{n+1} = \theta_{n+1} - \theta_n$$

we recover the standard map as an orbit that satisfies Lagrange's equations and so minimizes the sum A. The standard map is an example of a *monotone twist map* and these are a subject of Aubry-Mather theory.

Theorem (Aubry & Mather). A monotone twist map possesses orbits for every rotation number.

4.2 The map is area preserving

The map is derived from a Hamiltonian system so it must be area preserving. However we can also check that the determinant of the Jacobian must be 1.

To compute the Jacobian we write the map as

$$\begin{aligned} p_{n+1} &= p_n + K\sin\theta_n \\ \theta_{n+1} &= \theta_n + p_n + K\sin\theta_n \end{aligned}$$

where I have grouped values at iteration $n+1$ on the left and values for n on the right. The Jacobian

$$J = \det\begin{pmatrix} \dfrac{\partial\theta_{n+1}}{\partial\theta_n} & \dfrac{\partial\theta_{n+1}}{\partial p_n} \\ \dfrac{\partial p_{n+1}}{\partial\theta_n} & \dfrac{\partial p_{n+1}}{\partial p_n} \end{pmatrix} = \det\begin{pmatrix} 1 + K\cos\theta_n & 1 \\ K\cos\theta_n & 1 \end{pmatrix} = 1$$

4.3 Connection to the periodically perturbed pendulum and K as a resonance overlap parameter

Consider a Hamiltonian that has a function of the Dirac comb

$$H(p,\theta,t) = \frac{p^2}{2} + KT\cos\theta \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Take the Fourier transform of the Dirac comb

$$\begin{aligned} D_T(t) &= \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{m=\infty}^{\infty} e^{i2\pi m\frac{t}{T}} \\ &= \frac{1}{T}\sum_{m=1}^{\infty}\left(e^{i2\pi m\frac{t}{T}} + e^{-i2\pi m\frac{t}{T}}\right) + \frac{1}{T} \\ &= \frac{1}{T}\sum_{m=1}^{\infty} 2\cos\left(\frac{2\pi m t}{T}\right) + \frac{1}{T} = \frac{1}{T}\sum_{m=-\infty}^{\infty}\cos\left(\frac{2\pi m t}{T}\right) \\ &= \frac{1}{T}\sum_{m=-\infty}^{\infty}\cos(m\nu t), \end{aligned}$$

with frequency

$$\nu = \frac{2\pi}{T}.$$

The factor of $1/T$ may look mysterious, however integrate a delta function at $t=0$ and match it to the integral over time of one period of the Fourier series. Only the $m=0$ term contributes giving $\int_0^T dt = T$. We insert this form of the Dirac comb into the Hamiltonian

$$H(p,\theta,t) = \frac{p^2}{2} + K\cos\theta\sum_{m=-\infty}^{\infty}\cos\left(m\nu t\right). \tag{58}$$

We use the trigonometric relation $\cos a\cos b = \frac{1}{2}(\cos(a+b)+\cos(a-b))$. This gives

$$\begin{aligned} H(p,\theta,t) &= \frac{p^2}{2} + \frac{K}{2}\sum_{m=-\infty}^{\infty}\left[\cos\left(\theta+m\nu t\right)+\cos\left(\theta-m\nu t\right)\right] \\ &= \frac{p^2}{2} + K\sum_{m=-\infty}^{\infty}\cos\left(\theta+m\nu t\right). \end{aligned} \tag{59}$$

This is an infinite sum of resonance terms. The m-th resonance is at $p=\dot{\theta}\sim -m\nu$. This means that each resonance is separated by $\Delta p=\nu$ from its neighbors. The width of each resonance is $\Delta p=\sqrt{K}$. The resonance overlap condition occurs when $2\sqrt{K}\gtrsim\nu$.

When $T=1$, then $\nu=2\pi$ and this condition becomes $K\sim\pi^2$. This is is somewhat **off** from the $K\sim 1$ criterion for chaos dominating in the standard map. However, it might be about right for predicting when the two major resonant islands touch, but neglects when other smaller resonant islands touch.

We can think of this system as a collection of resonances separated by ν in frequency. This means that K acts like a resonance overlap parameter with large K corresponding to strong overlap. Unsurprisingly K large gives chaotic/area filling trajectories and K small gives trajectories that are 1 dimensional.

4.4 Structure in terms of periodic and irrational orbits

An orbit that is a line and contains all θ values is sometimes called a torus. Orbits are predominantly one-dimensional (lines) or area filling. A orbit that is a curved line that smoothly goes across the map would be called a torus (filling in between maps with time). An orbit can be a fixed point or a periodic orbit. Apparently there are also fractal like structures that are the remnants of tori, just after destruction that are called cantori.

Looking at the structure of the standard map you can see periodic orbits as islands of stability within a chaotic sea. A regions that divides periodic orbits is like a separatrix and becomes chaotic as K increases. Resonances that overlap cause a separatrix to become unstable.

As K is increased, which are the torri that are the last to dissolve into area filling or chaotic orbits? These are the most 'irrational' in the sense of a continued fraction expansion.

Periodic orbits (islands) can exist even at high K values. We see the dichotomy between periodic orbits which remain as islands of integrability and regions where resonances destroy stability.

Periodic orbits are sometimes called 'sticky' because orbits that are in their vicinity can stay near them for long periods of time. This gives intermittent behavior where an orbit can spend time in two different regions of phase space and behaves differently in each region, swithing back and forth at random times.

The area filling chaotic regions appear smooth but this is an artifact of the plot. If if you look at series of iterates, particularly near a region where there are torri, the orbit spends a lot of time in one spot and then can jump out of it. Often there are small islands of stability embedded within a chaotic sea.

4.5 Conjunction maps

It is much faster to iterate a map than integrate a set of differential equations. The approximation made here, where we assume that the perturbation is given as a delta function can be used to approximation some interesting dynamical systems. For example, consider a few planets in orbit around the Sun. Each planet primarily feels the effect of a nearby planet at closest approach or at conjunction. Instead of integrating the complete equations of motion, we can integrate the orbit of each planet between conjunctions and then apply a sharp perturbation to each planet. This is known as a conjunction map and gives dynamics qualitatively similar to the real system. The map can be made to be symplectic (or area preserving) and so will behave like a Hamiltonian system. The orbits of comets can be approximated in a similar fashion, by primarily perturbing the comet when it is at pericenter and assuming that its trajectory is unperturbed while it is in the outer Solar system.

4.6 Resonance Overlap and the 2/7-law for the chaotic zone at corotation

We consider a planet with mean motion n_p and an asteroid in orbit exterior to the planet with mean motion n. First order mean motion resonances in the form $jn_p = (j+1)n$. Corotation would be where $n = n_p$, an asteroid like a Trojan asteroid that is rotating with the planet. The ratio

$$\frac{n_p}{n} = \frac{j+1}{j}$$

Mean motions depend on semi-major axis to the 3/2 power.

$$\frac{n_p}{n} = \frac{j+1}{j} = \left(\frac{a_j}{a_p}\right)^{\frac{3}{2}}$$

where a_j is the semi-major axis of the $j : j+1$ resonance.

$$\frac{a_j}{a_p} = \left(\frac{j+1}{j}\right)^{\frac{2}{3}} = (1+1/j)^{\frac{2}{3}} \sim 1 + \frac{2}{3j}. \tag{60}$$

We can see that as j increases the resonances get closer and closer together. The resonances also get stronger as they are closer to the planet. The resonance strength depends on the planet mass. We can look for the j value that describes when the distance between resonances is the same size as the distance between resonances.

Distance between resonances we often describe with a frequency. In our case the frequency is that of the mean motion. Setting $GM = 1$ and $a_p = 1$

$$a_j = \left(\frac{j+1}{j}\right)^{\frac{2}{3}}$$

$$n_j = \frac{j}{j+1}$$

$$n_{j+1} - n_j = \frac{dn_j}{dj} = \frac{1}{j+1} - \frac{j}{(j+1)^2} \sim \frac{1}{j^2}$$

and this gives a distance between two nearby resonances approximately in units of energy or a_p.

How do we estimate the strength of the resonances? We want to estimate the libration frequency in the resonance. The width of the resonance (in units of semi-major axis or mean motion) is the range of mean motions that gives $|b| < \omega$ where ω is the libration frequency.

At a $j : j+1$ resonance the distance between planet and asteroid is

$$\delta = \frac{a_j}{a_p} - 1 = \frac{2}{3j} \sim \frac{1}{j}$$

where we work with distance in units of a_p and have used equation 60. The force on the asteroid as it swings by is $\frac{\mu}{\delta^2}$ where μ is the mass of perturbing planet compared to that of the central star. The length of time of encounter depends on the difference of mean motions or $n_j - n_p \sim j^{-1} \sim \delta$. The resulting kick is a velocity $\mu\delta^{-2}\delta^{-1}$. The velocity change is tangential and so the total orbital energy change is approximately the same thing and equal to $\mu\delta^{-3}$. We let this be the size of a cosine pendulum term and get a frequency that is the square root of this giving a libration frequency $\omega_0 = \mu^{\frac{1}{2}}\delta^{-\frac{3}{2}}$ and we set that equal to the frequency between resonances or $1/j^2 = \delta^2$. Solve for μ in terms of δ and we find

$$\delta = \mu^{\frac{2}{7}}$$

for resonance overlap and chaos.

This does not work!!!!!

A somewhat more reliable approach expands the potential perturbation (equation 56) from the planet in a Fourier series, extracts the relevant term and estimates the size of resonant libration from the strength of the perturbation term. The powerlaw form of the perturbation term affects the libration frequency estimate. The 2/7 law is sensitive to how the resonant libration frequencies depend on planet mass.

The resonance overlap criterion near a planet's corotation is a good predictor because the resonances have similar frequencies. Looking back to Chirikov's use of Melnikov's integral, the ratio $\nu/\omega_0 \sim 1$. The width of the chaotic region depends mostly on the perturbation strength μ and there is only a single timescale in the chaotic region that is the same as ω_0 or the resonance libration frequency. Thus the Lyapunov time is also approximately equal to the resonance libration period. In a setting where resonance separations and frequencies differ by a large amount, the Melnikov integral would be exponentially small and even if chaotic behavior is present, we may not care about it because it would be restricted to a very narrow region in phase space. The corotation region of a planet is a setting where the chaotic behavior is wide-spread.

The 2/7 law is due to Jack Wisdom, (1980).

4.7 Twist maps

A twist map is an area-preserving map on a rectangle with periodic boundary conditions in both dimensions.

$$T : (p, \theta) \to (p, \theta).$$

A twist map can be derived from a Poincaré surface of section of a 4-dimensional Hamiltonian system. Each time an orbit passes through a particular plane gives a new position on the plane. The sequence of positions on the plane is an orbit of the map.

A very boring twist map is

$$\begin{aligned} p_{n+1} &= p_n \\ \theta_{n+1} &= \theta_n + \Omega(p_n). \end{aligned} \tag{61}$$

We take momentum $p \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$ so the map is done modulo 2π. The frequency $\Omega(p)$ is assumed to be a smooth function. It's like a frequency if we think of each iteration of the map occurring after a time T. The orbits of the map conserve p. If we think of p like a radius, the map sends circles into circles. If $\Omega(p_n)$ increases, higher values of p give a larger shift in θ. This is described as a twist.

An additional requirement of a twist map is that

$$\left.\frac{\partial \theta_{n+1}}{\partial p_n}\right|_{\theta_n, p_n} \neq 0. \tag{62}$$

This condition implies that there is some twist everywhere. A twist map is **monotone** if there is a positive ($\exists \epsilon > 0$) such that

$$\left.\frac{\partial \theta_{n+1}}{\partial p_n}\right|_{\theta_n, p_n} > \epsilon \qquad \forall p_n, \theta_n \tag{63}$$

for all p_n, θ_n. This condition implies that the twist is always in the same direction. The shift in θ always increases.

How can we describe the average angular shift per map iteration in an orbit? Suppose we compute the average difference in angle after N iterations

$$\frac{1}{N} \sum_{n=0}^{N} (\theta_{n+1} - \theta_n).$$

The rotation number ν of an orbit is defined in terms of the limit

$$\nu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} (\theta_{n+1} - \theta_n) \tag{64}$$

What happens if θ_{n+1} is near 2π and θ_n is near 0? This problem arises because the map operates on a periodic space, $\theta \in [0, 2\pi]$. To better define the winding number we **lift** θ onto $\mathbb{R}$. In other words we define another version of the map, the lifted map, $\tilde{T}$, that does not take θ modulo 2π. The angle differences in equation 64 are computed using the lifted map.

4.8 Generating twist maps using a generating function

The twist map of equation 61 not very exciting. The standard map is a twist map and is more interesting.

A twist map can be generated using via a generating function, ensuring that it is area preserving. The generating function is a function of old and new coordinates $F(\theta_n, \theta_{n+1})$.

$$\begin{aligned} p_n &= -\frac{\partial F(\theta_n, \theta_{n+1})}{\partial \theta_n} \\ p_{n+1} &= \frac{\partial F(\theta_n, \theta_{n+1})}{\partial \theta_{n+1}}. \end{aligned} \tag{65}$$

The one form

$$dF = -p_n d\theta_n + p_{n+1} d\theta_{n+1} \tag{66}$$

is exact, ensuring that the map is area preserving. A common form for a twist map is

$$\begin{aligned} p_{n+1} &= p_n + \epsilon f(\theta_n) \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned} \tag{67}$$

where $f(\theta)$ is periodic. The associated generating function

$$F(\theta_n, \theta_{n+1}) = \frac{1}{2}\left(\theta_n - \theta_{n+1}\right)^2 + \epsilon V(\theta_n) \tag{68}$$

where

$$f(\theta) = \frac{\partial V(\theta)}{\partial \theta}. \tag{69}$$

The generating function, $F(\theta_n, \theta_{n+1})$, is also a Lagrangian. Some properties of the map may be derived from a sum of the Lagrangian that is like an action and is minimized on an orbit. Because of equation 65 for the momenta,

$$\frac{\partial}{\partial \theta_n}\left(F(\theta_{n-1}, \theta_n) + F(\theta_n, \theta_{n+1})\right) = 0 \tag{70}$$

where $\theta_{n-1}, \theta_n, \theta_{n+1}$ are consecutive values of the map. This equation looks similar to Lagrange's equations but for a Lagrangian that is not dependent upon velocities. Hence the generating function is called a *stationary* Lagrangian. The analogy is not straightforward as a stationary Lagrangian would be a function of coordinates at a particular time, and here the generating function is a function of coordinates at two consecutive iterations of the map.

Suppose we start at θ_j and end at θ_k. The sum of the stational Lagrangian with these two endpoints is

$$S = \sum_{n=j}^{k-1} F(\theta_n, \theta_{n+1}). \tag{71}$$

If you take the derivative with respect to an angle in the series you find equation 70. The sum S of the Lagrangian is minimized if the angles in the series are in an orbit of the map. This sum S is equivalent to an integral of the Lagrangian along a path so it's like an action.

4.9 Poincaré-Birkoff fixed point theorem

We consider an area preserving map T_0 in which the orbits are circles. A central circle C is preserved by the map $T_0(C) = C$, however points on this circle are rotated. We choose the circle C so that points on it have a rational winding number $\nu_C = N/M$ for prime integers N, M. We assume that the winding number is a smooth function of radius. A slightly larger circle $C+$ is also preserved by the map but has a higher and irrational winding number. A slightly smaller circle $C-$ is rotated by T_0 but has a smaller and irrational winding number; $\nu_{C+} > N/M, \nu_{C-} < N/M$.

Because the map's orbits are circles and because C has a rational winding number N/M, the circle C consists of fixed points of the map iterated M times; $T_0^M(x) = x$ for $x \in C$.

Now perturb the map $T_0 \to T_\epsilon$, where T_ϵ is also area preserving. We assume that for a sufficiently small perturbation to the map, the relative twist is maintained. That means that the perturbed map must have small winding numbers at small radius and larger winding numbers at larger radius.

Consider a ray from the origin that is at angle θ. At some point on this ray we must be able to find a point z where the angle of z is not changed by the map T_ϵ^M. Let $z = (r, \theta)$ and $T_\epsilon^M(z) = (r', \theta')$. At a particular angle θ we should be able to find a point z (at a particular radius r), such that $\theta' = \theta$. That does not necessarily mean that $r' = r$. Now take all rays originating from the origin and create a loop R that consists of the points for which the map T_ϵ^M that preserves the angle. Now look at the loop that is $R' = T_\epsilon^M(R)$. If the map $T_\epsilon \neq T_0$ then the loop R' is not the same as R. Because the two loops are area preserving this means that they must cross each other at least at two points. Where they cross we have fixed points, of the map T_ϵ^M. Fixed points must alternate between elliptic and hyperbolic.

Poincaré-Birkhoff theorem: For any curve C of an unperturbed system with rational winding number $\nu = N/M$ and N, M coprime of the area preserving map T_0, (and whose points are all fixed points of T_0^M), there will remain only an even number of fixed points under perturbation. That means there we remain only an even number of fixed points of the map T_ϵ^M.

Why did we define $C+$, $C-$? Orbits with irrational winding number only weakly deform under perturbation, whereas the orbit with ration winding number that gave lots of fixed points (for an iterated map) is more prone to changing shape.

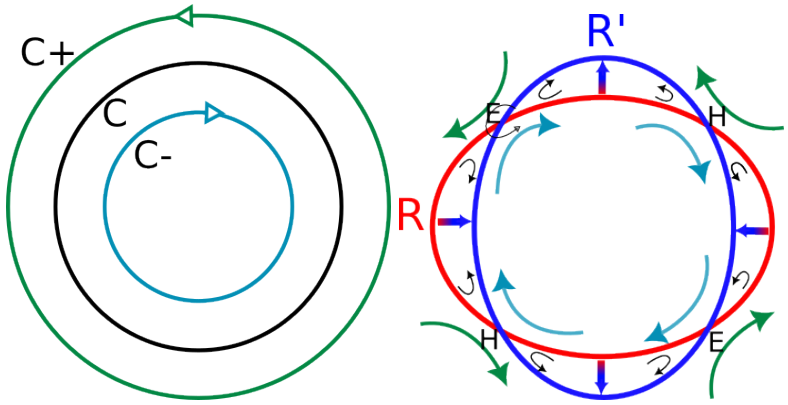

Figure 14: Illustrations relevant for the Pincaré-Birkhoff theorem. On left: The loops $C, C+, C-$ for the unperturbed map and how they rotate under the map T_0^M. The loop C has a rational winding number $\nu = N/M$. For the perturbed map $T_\epsilon^M(R) = R'$ we create a loop R in the following way. For each ray from the origin at angle θ we find a radius at which the map preserves the angle. The different points that are found at different ray angles give the loop R. On Right: How loops R and $T_\epsilon^M(R) = R'$ intersect. The points of intersection are fixed points of the iterated perturbed map T_ϵ^M. There are pairs of fixed points. The elliptic points are labelled with E and the hyperbolic ones are labelled with H.

Why is this theorem relevant? For twist maps it establishes that under perturbation, hyperbolic points appear that can then go unstable. In a twist map, the Poincaré-Birkhoff theorem assures the existence of a hierarchy of stable and unstable fixed points. All the hyperbolic fixed points lead to chaotic structures via perturbations on stable and unstable manifolds through the process shown in Figure 6 and in Figure **??**. Therefore, chaotic motion is present in a self-similar manner on all scales.

4.10 Destruction of KAM tori

The nonresonant tori (or orbits) that have not been destroyed by resonances are called KAM tori. On the standard map, or a monotone twist map and for small mapping parameters, resonance zones are separated from one another by KAM tori. The mechanism by which a KAM torus is destroyed and converted to a cantorus is universal to all area-preserving twist maps. Each KAM torus has an irrational winding number. Every irrational number

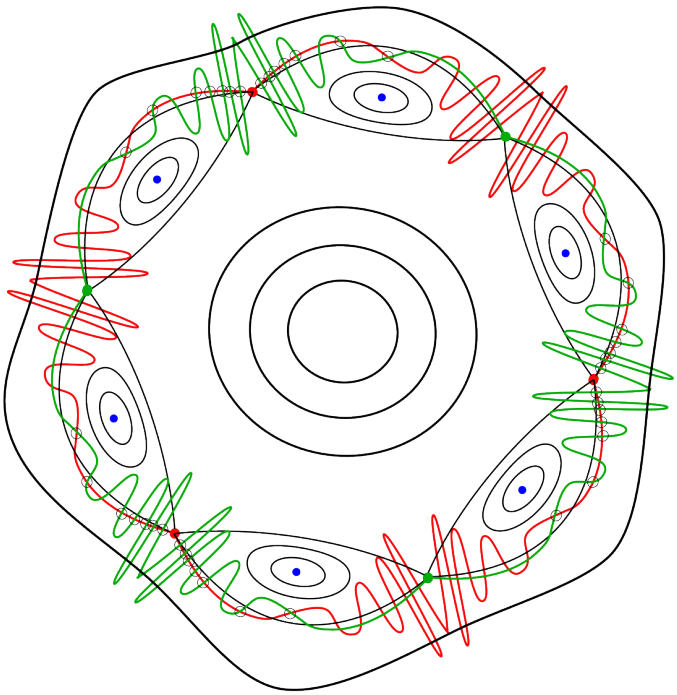

Figure 15: Pairs of hyperbolic fixed points going unstable. The elliptic fixed points are marked with small blue circles and the hyperbolic ones with green and red circles. Pairs of hyperbolic and elliptic fixed points are generated via the Poincaré-Birkhoff fixed point theorem. Stable and unstable manifolds are shown in red and green and cannot intersect, giving what is called the heteroclinic tangle.

can be approximated by a unique sequence of fractions, given by a continued fraction, that converges to the irrational number. Therefore, we can represent every winding number in terms of a unique continued fraction,

$$\nu = a_0 + \frac{1}{1 + a_1 + \frac{1}{1+a_2+\frac{1}{1+a_3\ldots}}} \tag{72}$$

where $a_0, a_1 ...$ are all integers. A rational approximation to an irrational ν is made by computing the continued fraction expansion out to a particular index (which a_i is the last one used in the continued fraction expansion). The higher the index, the more accurate the approximation. The most irrational number, that is most poorly approximated by the series of rational approximations is the irrational number computed from the sequence of 1s (where $a_i = 1$ for all i). This number is called the golden mean.

KAM tori are destroyed by resonances between degrees of freedom whose periods are rationally related. Thus, each rational approximate will be associated with a resonance in phase space and a corresponding island chain. As the parameter K of the standard map increases, these resonance regions grow and finally destroy their neighboring KAM tori. The last torus to be destroyed is the one that is the most distant from all resonances and that is the one that is most poorly approximated by rational approximations. The last KAM torus to be destroyed is the one with winding number similar to the golden mean.

5 Adiabatic variations

Consider a time dependent Hamiltonian system $H(p, q, \lambda(t))$, where $\lambda(t)$ varies slowly. We can look at the associated time independent system $H(p, q, \lambda_0)$ where λ_0 is fixed.

Suppose we have an orbit in phase space $p_0(t)$, $q_0(t)$ that is closed in the time independent system. When we allow λ to vary, the system is still a Hamiltonian system, which implies that volume in phase space is conserved. We assume that variation in H takes place *slowly* compared to the orbit's period. In fact, this is the condition for a variation of H to be 'adiabatic'.

Because the Hamiltonian varies slowly we can think of the system as a loop in phase space that is slowly changing shape. For example consider the red loop slowly deforming into the blue loop in Figure 16. The action is the integral over the closed orbit

$$I = \oint p dq.$$

Even if the orbit is changing shape, the area enclosed by the orbit should be constant. This means that the action I should be constant for adiabatically varying systems. If the system is integrable and adiabatically varying, then its behavior can be predicted by varying the frequencies, not the action variables. which should remain fixed.

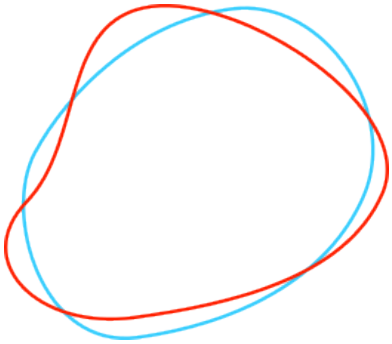

Figure 16: An orbit in phase space slowly changes shape, while keeping its area fixed. The red and blue lines show the orbit at different times for a slowly varying Hamiltonian system. The action variable remains fixed for an adiabatically varying system.

5.1 The drifting harmonic oscillator

We consider a harmonic oscillator

$$H(p,q) = \frac{p^2}{2} + \lambda(t)\frac{q^2}{2}$$

We start at a fixed time with $\lambda = \lambda_0$. We transform to action angle coordinates

$$H(I,\phi) = I\omega$$

with frequency

$$\omega = \sqrt{\lambda_0}.$$

The canonical transformation has $I = \lambda_0^{-1}p^2/2 + q^2/2$.

We now consider an adiabatic perturbation. Because the action variable is conserved, we infer that

$$H(I,\phi,t) = I\omega(t)$$

with

$$\omega(t) = \sqrt{\lambda(t)}.$$

What is the condition for the perturbation to be adiabatic? Notice that λ is in units of the square of frequency. We can put the drift rate in units of time by estimating $\frac{1}{\lambda}\frac{d\lambda}{dt}$. This should be slow compared to the oscillation frequency of the system itself. This means that

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \sqrt{\lambda}$$

for the drift rate to be adiabatic.

5.2 The drifting pendulum

The pendulum is interesting because it has an orbit, the separatrix, that has an infinite period. That means that no matter how slowly the system drifts, there is a location in phase space where the drift rate is not adiabatic. Instead of focusing on an action that can be conserved, we look at the rate that phase space volume crosses the separatrix. In other words, we assume that the drift is adiabatic most of the time and only consider what happens when there is a transition across a region, the separatrix, when the drift is not adiabatic. Note that, for a drifting Hamiltonian system, volume in phase space is conserved, even if the drift is not adiabatic.

We take a Hamiltonian for the pendulum

$$H(p,\theta) = a\frac{p^2}{2} + bp + \epsilon\cos\theta.$$

Orbits for a fixed pendulum are shown in Figure 17.

Any or all of the coefficients a, b, ϵ could be time dependent. With these coefficients time dependent, we can consider

- V_+ the rate that the upper separatrix sweeps up phase space volume.
- V_- the rate that the upper separatrix sweeps up phase space volume.
- $V_+ - V_-$ is the volume growth rate of the libration region.

Assuming that the libration region is growing, and that the resonance is moving upward, the probability that a particle is captured into the libration region is

$$P_{capture} = \frac{V_+ - V_-}{V_+}.$$

When is the drift of a pendulum adiabatic? To answer this question we need to understand characteristic timescales in the problem. First notice that b is in units of frequency. Notice that we can write

$$\frac{a}{2}p^2 + bp = \frac{a}{2}\left(p + \sqrt{\frac{b}{a}}\right)^2 + \text{constant}$$

A shift in b essentially shifts the momentum origin.

The characteristic libration frequency is $\omega_0 = \sqrt{\epsilon a}$. This can be seen by expanding about the stable fixed point. This frequency is also characteristic of the time-scale for divergence from the hyperbolic fixed point in the separatrix.

The width of the resonance is $\sqrt{2\epsilon/a}$ (units of momentum). A system is adiabatic if the time it takes to drift across the resonance is longer than the oscillation period.

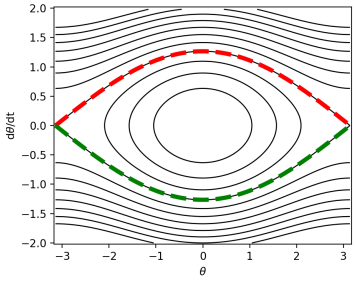

Figure 17: The rate that phase space volume is swept by the red dashed line, showing the top separatrix is V_+. The rate that phase space volume is swept by the bottom separatrix, the dashed green line, is V_-. The probability of capture into the libration region is $P_{capture} = \frac{V_+ - V_-}{V_+}$.

The time to drift across the resonance width is $a/\dot{b} \times \sqrt{\epsilon/a}$ and this should be larger than the oscillation period which is of order $1/\sqrt{\epsilon a}$. Altogether this gives

$$\dot{b} \ll \sqrt{\epsilon a}$$

for the drift in b to be adiabatic.

If only ϵ or a are drifting then b should be irrelevant. If only ϵ is drifting then a condition for adiabatic drift would be

$$\frac{1}{\epsilon}\dot{\epsilon} \ll \sqrt{\epsilon a}$$

If only a is drifting then a condition for adiabatic drift would be

$$\frac{1}{a}\dot{a} \ll \sqrt{\epsilon a}$$

All of these expressions are essentially dimensional!

For an illustration of a drifting system, see Figure 19. Example code is available at `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/drift_integrate.ipynb` or `https://astro.pas.rochester.edu/~aquillen/phy411/pylab/drift_integrate.html`.

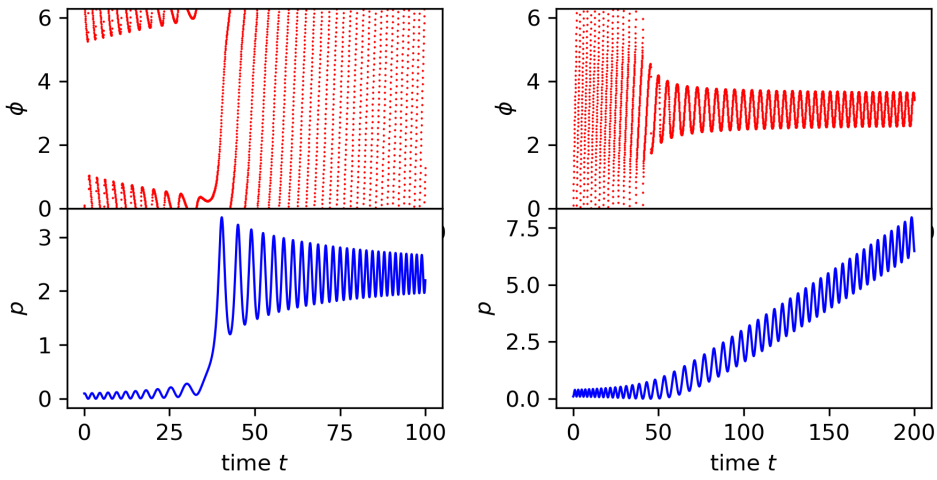

Figure 18: Illustrations of integrations of a drifting resonant system. Top panels are angle and bottom panels are momentum. The Hamiltonian integrated is $H(p,\phi) = p^2/2 + b(t)p + p^{\frac{1}{2}}\cos\phi$. The initial condition for both integrations is $\phi_0 = 0.1, p_0 = 0.1$. On the left $b(0) = -3$ and the drift rate $\dot{b} = 0.05$. On the right $b(0) = 3$ and $\dot{b} = -0.05$. On the left the system jumps across resonance. The system is initially librating near $\phi \sim 0$ and is circulating at later times. On the right the system is librating at later times but circulating at earlier times. After being captured into the resonance, the momentum continues to increase with time.

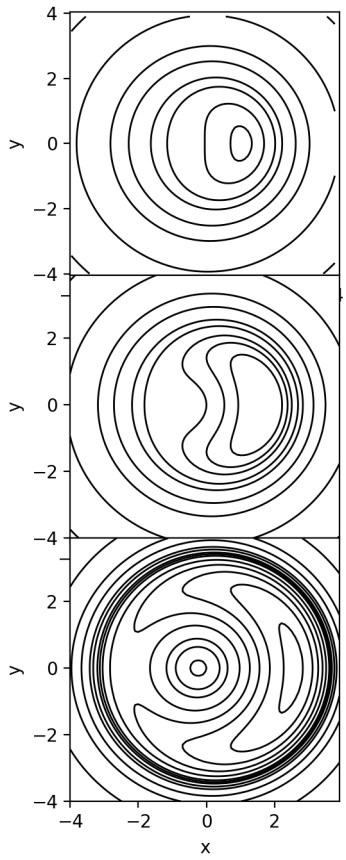

Figure 19: Illustrations of integrations of the level curves of the Hamiltonian $H(p,\phi) = p^2/2 + b - \epsilon\cos\phi$ with $\epsilon = 1$. The axes are $x = \sqrt{2p}\cos\phi$ and $y = \sqrt{2p}\sin\phi$. From top to bottom we show $b = 0, -1, -3$, respectively. A system with a varying b would go from top to bottom if b is decreasing. A libration region at negative $\phi \sim \pi$ is born as b decreases, and there is a separatrix in the bottom panel but not in the others. Particles either jump to this island or librate at $\phi \sim 0$ and are pushed to high momentum as b decreases.

6 Closed orbits and averaging

Suppose we have a nearly integrable Hamiltonian system

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}, \boldsymbol{\theta}).$$

Chose $\mathbf{I}$ such that H_0 gives a periodic orbit with duration T. This means that frequencies

$$\boldsymbol{\omega} = \frac{\partial H_0}{\partial \mathbf{I}}$$

satisfy

$$\boldsymbol{\omega} T = 2\pi \mathbf{k}, \tag{73}$$

where $\mathbf{k}$ is a vector of non-zero integers.

To zero-th order $\mathbf{I}$ is fixed and $\boldsymbol{\theta}(T) = \boldsymbol{\theta}(0)$ returns to its initial value after time T. To zero-th order

$$\theta_i = \theta_i(0) + \omega_i t = \theta_i(0) + 2\pi k_i \frac{t}{T}.$$

Equivalently

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + 2\pi \mathbf{k} \frac{t}{T}. \tag{74}$$

It is convenient to define an angle $\phi = 2\pi t/T$ which goes from 0 to 2π while t ranges from 0 to T. To zero-oth order

$$\theta_i = \theta_i(0) + k_i \phi. \tag{75}$$

We can also write this as

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \mathbf{k}\phi. \tag{76}$$

For the full solution how much does $\mathbf{I}$ change after time T? We can approximate this by averaging over time T. We define the function

$$f_i(\mathbf{I}, \boldsymbol{\theta}) = \frac{\partial H_1(\mathbf{I}, \boldsymbol{\theta})}{\partial \theta_i}$$

that is necessarily periodic with respect to each angle. Because $\dot{\mathbf{I}} = -\frac{\partial H}{\partial \boldsymbol{\theta}}$,

$$\dot{I}_i \sim -\frac{\epsilon}{T} \int_0^T f_i(\mathbf{I}, \boldsymbol{\theta}) dt$$

We insert the zero-th order solution

$$\begin{aligned} \dot{I}_i &\sim -\frac{\epsilon}{T} \int_0^T f_i(\mathbf{I}, \boldsymbol{\theta}(t=0) + 2\pi \mathbf{k} t/T) dt \\ &\sim -\frac{\epsilon}{2\pi} \int_0^{2\pi} f_i(\mathbf{I}, \boldsymbol{\theta}_0 + \mathbf{k}\phi) d\phi \end{aligned} \tag{77}$$

This depends on an average over the zeroth order periodic orbit.

The function has to start and end at the same value because the orbit is periodic. Take the Fourier transform of H_1 giving

$$H_1(\mathbf{I}, \boldsymbol{\theta}) = \sum_{\mathbf{j}} a_j(\mathbf{I}) \cos(\mathbf{j} \cdot \boldsymbol{\theta} + \theta_{0,j}) \tag{78}$$

$$f_i(\mathbf{I}, \boldsymbol{\theta}) = \frac{\partial H_1}{\partial \theta_i} = -\sum_{\mathbf{j}} a_j(\mathbf{I}) j_i \sin(\mathbf{j} \cdot \boldsymbol{\theta} + \theta_{0,j}) \tag{79}$$

Insert this into equation 77

$$\dot{I}_i \sim \frac{\epsilon}{2\pi} \int_0^{2\pi} d\phi \sum_j a_j(\mathbf{I}) j_i \sin(\mathbf{j} \cdot \mathbf{k}\phi + \mathbf{k} \cdot \boldsymbol{\theta}_0 + \theta_{0,j}) \tag{80}$$

As both $\mathbf{j}, \mathbf{k}$ are vectors of integers, this is equal to zero. A constant term cannot contribute because H_1 itself must be periodic and depend upon the angles. The constraint that the orbit is periodic implies that the momenta cannot change very much.

To do better than this approximation, we need to insert the zero-th order solution for $\boldsymbol{\theta}(t)$ (from equation 74) back into the Hamiltonian and do an expansion to first order in ϵ.

Suppose we relax the assumption that the unperturbed system is at a periodic orbit to assuming that it is **near** a periodic orbit. Equation becomes

$$\boldsymbol{\omega} T = 2\pi \mathbf{k} + \boldsymbol{\delta}, \tag{81}$$

where $\boldsymbol{\delta}$ is a vector of small quantities. Equation 74 becomes

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + (2\pi \mathbf{k} + \delta) \frac{t}{T} \tag{82}$$

$$= \boldsymbol{\theta}_0 + \left(\mathbf{k} + \frac{\boldsymbol{\delta}}{2\pi} \right) \phi \tag{83}$$

Equation 80 becomes

$$\begin{aligned} \dot{I}_i &\sim \frac{\epsilon}{2\pi} \int_0^{2\pi} d\phi \sum_j a_j(\mathbf{I}) j_i \sin\left(\mathbf{j} \cdot \mathbf{k}\phi + \frac{\mathbf{j} \cdot \boldsymbol{\delta}}{2\pi} \phi + \mathbf{k} \cdot \boldsymbol{\theta}_0 + \theta_{0,j} \right) \\ &\sim \frac{\epsilon}{2\pi} \sum_j a_j(\mathbf{I}) j_i \mathbf{j} \cdot \boldsymbol{\delta} \end{aligned} \tag{84}$$

I assumed $j \cdot \delta < 2\pi$, but this might not have been a good idea.

It is always possible to find a nearby periodic orbit for the zero-th order system for any initial I if T is chosen to be sufficiently large. The size of the chosen T givse a limit on the size of the frequencies $\boldsymbol{\delta}$ and the size of the integers $\mathbf{k}$. For larger T you need larger integers but frequencies errors are smaller. The change in I after duration time T can be estimated from the Fourier series. If using a Fourier series to place a limit, it is handy to place a limit on a_j based on $|\mathbf{j}|$ and using the smoothness of the function H_1.

This discussion is relevant to the idea that periodic orbits are sticky and the Nekhoroshev theorem that can be discussed in terms of how periodic orbits place limits on the extent of drifting possible in a weakly perturbed Hamiltonian system. (See work by Pierre Lochak).