PHY411 Lecture notes Part 14

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December 25, 2023

Contents

1 Introduction

We ask about the relation between the dynamics of a classical system, and that of an associated quantized system. Whether or not a classical system is chaotic affects the statistics of the energy spectrum of the related quantum system. This relationship is made more precise via two conjectures.

1.1 Berry-Tabor conjecture

Berry-Tabor conjecture: In the limit of large energies (semiclassical limit), the statistical properties of the quantum spectra of classically integrable systems correspond to the prediction for randomly distributed energy levels.

Assume a random distribution of ordered numbers on the real line. We call each number an event. The number of events per unit time we take to be r. In any interval of time t

the number of events is given by the Poisson probability distribution

$$P(k \text{ events in interval } t) = \frac{(rt)^k e^{-rt}}{k!}. \tag{1}$$

To find the probability of times between events we need to know the probability that there are no events out to time t, which is $P(k = 0 \text{ events in interval } t) = e^{-rt}$ (using equation 1). Let $g(s)$ be the probability of a spacing s between two events. If we integrate $\int_0^t g(s)ds$ this should equal the probability of getting a subsequent event before time t after an event at $t = 0$. This should be equal to $1 - e^{-rt}$, where we subtract the probability of getting no events in time interval t. Setting the two expressions to be equal gives

$$\int_0^t g(s)ds = 1 - e^{-rt}.$$

Taking the derivative of both sides gives $g(t) = re^{-rt}$. This gives the probability distribution for the spacing s between events;

$$g(s) = re^{-rs}. \tag{2}$$

For a quantum system that is derived from an integral classical system, the Berry-Tabor conjecture implies that the differences between energy levels are described with a Poisson probability distribution. The probability distribution of differences between energy levels would look like equation 2.

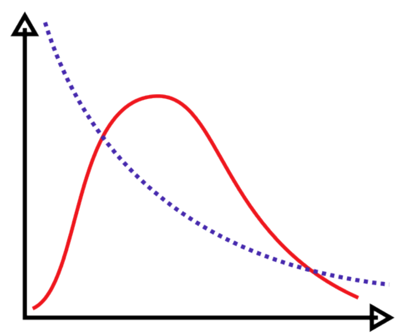

Figure 1: The Poisson (dashed blue line) and Wigner (solid red line) distributions. For quantum systems derived from integrable classical systems, the distribution of energy level spacings is Poisson. For quantum systems derived from chaotic classical systems, the distribution is a Wigner distribution.

1.2 Bohigas-Giannoni-Schmit conjecture

Bohigas-Giannoni-Schmit conjecture: The eigenvalues of a quantum system, whose classical analogue is fully chaotic, obey the same universal statistics of level spacings as predicted for the eigenvalues of Gaussian random matrices. The distribution is called the Wigner-Dyson distribution.

We describe the distribution for differences between energy levels for what is known as the GUE which is the Gaussian Unitary Ensemble. The GUE describes randomly chosen matrices. Note that here Unitary refers to transformations preserving the probability function, not a property of the random matrix itself.

We assume that the system is described by a Hermitian operator or Hamiltonian, H. The probability function $p(H)$ tells us how likely we are to have any particular Hamiltonian.

We find a basis for the quantum space. We choose the matrix elements in this basis. We assume the number of possible states is large.

We assume that the matrix elements are independent of each other and that the matrix elements are each chosen with probability distributions.

We assume that the probability $p(H)$ is independent of basis. This means that the probability $p(U^\dagger H U) = p(H)$ for all unitary transformations U.

For H an $n \times n$ Hermitian matrix, the Gaussian Unitary Ensemble has probability of H given by the function

$$p(H) = \frac{1}{Z_{\text{GUE}}} e^{-\frac{n}{2} \operatorname{tr} H^2}. \tag{3}$$

The normalization constant

$$Z_{\text{GUE}} = 2^{n/2} \left(\frac{\pi}{n}\right)^{\frac{1}{2}n^2}. \tag{4}$$

The normalization constant is set by integrating the probability distribution over all possible values for the matrix elements and setting the result to 1.

Suppose we order the eigenvalues of H, with $\lambda_1 \leq \lambda_2 .... \leq \lambda_n$. We define the spacing between consecutive energy levels as $s_i = (\lambda_{i+1} - \lambda_i)/\langle s \rangle$. For the GUE the distribution of energy level spacings is

$$p(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi} s^2}. \tag{5}$$

Notice that this has a Gaussian form rather than an exponentially decaying form as in equation 2. With the Poisson distribution, energy spacings are likely to be small. With the GUE, energy levels don't tend to be near each other. This is consistent with the idea that perturbations tend to cause degenerate energy levels to repel rather than cross.

The tendency of quantized chaotic systems to have energy level differences distributions like that of the GUE is consistent with the idea that these systems are disordered.

Numerical studies and experimental studies of classical billiard systems and their quantum counterparts support these two conjectures.

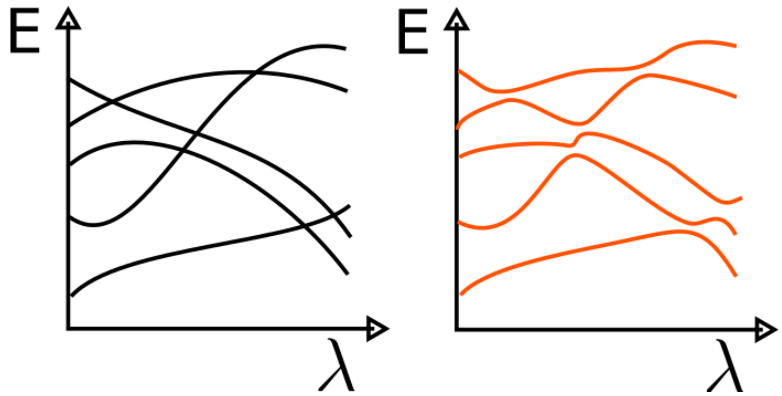

Figure 2: On the left an integrable system has a high degree of symmetry. The parameter λ is varied and this causes variations in the energies of the eigenstates. Energy levels can be degenerate. On the right the system is not integrable and energy levels repel each other.

1.3 Measures and metrics for random matrices

We need a way to integrate the probability distribution for square matrices. We need a measure. First we will define a notion for a distance between matrices. Then we will constrain the matrix via symmetries. Then we will make a volume element that we can use to integrate over all matrices that obey those symmetrices.

For an N dimensional Hilbert space, a linear transformation has N^2 elements. We can specify the distance between two matrices W_1, W_2 with the trace distance

$$d = \mathrm{tr}\left(W_1 W_2^\dagger\right)$$

A metric for the matrix dW is

$$ds^2 = \mathrm{tr}\left(dW dW^\dagger\right) = \sum_{ij} |dw_{ij}|^2$$

where dw_{ij} is the complex number describing the variation in the i,j-th element of the matrix. The trace distance is invariant under unitary transformations.

If there is a symmetry, then not all matrix elements of dW are independent. We denote dx_a as a list of independent elements. Then with those alone, we rewrite the metric, $ds^2 = \sum_a g_a dx_a^2$. A volume element for integrating quantities that depend on these independent variables, known as a **measure** is

$$d\Omega = \sqrt{|\mathrm{Det} g|} \prod_a dx_a \tag{6}$$

With a Hermitian matrix, each diagonal element h_{ii} is real. Off diagonal elements have real and complex parts but each element is related to its transpose element; $h_{ij} = h_{ji}^*$ (for

$i \neq j$). Since each matrix element is independent of any other element, we can describe the probability distribution of the whole matrix as a product of probability functions, one for each diagonal element and two for the off-diagonal elements on one side of the diagonal $p(H) = \prod_{i=1}^{n} f(h_{ii}) \prod_{i<j,\in 1\ldots n} g(\text{Re}[h_{ij}])g'(\text{Im}[h_{ij}])$. Each of the functions f, g, g' are normalized probability functions. In this setting the metric is

$$ds^2 = \sum_j dh_{jj}^2 + 2 \sum_{i<j} dh_{ij} dh_{ij}^*$$

where the first term contains real diagonal elements and the second term contains off diagonal elements. The factor of 2 is because we used the trace distance to calculate this! For the off diagonal elements, we write $dh_{jk} = dh_{jk}^R + i dh_{jk}^I$ in terms of real and imaginary parts. The invariant measure is

$$d\Omega = 2^{N(N-1)/2} \prod_j dh_{jj} \prod_{i<j} dh_{ij}^R dh_{ij}^I. \tag{7}$$

2 Billiards

Consider a particle in a 2D rink (or stadium) or a 3D cavity. Call the space inside the rink $\Omega \in \mathbb{R}^2$ and the boundary of the rink or cavity $\partial\Omega$. The Hamiltonian for a classical problem has Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

with $V(q) = 0$ for $q \in \Omega$ inside the rink and $V(q) = \infty$ for $q \notin \Omega$ outside the rink. When a particle hits the boundary, the boundary condition causes it to reflect off of it elastically. The physical setting is that of a free particle in a bounded region that reflects elastically when it encounters the boundary. Within the cavity, momentum p is conserved. At at an impact with the boundary, the particle velocity reflects about the vector normal to the boundary curve or surface. In other words, the angle between the particle trajectory and the curve normal vector flips sign after the encounter. The change of direction at for an elastic bounce is called the specular reflection rule as it is equivalent to a light ray bouncing off a mirror.

2.1 Classical Billiards

The particle position on the boundary is given by the parameter $s \in \partial\Omega$. The parameter s is normalized by the length of the boundary so that $s \in [0, 1]$. For a 2D stadium, as shown in Figure 3, the particle direction of motion is given by an angle θ. The particle trajectory can be described with a map between pairs of these two quantities. This type of billiard map is attributed to Birkhoff.

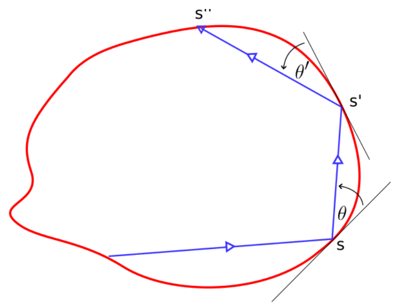

Figure 3: A classical billiard dynamics problem. A small particle, confined to move on a frictionless plane, bounces off a rigid loop boundary without loosing any kinetic energy.

The angle $\theta \in [0, \pi]$ with $\theta = 0$ tangent to the surface. The direction that s increases on the loop defines a particle direction from the point of impact. The angle of a trajectory has $\theta = 0$ if the particle is grazing the boundary and moving in the direction of increasing s. Equivalently we can take $\mu = \cos\theta$ with $\mu \in [-1, 1]$. If the boundary is a closed curve (a loop), then s is periodic.

We call X the space with points (s, μ). The billiard map is

$$T : (s, \mu) \to (s', \mu')$$

mapped from just after a bounce to just after the next bounce. As $\mu \in [-1, 1]$, and $s \in [0, 1]$ and periodic, the space is equivalent to an annulus. We assume that $T(s, -1) = T(s, -1)$ are fixed points and $T(s, 1) = T(s, 1)$ are also fixed points. The fixed points make sense if you consider the limit of trajectories that have angle closer and closer to grazing the boundary. If the trajectory is parallel to the boundary when it hits the boundary, then it stays stuck on the boundary. If the boundary is smooth, these limits along with the specular reflection condition cover all possible bounces.

The specular reflection condition depends upon the tangent of the boundary curve at the impact point. If the boundary has a corner, then the tangent at the vertex doesn't exist. In this case, one must decide what happens if a trajectory directly hits the corner; (does it bounce back? if so at what angle? or does it simply stop?). If there is a sharp corner orbits can bounce and infinite number of times back and forth as they approach the vertex.

We define $l(s, s')$ to be the Euclidean length of the segment connecting points s, s' on the boundary. Consider small variations in the position of s and the position of s' as shown in Figure 4. The angles θ, θ' are defined by the tangent lines to the boundary at s, s'. The change in length of the segment

$$dl = -ds\cos\theta + ds'\cos\theta'. \tag{8}$$

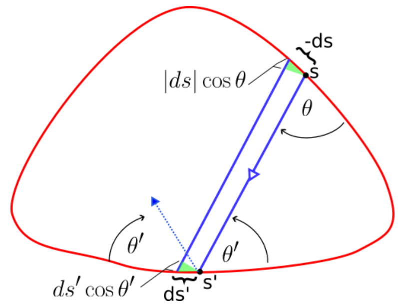

Figure 4: The length of the segment connecting s and s' is $l(s,s')$. The segment length l varies as $dl = |ds|\cos\theta + ds'\cos\theta'$.

If the slope $\theta = \pi/2$ then $\frac{\partial l(s,s')}{\partial s} = 0$. As $\theta \to 0$ we see that $\left|\frac{\partial l(s,s')}{\partial s}\right| = 1$, as expected. If the boundary is smooth, and the tangent to it defined so that the angles θ and θ' are defined,

$$\begin{aligned} \frac{\partial l(s,s')}{\partial s} &= -\cos\theta = -\mu \\ \frac{\partial l(s,s')}{\partial s'} &= \cos\theta' = \mu'. \end{aligned} \tag{9}$$

The billiard map can be constructed using $-l(s,s')$ as a generating function. Assume we have canonical variables s,μ. Construct a canonical transformation using generating function of new and old coordinates $F_1(s,s') = -l(s,s')$. The new and old momenta are $\mu = \partial_s F_1$ and $\mu' = -\partial_{s'} F_1$ and consistent with equations 9. The system has a 2-form $\omega = ds \wedge d\mu$ that is preserved by the map. If the boundary is smooth and convex, the map is symplectic and area preserving.

As $-l(s,s')$ is a generating function, it gives a Lagrangian. Orbits of the map also minimize a path distance.

An orbit is all points $y : \{T^n(x_0) = y\}$ for non-negative integers n and initial condition x_0.

A periodic orbit is one that gives $T^n(x) = x$ for an integer iteration $n > 1$ and $x \in X$ is a point (s,μ).

An orbit is a finite set of points if is a fixed point or if it is a periodic orbit.

One definition of **integrability** is that there exists a function f (piecewise continuous) $f : X \to \mathbb{R}$ such that each set $S_c : \{s,\mu\}$ defined by $f(s,\mu) = c$ is either a union of points or lines. Here X is the two dimensional space for position on the boundary and trajectory angle, with a point (s,μ).

Another definition for **ergodic**: All invariant sets (aka orbits) of the map either have measure 0 or measure equal to that of the full space. In other words, the map is ergodic if the orbits fill an area and the map is integrable if the orbits are either points or curves in the space s, μ.

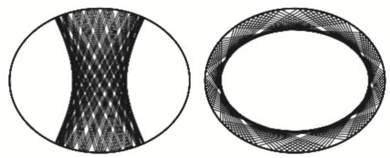

Figure 5: Types of orbits for the ellipse billiard. On the left the caustic is a hyperbola. On the right the caustic is an ellipse. The type of orbit depends upon whether the trajectory passes inside or outside the ellipses two focal points. Taken from the book by U. A. Rozikov, *An Introduction to Mathematical Billiards*.

The ellipse billiard system is integrable. Consider two points $T(s_1, \mu_1) = (s_2, \mu)$. T the segment connecting two points in an orbit are tangent to an ellipse that is inside the boundary or are tangent to a hyperbola. The distance between the focal points of this inner ellipse or hyperbola give a smooth function that can be used to show that the billiard system is integrable. A convex curve lying in the interior of the billiard table with the property that any billiard trajectory tangential to it stays tangential after reflection at the boundary, will be called a caustic. Existence of caustics gives rise to existence of invariant curves for the billiard map.

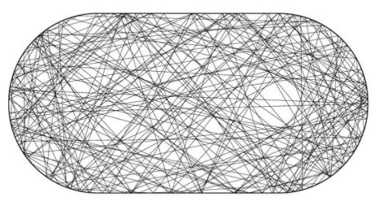

Figure 6: Example of a chaotic orbit in the Bunimovich stadium billiard system.

The Bunimovich stadium is a 2D rink comprised of a rectangle with two semicircles added at each end. Even though this stadium is convex everywhere, there are ergodic orbits. Chaos and order can co-exist. Trajectories in billiards may be chaotic or regular (quasiperiodic), depending on the initial condition. Systems can display intermittency. When chaotic trajectories approach the regular region they can spend a very long time close to them, acting as if they were not chaotic orbits, before visiting again the rest of the chaotic region. Rinks with cusps (a triangle) or that are not convex (a mushroom shape) are particularly prone to chaos.

2.1.1 Notes

Related dynamical problems allow trajectories to be curved as they travel across the space (as in Artin's and Hadamard's billiard problems). For example, a ball rolling in a rink that does not have a flat bottom. Sinai's billiard is a free space billiard on a square with a disk removed from its interior. In this system most trajectories are ergodic.

There is a KAM-like theory for billiard systems by Lazutkin. If the boundary is sufficiently smooth then orbits have caustics and are integrable. If the boundary has a discontinuity, then orbits tend to be ergodic.

Writing code to integrate trajectories for an smooth loop bounday on a 2d rink requires some effort. Suppose you start with a parametric form $x(t), y(t)$ for the boundary with $t \in [0, T]$. Firstly you need a function for the the distance along the boundary to relate s to t where $s \in [0, 1]$.

$$s(t) = \frac{1}{T} \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

The orientation of the tangent vector at a point $(x_0, y_0) = (x(t_0), y(t_0))$ depends on angle $\phi = \text{atan2}(y'(t_0), x'(t_0))$. With the tangent vector orientation angle ϕ and the trajectory angle $\mu = \cos\theta$, the linear trajectory leaving the point can be specified $(x(u), y(u)) = (\cos(\phi + \theta), \sin(\phi + \theta))u + (x_0, y_0)$ where u specifies the distance from x_0, y_0 of each point on the line. To iterate the map, the next step is to determine where this line intersects the boundary at a location other than (x_0, y_0). This requires either minimization or root finding. If you choose root finding, then you need to bracket the root with a function that is positive at one side of the bracket and negative on the other side. It is clearer why people often choose rinks that are described by simple geometric shapes so that intersections can be more easily computed.

It might be interesting to explore billiards with slowly varying boundaries. For example, Bunimovich stadium could be varied by increasing its length, while decreasing the diameter of its caps to maintain a constant perimeter length. The position of the transition could be drifted. II would carry out the change in shape, while maintaining s, μ, then compute the next bounce with the new stadium shape. Other stadiums that could be varied are the D and ripple stadiums. The ripple stadium can be derived semi-analytically in the associated quantum system via a coordinate transformation.

The ripple stadium can be defined as an area with internal x, y with horizontal $x \in [-\pi, \pi]$, the left wall at $x = -\pi$, the right wall at $x = \pi$, the bottom wall at $y = 0$ and the top wall at $y = d + a\cos x$. The shape of the stadium depends on the ratio a/d. With small a the system is integrable and with large a, the system is chaotic. Following Luna-Acosta

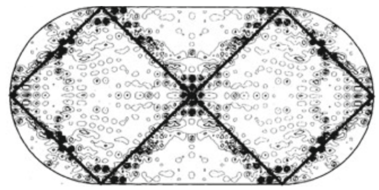

Figure 7: The Bunimovich stadium is a chaotic classical billiard system. A periodic orbit is shown with a solid black line. In the quantized system, there is an eigenstate with wave function that resembles the periodic orbit. This is a piece of a figure by Heller 1984, but taken from the textbook by Wimberger.

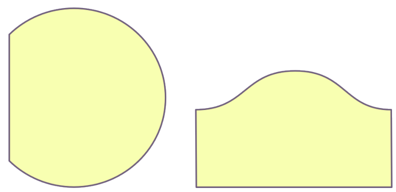

Figure 8: The D billiard on the left is a truncated circle. The ripple billiard is on the right and has a sinusoidal upper wall.

et al. (1996)[1], the system is not plotted from bounce to bounce, but rather only at times when it touches the $y = 0$ bottom wall. The coordinate system used is $(x, \sin\alpha)$ where x is the position (in $[-\pi, \pi]$ along the bottom wall) and α is the angle between surface normal and trajectory. Since the wall is horizontal at this bounce, the x component of momentum is preserved by the bounce. So the map is essentially a map between points (x, p_x). Why is this canonical? We could start with x, y, p_x, p_y with standard canonical two-form $x \wedge p_x + y \wedge p_y$, but if we choose only to plot at $y = 0$, we are essentially constructing a surface a section every time the system goes through $y = 0$ with two-form $x \wedge p_x$. Examples of orbits for the ripple stadium are shown in Figure 10. It was moderately tedious to get this working. There were a bunch of cases to consider for each bounce and finding where the trajectories intersected the top wall involved calling a root finding routine. The trickiest issue involved dealing with trajectories that bounced off the top wall twice.

Note since surfaces of section were plotted for the classical system, the comparison Hussimi function for the associated quantum system was computed for the associated quantized system near the lower boundary (see the discussion in Linda Reichl's book).

[1]Luna-Acosta GA, Na K, Reichl LE, Krokhin A (1996) Phys Rev E 53:3271

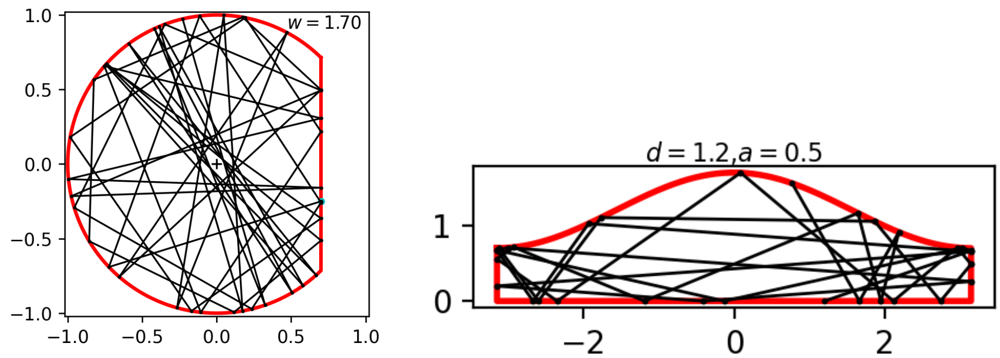

Figure 9: Left: an orbit for the D-billiard with radius 1 and horizontal width w. I think that orbits that bounce off the segment are either periodic or chaotic. Orbits that never bounce off the segment must be periodic (they are the same as in a circle rink which is integrable). All chaotic orbits must bounce off the segment. Right: The ripple billiard has a mixed phase space with chaotic region size that depends upon the amplitude of the ripple perturbation and the height of the rectangular region below. Orbits that bounce off the side walls are either periodic or chaotic. Orbits that never bounce off the side walls can be quasi-periodic or periodic. I think all chaotic orbits bounce off the side walls. The ripple billiard has a semi-analytical form for its quantum counterpart. Here we assume that the linear walls are hard and reflecting. The top wall is $y = d + a \cos x$ with $x \in [-\pi, \pi]$ along the horizontal direction.

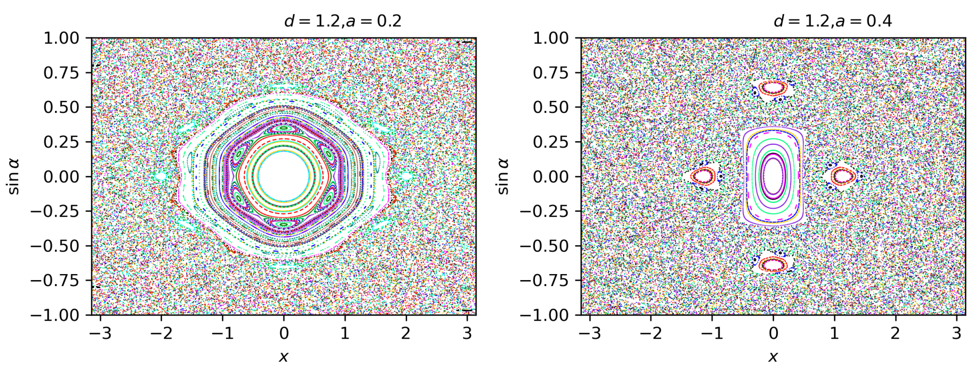

Figure 10: Some surfaces of section for the ripple billiard. These are plotting bounces on the lower rink wall only. The y axis gives $\sin \alpha$ where α is the angle between trajectory and vertical line during the bounce. The horizontal axis is the location on the lower wall where the bounce takes place. Chaos is restricted to larger x and extreme slopes. With $a > d$ larger (on the right, compared to the left), the chaotic region is larger.

2.2 Quantum billiards

A related quantum mechanical system has a wave function $\psi(\mathbf{q})$ where $\mathbf{q}$ gives the coordinate inside the rink or stadium. The Hamiltonian is $H(p,q) = \frac{p^2}{2m} + V(q)$ and the potential is infinity outside the rink and 0 inside it. Shrödinger's equation for an eigenstate with energy E_n is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E_n\psi \tag{10}$$

Because $V(q) = \infty$ for $q \notin \Omega$, the wave function $\psi(q) = 0$ for $q \notin \Omega$. The system has a Dirichlet boundary condition on the boundary, $\psi(q) = 0$ with $q \in \partial\Omega$. The free field Shrödinger equation is the same as the Helmholtz equation,

$$\left(\nabla^2 + k^2\right)\psi = 0.$$

For an eigenstate of the Laplacian operator with $\nabla^2\psi = k^2\psi$,

$$\left(\frac{k^2}{2m} + E_n\right)\psi = 0 \tag{11}$$

giving the relation between E_n and k, of

$$k^2 = 2mE_n/\hbar^2. \tag{12}$$

This means the quantum billiard problem is equivalent to a wave problem within a cavity. The quantum mechanical version of a billiard problem can be studied experimentally with radar, though you need to specify the type of EM modes (transverse magnetic?) to be consistent with the Dirichlet boundary condition. The mathematical problem is called the Dirichlet-Laplacian boundary value problem where the goal is to find the spectrum or values λ that satisfy $(\nabla^2 - \lambda)\phi(x) = 0$ for $x \in \Omega$ and $\phi(x) = 0$ for $x \in \partial\Omega$.

The Hilbert space is that given by basis $|\mathbf{q}\rangle$ in $\mathbb{R}^n$ where n is the dimension of $\mathbf{q}$. The problem studied is how the shape of the boundary affects the eigenfunctions and associated k values of the Helmholtz equation. Planck's constant does not affect the associated quantized system as it only determines the energy from k.

2.3 What is the relation between the dynamics of a classical billiard and the related quantum system?

The spectrum of the eigenstates of the quantum system and the morphology of the eigenstates are sensitive to the classical trajectories.

Quantum ergodicity. A quantum system that is **ergodic** has eigenfunctions $\psi_n(q)$ which for large n

$$\int |\psi_n|^2 g(q)dq \sim \frac{1}{A}\int dp\ dq\ g(q)$$

for $g(q)$ a smooth bounded function. This means that the wave functions are distributed evenly within the stadium, particularly at large n.

For an integrable classical system:

- The associated quantum eigenstates concentrate on structures that can be described as invariant tori.
- The energy levels of the quantum system are distributed so that their differences obey a Poisson distribution function.

For a chaotic classical system:

- The associated quantum eigenstates are ergodic (in the sense mentioned above).
- The energy levels of the quantum system are distributed so that differences obey a Wigner distribution function. In this way the spectrum resembles that of a random matrix model known as the Gaussian unitary ensemble.

For a mixed classical system, eigenstates concentrate on chaotic regions or they concentrate on regions with invariant tori.

The Fourier transform of the density of states has peaks at the periods of the periodic orbits in the classical system.

There are special eigenstates, called scars, in the quantum system, that are concentrated at hyperbolic periodic orbits in the associated classical system.

2.4 Numerically computing the spectrum for a quantum billiard problem

The Helmholtz problem is to solve

$$(\nabla^2 + k^2)\psi(\mathbf{q}) = 0 \qquad \text{for} \quad \mathbf{q} \in \Omega \tag{13}$$

$$\psi(\mathbf{q}) = 0 \qquad \text{for} \quad \mathbf{q} \in \partial\Omega. \tag{14}$$

The technique used to find eigenvalues and eigenvectors is an integral differential equation method.

The Green's function

$$(\nabla^2 + k^2)G_k(\mathbf{q}, \mathbf{q}') = \delta(\mathbf{q} - \mathbf{q}') \tag{15}$$

where ∇^2 operates on $\mathbf{q}$.

Reverse the order so that the derivative is w.r.t to $\mathbf{q}'$ and multiply this by $\psi(\mathbf{q}')$ to get

$$\psi(\mathbf{q}')(\nabla'^2 + k^2)G_k(\mathbf{q}, \mathbf{q}') = \psi(\mathbf{q}')\delta(\mathbf{q}' - \mathbf{q}) \tag{16}$$

Multiply the Green's function by equation 13

$$G_k(\mathbf{q}, \mathbf{q}')(\nabla'^2 + k^2)\psi(\mathbf{q}') = 0 \tag{17}$$

Subtract the two equations

$$\psi(\mathbf{q}')\nabla'^2 G_k(\mathbf{q},\mathbf{q}') - G_k(\mathbf{q},\mathbf{q}')\nabla'^2\psi(\mathbf{q}') = \psi(\mathbf{q}')\delta(\mathbf{q}'-\mathbf{q}). \tag{18}$$

We integrate over the domain,

$$\int_\Omega \left[\psi(\mathbf{q}')\nabla'^2 G_k(\mathbf{q},\mathbf{q}') - G_k(\mathbf{q},\mathbf{q}')\nabla'^2\psi(\mathbf{q}')\right] d^2\mathbf{q}' = \int_\Omega \psi(\mathbf{q}')\delta(\mathbf{q}'-\mathbf{q}) d^2\mathbf{q}' \tag{19}$$

How we use the divergence theorem on $\nabla\cdot(\psi\nabla G - G\nabla\psi) = \psi\nabla^2 G - G\nabla^2\psi$ (also known as Green's second theorem) to find

$$\int_{\partial\Omega} ds' \left[\psi(\mathbf{q}')\frac{\partial G_k(\mathbf{q},\mathbf{q}')}{\partial \mathbf{n}'} - \frac{\partial\psi(\mathbf{q}')}{\partial\mathbf{n}'}G(\mathbf{q},\mathbf{q}')\right] = \psi(\mathbf{q}) \text{ for } \mathbf{q}\in\Omega, \notin \partial\Omega \tag{20}$$

Here $\mathbf{n}'$ is normal to the boundary curve. The left term is zero because $\psi = 0$ on the boundary. This leaves

$$\int_{\partial\Omega} u(s')G_k(\mathbf{q},\mathbf{q}')ds' = 0 \text{ for } \mathbf{q}\in\partial\Omega \tag{21}$$

where

$$u(s) = \frac{\partial}{\partial\mathbf{n}}\psi(\mathbf{q}(s)) \tag{22}$$

is the normal component of the gradient of the wave function at the boundary.

Then discretize the boundary to get a matrix equation for u which depends upon k. This apparently can be solved for both u and k, but nontrivially. See *Numerical Aspects of Eigenvalue and Eigenfunction Computations for Chaotic Quantum Systems* by Arnd Bäcker `https://arxiv.org/abs/nlin/0204061`.

For the ripple billiard, a coordinate transformation helps in solving the system. Oddly for this system, the surface of section computed in the classical system is compared to a Husimi function computed from the wave function near the lower boundary.

3 The quantized periodically kicked rotor

There are reviews devoted to the quantized kicked rotor. See for example, Santhanam et al. (2022)[2], I enjoyed the more accessible lecture notes by Dominique Delande[3]. The textbooks by Linda Reichl (*The Transition to Chaos; Conservative Classical and Quantum Systems*; 3rd edition is 2021) and Sandro Wimberger (*Non-linear Dynamics and Quantum Chaos*; 2014) have nice introductions.

[2] *Quantum kicked rotor and its variants: Chaos, localization and beyond,* M.S. Santhanam, S. Paul, and J. B. Kannan, 2022, Physics Reports, 956, 1-8. – Unfortunately there is no arXiv version

[3] Kicked rotor and Anderson localization, Boulder school on Condensed Matter Physics, 2013, Dominique Delande, `https://boulderschool.yale.edu/sites/default/files/files/Delande-kicked_rotor_lectures_1_and_2.pdf` `https://boulderschool.yale.edu/sites/default/files/files/Delande-kicked_rotor_lecture_3.pdf`

3.1 The classical periodically kicked rotor

The standard map is generated from the single particle classical Hamiltonian

$$H(p, \theta, t) = \frac{p^2}{2I} + V(\theta, t) \tag{23}$$

where $V(\theta, t)$ is periodic in t with period T, the angle $\theta \in [0, 2\pi]$, p is momentum, and I is the moment of inertia. Here p, θ are a canonical pair but the Hamiltonian is time dependent. Note that p is actually an angular momentum as θ is assumed to be angle. For momentum p one can either assume a periodic boundary condition or allow p to extend to $\pm\infty$. To be consistent with the standard map the potential

$$V(\theta, t) = k \cos\theta \sum_{j=\infty}^{\infty} \delta(t - jT). \tag{24}$$

The standard map is the map between p, θ values at consecutive values of time separated by period T. It would look like k is in units of energy, but actually k is in units of energy per unit time because you need to integrate over one of the delta functions to get a unit of energy.

Using Hamilton's equations $\dot{p} = -\frac{\partial H}{\partial \theta}$. We integrate this across one of the kicks to find the change in momentum after a kick

$$\Delta p = \int_{-\epsilon}^{\epsilon} dt\ k \sin\theta\ \delta(t).$$

If we label each period = with index n, this gives the momentum change

$$p_{n+1} = p_n + k \sin\theta_n.$$

Between kicks the momentum is fixed and

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{p}{I}.$$

This means that

$$\theta_{n+1} = \theta_n + p_{n+1}\frac{T}{I}$$

across a time that is duration T. We can define a dimensionless momentum with

$$\tilde{p} = p\frac{T}{I}$$

giving

$$\tilde{p}_{n+1} = \tilde{p}_n + \frac{kT}{I}\sin\theta_n \tag{25}$$

$$\theta_{n+1} = \theta_n + \tilde{p}_{n+1}. \tag{26}$$

The properties of the map are given by the dimensionless parameter

$$K = \frac{kT}{I} \tag{27}$$

which is used to characterize the standard map.

3.2 The quantum space

We relate phase space p, θ to a 1-dimensional Hilbert space for $\theta \in [0, 2\pi]$ with wave functions that can be described in a basis $|\theta\rangle$.

$$|\psi\rangle = \psi(\theta)\,|\theta\rangle\,.$$

Here $\psi(\theta) \in \mathbb{Z}$ gives a complex number and could be a function of time. The wave function $\psi(\theta)$ is normalized so that

$$\langle\psi|\psi\rangle = \int_0^{2\pi} d\theta\ \psi(\theta)^*\psi(\theta) = 1.$$

We construct an operator $\hat{\theta}$ which has eigenstates $|\theta\rangle$ so that $\hat{\theta}\,|\theta\rangle = \theta\,|\theta\rangle$. We need to create a momentum operator $\hat{p}$ with eigenstates $|p\rangle$, with $\hat{p}\,|p\rangle = p\,|p\rangle$, that should be related to the $|\theta\rangle$ basis via a Fourier transform. Because we are working in the interval $\theta \in [0, 2\pi]$ we use a Fourier series to relate the basis $\{|m\rangle\}$ with $m \in \mathbb{Z}$ and the basis $\{|\theta\rangle\}$ with $\theta \in [0, 2\pi]$;

$$|m\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta\ e^{im\theta}\,|\theta\rangle \tag{28}$$

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\theta}\,|m\rangle\,. \tag{29}$$

A delta function on the interval $x \in [0, 2\pi]$

$$\delta(x) = \frac{1}{2\pi} \sum_{m=\infty}^{\infty} e^{imx}. \tag{30}$$

For integers m, m'

$$\delta_{mm'} = \frac{1}{2\pi} \int_0^{2\pi} d\theta\ e^{i\theta(m-m')}. \tag{31}$$

What does the operator $-i\partial_\theta$ do?

$$\begin{aligned} -i\partial_\theta\,|m\rangle &= -i\partial_\theta \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta\ e^{im\theta}\,|\theta\rangle \\ &= -i\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta\ (im)e^{im\theta}\,|\theta\rangle \\ &= m\,|m\rangle\,. \end{aligned}$$

The states $|m\rangle$ are eigenstates of the operator $-i\partial_\theta$.

The states $|m\rangle$ are also eigenstates of $-i\hbar\partial_\theta$ but with eigenvalue $\hbar m$.

We define

$$\hat{p} \equiv -i\hbar\partial_\theta \tag{32}$$

which obeys the expected commutation relation

$$[\hat{\theta}, \hat{p}] = i\hbar. \tag{33}$$

The eigenstates of $\hat{p}$ are

$$\hat{p}\,|m\rangle = \hbar m\,|m\rangle\,. \tag{34}$$

It is handy to have identity operators in both bases

$$\begin{aligned}\mathbf{I} &= \int_0^{2\pi} d\theta\ \ |\theta\rangle\,\langle\theta| \\ &= \sum_{m=-\infty}^{\infty} |m\rangle\,\langle m|\,. \end{aligned}\tag{35}$$

Also the following are useful.

$$\langle\theta|m\rangle = \frac{1}{\sqrt{2\pi}}e^{im\theta} \qquad \langle m|\theta\rangle = \frac{1}{\sqrt{2\pi}}e^{-im\theta}. \tag{36}$$

3.3 The quantized kicked rotor

The classical Hamiltonian has kinetic energy term $p^2/(2I)$. Using $\hat{p} = -i\hbar\partial_\theta$, the kinetic energy becomes $\hat{T} = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}$.

We convert the classical Hamiltonian into one that operates on the Hilbert space on $[0, 2\pi]$ by converting momentum and coordinate to operators; $p \to \hat{p}$ and $\theta \to \hat{\theta}$;

$$\hat{H} = \frac{\hat{p}^2}{2I} + k\cos\hat{\theta}\sum_{j=-\infty}^{\infty}\delta(t - jT). \tag{37}$$

Shrödinger's equation becomes

$$-i\hbar\frac{\partial}{\partial t}\psi(\theta, t) = \left[-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2} + k\cos\theta\sum_{j=-\infty}^{\infty}\delta(t - jT)\right]\psi(\theta, t). \tag{38}$$

On the order of operators: The Hamiltonian is the sum of a term that only depends upon $\hat{p}$ and a term that only depends on $\hat{\theta}$. This is a clean situation as there is no ambiguity on the order of the operators. If the classical Hamiltonian had terms that depended upon

$p\theta$ or pq then the order of the operators must be specified for the associated quantized system.

With a Hamiltonian $\hat{H}$ that is independent of time $|\psi(t)\rangle = e^{-i\hbar Ht/\hbar} |\psi(0\rangle$. The unitary operator

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}}$$

is a propagator in the sense that the wavefunction at a future time $\psi(t) = \hat{U}\psi(0)$.

We construct a unitary operator for evolution from kick to kick (when the delta functions are applied) for the time dependent Hamiltonian of equation 37. Between kicks, evolution is that of a free particle with propagator $e^{-i\hat{p}^2 t/(\hbar 2I)}$. We apply this for a time T giving propagator

$$\hat{U}_{free} = e^{-\frac{iT}{\hbar 2I}\hat{p}^2}.$$

For the kick, the propagator is integrated over the delta function in time giving

$$\hat{U}_{kick} = e^{-\frac{ik\cos\hat{\theta}}{\hbar}}.$$

Together the propagation of the wave function from $t = 0$ to $t = T$ including free evolution and kick is the product of these two propagators

$$\hat{U} = \hat{U}_{kick}\hat{U}_{free} = e^{-\frac{ik}{\hbar}\cos\hat{\theta}} e^{-\frac{iT}{2\hbar I}\hat{p}^2}. \tag{39}$$

Given $|\psi(t_0)\rangle$ where t_0 is just after a kick

$$|\psi(t_0 + T)\rangle = \hat{U}_{kick}\hat{U}_{free} |\psi(t_0)\rangle$$

We use an identity involving the Bessel function $J_r(z)$ (see Bessel's integral `https://dlmf.nist.gov/10.9`)

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz\cos\theta} \cos(n\theta) d\theta \tag{40}$$

It is convenient to work in the Fourier basis $\{|m\rangle\}$. In this basis

$$e^{-\frac{iT}{2\hbar I}\hat{p}^2} |m\rangle = e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} |m\rangle .$$

In the Fourier basis we compute the elements of the propagator of equation 39

$$\begin{aligned} U_{m'm} &= \langle m'| \hat{U}_{kick}\hat{U}_{free} |m\rangle \\ &= \langle m'| e^{-\frac{ik}{\hbar}\cos\hat{\theta}} e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} |m\rangle . \end{aligned} \tag{41}$$

We insert the identity in the $|\theta\rangle$ basis

$$
\begin{aligned}
U_{m'm} &= \int_0^{2\pi} d\theta' \left\langle m' \right| e^{-\frac{ik}{\hbar}\cos\hat{\theta}} \left|\theta'\right\rangle \left\langle\theta'\right| e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} \left|m\right\rangle \\
&= \int_0^{2\pi} d\theta' \left\langle m' \middle| \theta' \right\rangle e^{-\frac{ik}{\hbar}\cos\theta'} \left\langle \theta' \middle| m \right\rangle e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta' \; e^{-im'\theta'} e^{-\frac{ik}{\hbar}\cos\theta'} e^{im\theta'} e^{-\frac{iT}{2\hbar I}\hbar^2 m^2}.
\end{aligned}
$$

where in the last step we have used expressions from equation 36.

The integral is over values of $\theta \in [0, 2\pi]$. We define $\theta' = \theta + \pi$ which gives $\cos\theta = -\cos\theta'$ and $e^{im\theta} = -e^{im\theta'}$.

$$
U_{m'm} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \; e^{i(m-m')\theta} e^{\frac{ik}{\hbar}\cos\theta} e^{-\frac{iT}{2\hbar I}\hbar^2 m^2}. \tag{42}
$$

Lastly we need to apply equation 40 so that we can write the expression in terms of a Bessel function. Let's try to split the θ dependent part into two pieces.

$$
\begin{aligned}
&\int_0^{\pi} d\theta \; e^{i(m-m')\theta} e^{\frac{ik}{\hbar}\cos\theta} + \int_{-\pi}^{0} d\theta \; e^{i(m-m')\theta} e^{\frac{ik}{\hbar}\cos\theta} \\
&= \int_0^{\pi} d\theta \; e^{i(m-m')\theta} e^{\frac{ik}{\hbar}\cos\theta} + \int_0^{\pi} d\theta \; e^{-i(m-m')\theta} e^{\frac{ik}{\hbar}\cos\theta} \\
&= \int_0^{\pi} d\theta \; e^{\frac{ik}{\hbar}\cos\theta} \big(e^{i(m-m')\theta} + e^{-i(m-m')\theta}\big) \\
&= \int_0^{\pi} d\theta \; e^{\frac{ik}{\hbar}\cos\theta} 2\cos\big((m-m')\theta\big) \\
&= i^{m-m'} 2\pi J_{m-m'}(k/\hbar).
\end{aligned}
$$

Equation 42 becomes

$$
U_{m'm} = i^{m-m'} J_{m-m'}\left(\frac{k}{\hbar}\right) e^{-\frac{iT\hbar m^2}{2I}}. \tag{43}
$$

Happily this is consistent with equation 14 in lecture notes (1 and 2) by Delande and equation 33 in the review by Santhanam et al.(2022). Note that $m \in \mathbb{Z}$ is a quantum number, not a mass.

Because of Plank's constant, the matrix elements are characterized by two dimensionless parameters instead of 1 (as was the case for classical standard map)! From equation 43 we see two dimensionless parameters, $k/\hbar$ and $T\hbar/I$. If we multiply these together we find $K = kT/I$ which is the dimensionless parameter we defined for a similar Hamiltonian giving the K parameter of the classical standard map (equation 27). It is popular to define a dimensionless number

$$\alpha \equiv \frac{T\hbar}{I} \tag{44}$$

which is sometimes called a rescaled Plank's constant. As the second parameter we choose

$$K \equiv \frac{kT}{I} \tag{45}$$

(as before in equation 27). With these dimensionless parameters, 43 becomes

$$U_{m'm} = i^{m-m'} J_{m-m'} \left(\frac{K}{\alpha} \right) e^{-\frac{i\alpha m^2}{2}}. \tag{46}$$

With this choice of dimensionless parameters, it is clear how to relate this quantum system to that of the associated classical one. Furthermore we can see that a smaller α corresponds to a smaller $\hbar$ which should recover the classical limit. The Bessel function $J_n(k)$ approaches zero rapidly when $k > n$ so elements in the matrix that are distant from the diagonal are very small.

Because the propagator U is unitary, its eigenvalues should be in the form $e^{i\phi}$ with phases ϕ. If we define $\epsilon = \hbar\phi/T$ then the eigenvalues are in the form $e^{i\epsilon T/\hbar}$ with ϵ resembling an energy. The phases ϕ can be considered quasi-energies of the Hamiltonian $\hat{H}$.

One approach to studying the system numerically is to use equation 46 to fill a matrix for $\hat{U}$ with values up to some limiting $|m|$ value, and then numerically compute its eigenvalues. However I find that eigenvalues are not all on the unit circle, indicating that the resulting matrix is not unitary. It's not obvious from equation 46 that $\hat{U}$ is unitary! Nor is it obvious how to correctly generate a unitary matrix from equation 46 that fits in a finite dimensional space. If we simply ignore the problem by normalizing the eigenvalues to the unit circle, then it is possible to show that the spacings (in phase) between the eigenvalues at large K approach a Wigner distribution, though I find that you need to have quite a large K to see this happen.

In parameter regimes where the classical delta-kicked rotor is chaotic, its quantum analog exhibits dynamic Anderson localization. The Floquet states of the quantum system can be mapped onto the tight-binding model for Anderson localization in condensed matter systems (Grempel et al. 1984).

Question: Notice that here the eigenstates of the propagator exhibit Anderson localization. Whereas, for the quantum billiard system, there is a notion of quantum ergodicity where the eigenfunctions tended to cover the space. Why is it that quantum chaos is associated with different area filling properties in the two situations?

Note: The standard map conventionally has p periodic as well as θ periodic. Here however, we assume θ periodic but did not put a constraint on p, though to calculate the propagator, often it is truncated at high $|m|$. When p is not periodic, the standard map allows orbits to go to very large p values. If you truncate the propagator, that's equivalent to approximating the system with a finite dimensional Hilbert space. Note that the propagator given by 46, after truncation, is not unitary.

3.4 Approximating the system with a finite dimensional Hilbert space

Rather than finding the exact propagator for the quantum mechanical system, and then truncating it so that it can be computed numerically, we could instead start by approximating the Hilbert space with a finite dimensional system. This is equivalent to approximating phase space with a discrete grid.

Instead of $\theta \in [0, 2\pi]$ we let θ takes on discrete values $\{j2\pi/N\}$ with $j \in \mathbb{Z}_N$. Let us call $|j\rangle$ the state corresponding to $\theta = 2\pi j/N$. The Fourier transform becomes the discrete Fourier transform

$$|m\rangle_F = \frac{1}{\sqrt{N}} \sum_{j=0}^{N} e^{im\frac{2\pi j}{N}} |j\rangle \tag{47}$$

$$|j\rangle = \frac{1}{\sqrt{N}} \sum_{m=0}^{N} e^{-im\frac{2\pi j}{N}} |m\rangle_F . \tag{48}$$

Recall that the sum of all N of the N-th complex roots of unity is 0. It is handy to compute

$$\langle j|m\rangle_F = \frac{1}{\sqrt{N}} e^{im\frac{2\pi j}{N}} = \frac{\omega^{mj}}{\sqrt{N}} \tag{49}$$

with $\omega = e^{2\pi i/N}$. We define the Fourier transform with the unitary operator

$$\hat{Q}_{FT} = \frac{1}{\sqrt{N}} \sum_{mn} \omega^{mn} |m\rangle \langle n| . \tag{50}$$

The Fourier transform can be used to change basis. For example for a wave vector in the conventional basis

$$\begin{aligned} |\psi\rangle &= \sum_j a_j |j\rangle \\ &= \sum_{jm} |m\rangle_F \langle m|_F |j\rangle a_j \\ &= \sum_m \sum_j \frac{\omega^{-mj}}{\sqrt{N}} a_j |m_F\rangle . \end{aligned}$$

To change basis from the $|j\rangle$ to $|m\rangle_F$ basis, operate on a wave vector in the $|j\rangle$ basis with $\hat{Q}_{FT}^\dagger$.

We can define an operator $\hat{\theta}$ which gives $\hat{\theta}\,|j\rangle = \frac{2\pi j}{N}\,|j\rangle$. We can also write

$$\hat{\theta} = \sum_j \frac{2\pi j}{N}\,|j\rangle\,\langle j|\,. \tag{51}$$

The operator $\hat{m}$ gives $\hat{m}\,|m\rangle_F = m\,|m\rangle_F$. In analogy to equation 34 we could choose $\hat{p} = \hbar\hat{m}$. In the m basis

$$\hat{p} = \sum_{m=0}^{N} \hbar m\,|m\rangle_F\,\langle m|_F\,. \tag{52}$$

However if we choose $\hat{p}$ in this way, the commutator $[\hat{\theta},\hat{p}] \neq i\hbar\mathbf{I}$, with $\mathbf{I} = \sum_m |m\rangle\,\langle m| = \sum_j |j\rangle\,\langle j|$. However, it is likely that the limit $\lim_{N\to\infty} \langle\theta|\,[\hat{\theta},\hat{p}]\,|\theta'\rangle = i\hbar\delta(\theta-\theta')$. This is related to the difficulty of quantizing a classical system.

The Hamiltonian of 37 is unchanged. The propagation of the wave function from $t=0$ to $t=T$ including free evolution and kick is the product of these two propagators (repeating equation 39)

$$\hat{U} = \hat{U}_{kick}\hat{U}_{free} = e^{-\frac{ik}{\hbar}\cos\hat{\theta}} e^{-\frac{iT}{2\hbar I}\hat{p}^2}. \tag{53}$$

However, $\hat{\theta}$ is now given by equation 51 and $\hat{p}$ is from equation 52. This gives

$$\begin{aligned}
U_{m'm} &= \langle m'|\,\hat{U}\,|m\rangle \\
&= \langle m'|\sum_j e^{-\frac{ik}{\hbar}\cos\left(\frac{2\pi j}{N}\right)}\,|j\rangle\,\langle j|\sum_{m''} e^{-\frac{iT}{2\hbar I}\hbar^2 m''^2}\,|m''\rangle\,\langle m''|\,|m\rangle \\
&= \langle m'|\sum_j e^{-\frac{ik}{\hbar}\cos\left(\frac{2\pi j}{N}\right)}\,|j\rangle\,\langle j|\,e^{-\frac{iT}{2\hbar I}\hbar^2 m^2}\,|m\rangle \\
&= \sum_j \langle m'|\,|j\rangle\,e^{-\frac{ik}{\hbar}} e^{\cos\left(\frac{2\pi j}{N}\right)} e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} e^{i2\pi jm/N} \\
&= \sum_{j=0}^{N-1} e^{i2\pi j(m-m')/N} e^{-\frac{ik}{\hbar}\cos\left(\frac{2\pi j}{N}\right)} e^{-\frac{iT}{2\hbar I}\hbar^2 m^2}.
\end{aligned} \tag{54}$$

Happily this looks consistent with equation 42.

The sum in the above result looks ugly however it is simpler when written in terms of the Fourier transform. Using this operator $\hat{p} = Q_{FT}^\dagger \Lambda Q_{FT}$ in the θ representation, where $\Lambda = \sum \hbar m\,|m\rangle\,\langle m|$ is a diagonal matrix with integers along the diagonal. Let's check the signs.

$$
\begin{aligned}
\hat{p} &= \sum_m \hbar m \left|m\right\rangle_F \left\langle m\right|_F \\
&= \sum_{mjj'} \hbar m \left|j\right\rangle \left\langle j\right| \left|m\right\rangle_F \left\langle m\right|_F \left|j'\right\rangle \left\langle j'\right| \qquad \text{recall } \langle j|m\rangle = \omega^{jm} \\
&= \sum_{mjj'} \left|j\right\rangle Q_{FT,jm} \left\langle m\right| \left|m\right\rangle \hbar m \left\langle m\right| \left|m\right\rangle_F Q^\dagger_{FT,jm'} \left\langle j'\right| \\
&= Q_{FT} \Lambda Q^\dagger_{FT}
\end{aligned}
$$

Likewise in the $|m\rangle_F$ basis

$$\hat{\theta} = Q^\dagger_{FT} \frac{2\pi j}{N} \left|j\right\rangle \left\langle j\right| Q_{FT}$$

We can write (in the $|m\rangle_F$ basis) the propagator

$$\hat{U} = Q^\dagger_{FT} \Lambda_A Q_{FT} \Lambda_B \tag{55}$$

with diagonal matrices

$$
\begin{aligned}
\Lambda_A &= \sum_j e^{-\frac{ik}{\hbar}\cos(2\pi j/N)} \left|j\right\rangle \left\langle j\right| \\
\Lambda_B &= \sum_m e^{-\frac{iT}{2\hbar I}\hbar^2 m^2} \left|m\right\rangle \left\langle m\right| .
\end{aligned} \tag{56}
$$

This should be equivalent to equation 54. It appears simpler as it involves diagonal matrices and Fourier transforms, that can be efficiently computed using the FFT if N is a power of 2. However one chooses to construct the matrix U, either with equation 54 or 55, after filling it, one can find its eigenvalues and eigenvectors numerically. I show my attempt to do this in Figure 11.

4 The Wigner function and the Weyl transformation

The Wigner function is used to show a quasi probability distribution for quantum systems in phase space. The Weyl transformation is used to create quantum operators from functions that depend on p, q in phase space. Both are used to create an alternative formulation for quantum mechanics.

The Wigner function can be used to illustrate the behavior of quantum systems and is sometimes used in tomography. It may be useful for describing information in quantum systems.

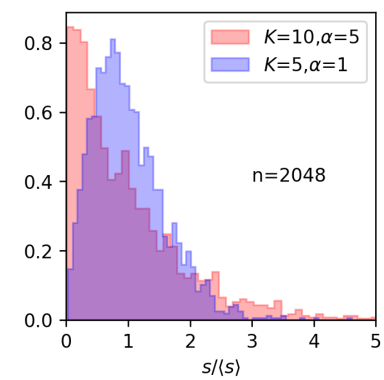

Figure 11: The distribution of eigenvalue spacings for the quantized rotator with $\alpha = 5$, $K = 10$ (in blue) and $\alpha = 1, K = 5$ (in red), computed an $N = 2048$ dimensional Hilbert space as described below. The higher K and α system resembles a Wigner distribution and the other one a Poisson distribution. Here s is the difference in consecutive phases of eigenvalues of the propagator U, and the x axis shows $s/\langle s\rangle$ where $\langle s\rangle$ is the mean of the distribution.

4.1 The space

Consider a classical system that lives in phase space $p, q \in \mathbb{R}$. In the related quantum system, we expect an uncertainty relation between p, q. In this setting p, q are operators that do not commute. One way to specify both q, p simultaneously in a quantum system is to find a way to *smooth* the system. The Wigner function gives a way to take a function and turn it into a generalized probability function as a function of p, q. The Weyl transform is a way to create operators.

We can think of a Hilbert space described via spatial eigenstates $|q\rangle$. Here q gives us a basis for our Hilbert space. We can construct an operator $\hat{q}$ that has eigenstates $|q\rangle$ and eigenvalues q. In other words $\hat{q}\,|q\rangle = q\,|q\rangle$. A wavefunction can be described via a spatial function $\psi(q)$ with $|\psi\rangle = \int dq\ \psi(q)\,|q\rangle$ normalized so that

$$
\begin{aligned}
\langle\psi|\psi\rangle &= \int dq'\psi^*(q')\,\langle q'| \times \int dq\ \psi(q)\,|q\rangle \\
&= \int dqdq'\psi^*(q')\psi(q)\delta(q-q') \\
&= \int dq\ \psi^*(q)\psi(q) \\
&= 1.
\end{aligned}
$$

We used the fact that our Hilbert space basis is orthogonal in the sense that $\langle q|q'\rangle = \delta(q-q')$. Integrals are from $-\infty$ to ∞. The position and identity operators in the $|q\rangle$ basis are

$$\hat{q} = \int dq\ q \left|q\right\rangle \left\langle q\right| \qquad \mathbf{I} = \int dq\ \ \left|q\right\rangle \left\langle q\right|.$$

The Fourier transform of a wave function (in one dimension)

$$\psi(q) = \frac{1}{\sqrt{2\pi}} \int dk\ e^{iqk} \psi(q). \tag{57}$$

The inverse Fourier transform

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int dq\ e^{-iqk} \psi(q). \tag{58}$$

The normalization can differ from this convention, but the signs of the exponent are conventional. It is useful to recall the form of a delta function

$$\delta(x - x') = \frac{1}{2\pi} \int dk\ e^{ik(x-x')}. \tag{59}$$

Taking

$$\begin{aligned} \left|\psi\right\rangle &= \int dq\ \psi(q) \left|q\right\rangle \\ &= \int dq\ \frac{1}{\sqrt{2\pi}} \int dk\ \tilde{\psi}(k) e^{iqk} \left|q\right\rangle \\ &= \int dk\ \tilde{\psi}(k) \frac{1}{\sqrt{2\pi}} \int dq\ e^{iqk} \left|q\right\rangle. \end{aligned} \tag{60}$$

It is natural to create a basis or representation

$$\left|k\right\rangle \equiv \frac{1}{\sqrt{2\pi}} \int dq\ e^{iqk} \left|q\right\rangle \tag{61}$$

which gives

$$\left|\psi\right\rangle = \int dq\ \psi(q) \left|q\right\rangle = \int dk\ \tilde{\psi}(k) \left|k\right\rangle. \tag{62}$$

The inverse transform

$$\left|q\right\rangle = \frac{1}{\sqrt{2\pi}} \int dk\ e^{-ikq} \left|k\right\rangle. \tag{63}$$

The momentum basis is constructed from the spatial basis using the same Fourier transform

$$\left|p\right\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dq\ e^{ipq/\hbar} \left|q\right\rangle. \tag{64}$$

The inverse transform

$$|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{-ipq/\hbar} \, |p\rangle \, . \tag{65}$$

In the momentum basis or representation we define an operator $\hat{p}$ that obeys $\hat{p}\,|p\rangle = p\,|p\rangle$.

The associated delta function

$$\delta(q - q') = \frac{1}{2\pi\hbar} \int dp \; e^{ip(q-q')/\hbar}. \tag{66}$$

The momentum basis is related to the $|k\rangle$ basis (of equation 63) via $p = \hbar k$. The momentum and identity operators in the $|p\rangle$ basis are

$$\hat{p} = \int dp \; p\,|p\rangle\,\langle p| \qquad \mathbf{I} = \int dp \;\; |p\rangle\,\langle p| \, .$$

Equations 64 and 65 imply that

$$\begin{aligned}
\langle q|p\rangle &= \frac{1}{\sqrt{2\pi\hbar}} e^{ipq/\hbar} \\
\langle p|q\rangle &= \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}.
\end{aligned} \tag{67}$$

How do you evaluate operator $\hat{p}$ if you are given wavefunction $\psi(q)$?

$$\begin{aligned}
\hat{p}\,|\psi\rangle &= \hat{p} \int dq \; \psi(q)\,|q\rangle \\
&= \int dp \; p\,|p\rangle\,\langle p| \int dq \; \psi(q)\,|q\rangle \\
&= \int dp \; dq \; p\,|p\rangle\,\psi(q) \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar} \qquad \text{using eqn 67} \\
&= \int dp \; \frac{p}{\sqrt{2\pi\hbar}} \int dq \; \psi(q) \frac{\partial_q e^{-ipq/\hbar}}{-ip/\hbar} \,|p\rangle \\
&= i\hbar \int dp \; \frac{1}{\sqrt{2\pi\hbar}} \int dq \; \psi(q)(\partial_q e^{-ipq/\hbar})\,|p\rangle \quad \text{integrate by parts} \\
&= -i\hbar \int dp \; dq \; \frac{1}{\sqrt{2\pi\hbar}} (\partial_q \psi(q)) e^{-ipq/\hbar}\,|p\rangle \\
&= -i\hbar \int dq \; (\partial_q \psi(q)) \frac{1}{\sqrt{2\pi\hbar}} \int dp \; e^{-ipq/\hbar}\,|p\rangle \\
&= -i\hbar \int dq \; \partial_q \psi(q)\,|q\rangle \, .
\end{aligned}$$

When integrating by parts we assume that $\psi(q) \to 0$ at large values of $\pm q$. The operator

$$\hat{p} = -i\hbar \partial_q.$$

and the commutator

$$[\hat{q}, \hat{p}] = i\hbar,$$

where the right hand side implicitly contains the identity operator.

In a d dimensional system

$$|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{d/2}} \int d^d\mathbf{q}\ e^{i\mathbf{p}\cdot\mathbf{q}/\hbar} |\mathbf{q}\rangle .$$

From a wave function $|\psi\rangle$ we can construct a density operator $\hat{\boldsymbol{\rho}} = |\psi\rangle \langle\psi|$. This corresponds to a density operator for a pure state. Taking a spatial basis, we have function $\psi(q)$ so that $|\psi\rangle = \int dq \psi(q) |q\rangle$. The density operator for the pure state associated with $|\psi\rangle$ would be

$$\hat{\boldsymbol{\rho}} = |\psi\rangle \langle\psi| = \int dq\ dq' \psi(q')^* \psi(q) |q'\rangle \langle q| . \tag{68}$$

4.2 The Wigner function

If we have a density operator $\hat{\boldsymbol{\rho}}$ describing the probability that a system is in a set of quantum states, then the associated Wigner function is

$$W_{\hat{\rho}}(\mathbf{p}, \mathbf{q}) \equiv \frac{1}{(2\pi)^d} \int d^d\mathbf{x} \left\langle \mathbf{q} + \frac{\mathbf{x}}{2} \right| \hat{\boldsymbol{\rho}} \left| \mathbf{q} - \frac{\mathbf{x}}{2} \right\rangle e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}. \tag{69}$$

The function $W_{\hat{\rho}}(\mathbf{p}, \mathbf{q})$ is a generalized probability density in phase space. With the substitution $\mathbf{y} = \mathbf{x}/\hbar$ the Wigner function can equivalently be written as

$$W_{\hat{\rho}}(\mathbf{p}, \mathbf{q}) \equiv \frac{1}{(2\pi)^d} \int d^d\mathbf{y} \left\langle \mathbf{q} + \frac{\mathbf{y}\hbar}{2} \right| \hat{\rho} \left| \mathbf{q} - \frac{\mathbf{y}\hbar}{2} \right\rangle e^{-i\mathbf{p}\cdot\mathbf{y}} \tag{70}$$

If $\hat{\boldsymbol{\rho}} = |\psi\rangle \langle\psi|$ is the density operator for a pure state, with $|\psi\rangle = \psi(q) |q\rangle$ then

$$W_{\hat{\rho}}(\mathbf{p}, \mathbf{q}) \equiv \frac{1}{(2\pi)^d} \int d^d\mathbf{y}\ \psi^* \left(\mathbf{q} + \frac{\mathbf{y}\hbar}{2} \right) \psi \left(\mathbf{q} - \frac{\mathbf{y}\hbar}{2} \right) e^{-i\mathbf{p}\cdot\mathbf{y}}. \tag{71}$$

The Wigner function is real but not necessarily positive everywhere. However it is normalized so that when integrated over all phase space, $\int W(p, q)\ dq\ dp = 1$.

In the classical limit the exponential term oscillates rapidly unless x is small. In this limit the Wigner function localizes to a specific p, q and the classical function is recovered.

4.3 Properties of the Wigner function

- There are various ways to write it, for example, flipping the sign of x,

$$W_{\hat{\rho}}(\mathbf{p}, \mathbf{q}) \equiv \frac{1}{(2\pi)^d} \int d^d\mathbf{x} \left\langle \mathbf{q} - \frac{\mathbf{x}}{2} \right| \hat{\boldsymbol{\rho}} \left| \mathbf{q} + \frac{\mathbf{x}}{2} \right\rangle e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}. \tag{72}$$

- It can be written in terms of a point operator that depends on a point in phase space $\mathbf{q}, \mathbf{p}$. If $\hat{\rho} = |\psi\rangle \langle\psi|$ then

$$\begin{aligned} W_{\hat{\rho}}(\mathbf{p}, \mathbf{q}) &= \frac{1}{(2\pi)^d} \int d\mathbf{x}^d \left\langle \mathbf{q} - \frac{\mathbf{x}}{2} \middle| \psi \right\rangle \left\langle \psi \middle| \mathbf{q} + \frac{\mathbf{x}}{2} \right\rangle e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \\ &= \langle\psi| \frac{1}{(2\pi)^d} \int d\mathbf{x}^d \left| \mathbf{q} - \frac{\mathbf{x}}{2} \right\rangle \left\langle \mathbf{q} + \frac{\mathbf{x}}{2} \right| e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} |\psi\rangle \\ &= \langle\psi| \hat{A}(\mathbf{q}, \mathbf{p}) |\psi\rangle \end{aligned} \tag{73}$$

  with point operator

$$\hat{A}(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int d\mathbf{x}^d e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \left| \mathbf{q} - \frac{\mathbf{x}}{2} \right\rangle \left\langle \mathbf{q} + \frac{\mathbf{x}}{2} \right|. \tag{74}$$

- It can be transformed so that it is integrated in the momentum basis

$$W_{\hat{\rho}}(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^d} \int d^d\mathbf{s} \left\langle \mathbf{p} + \frac{\mathbf{s}}{2} \right| \hat{\boldsymbol{\rho}} \left| \mathbf{p} - \frac{\mathbf{s}}{2} \right\rangle e^{\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{s}}. \tag{75}$$

- If you integrate over $\mathbf{q}$ you find the probability of your particle being at $\mathbf{p}$ and vice versa;

$$\begin{aligned} \int d^d\mathbf{q}\ W_{\hat{\rho}}(\mathbf{q}, \mathbf{p}) &= |\psi(\mathbf{p})|^2 \\ \int d^d\mathbf{p}\ W_{\hat{\rho}}(\mathbf{q}, \mathbf{p}) &= |\psi(\mathbf{q})|^2. \end{aligned} \tag{76}$$

- The previous item implies that $\int d^d\mathbf{q} \int d^d\mathbf{p}\ W_{\hat{\rho}}(\mathbf{q}, \mathbf{p}) = 1$. The integral over phase space is normalized even though the function $W_\rho(q, p)$ itself can take on negative values.

- When integrated along any direction in phase space it yields another probability distribution. Let the operator

$$\hat{q}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta/\hbar.$$

  This operator is Hermitian so it is an observable. The probability distribution for this observable is

$$p(q_\theta) = \int W_{\hat{\rho}}(q\cos\theta - p\sin\theta/\hbar, q\sin\theta + p\cos\theta/\hbar)dp \tag{77}$$

- Wigner function is translationally covariant. With

$$\hat{\rho}' = e^{i(a\hat{p} - b\hat{q})/\hbar} \hat{\rho} e^{-i(a\hat{p} - b\hat{q})/\hbar} \tag{78}$$

$$W_{\hat{\rho}}(q, p) = W_{\hat{\rho}'}(q - a, p - b). \tag{79}$$

  Here a is a shift in q and b is a shift in p.

- The expectation of an observable $\hat{\mathbf{A}}$ which is a Hermitian operator

$$\langle \hat{\mathbf{A}} \rangle = \mathrm{tr}\Big(\hat{\mathbf{A}}\hat{\boldsymbol{\rho}}\Big) = \int d^d\mathbf{q} d^d\mathbf{p}\ A_W(\mathbf{q},\mathbf{p}) W_\rho(\mathbf{q},\mathbf{p}), \tag{80}$$

  where $A_W(\mathbf{q},\mathbf{p})$ is the Wigner function associated with the Hermitian operator $\hat{\mathbf{A}}$.

- In the semiclassical limit, $\frac{d\hat{\boldsymbol{\rho}}}{dt} = -\frac{i}{\hbar}[\hat{\boldsymbol{\rho}}, \hat{H}]$ reduces to the classical Poisson bracket $\frac{dW_\rho}{dt} = \{W_\rho, H\}$.
- For classically integrable systems, the semiclassical density operator, and hence also the Wigner function, localizes around the corresponding integrable tori.

These last two items require some effort to illustrate. **to work on!**

The Wigner function is not necessarily easy to compute. Hence the focus on coherent states.

4.4 Weyl-Wigner Transformation

Suppose we have quantum operators $\hat{Q}, \hat{P}$ and they satisfy the expected commutation relation; $[\hat{Q}, \hat{P}] = i\hbar$. We can transform a function $f(q,p)$ in phase space into an operator via the Weyl transformation

$$\hat{\Phi}[f] \equiv \int da\ db\ e^{ia\hat{Q}+ib\hat{P}} \tilde{f}(a,b), \tag{81}$$

where $\tilde{f}$ is the Fourier transform of f,

$$\tilde{f}(a,b) = \int dp\ dq\ e^{-iaq-ibp} f(q,p). \tag{82}$$

The units seem odd for the Weyl-Wigner transform. The quantity a should be in units of $1/q$ and b should be in units of $1/p$.

Because $\hat{P}, \hat{Q}$ are operators, the Weyl-Wigner transform gives a map between functions in phase space and operators on a quantum system.

Wigner transform :	operator $\to$ function in phase space
Weyl transform :	function in phase space $\to$ operator

The Weyl-Wigner transform is opposite to the Wigner transform, which takes an operator to a function in phase space.

4.5 The Groenewold-van Hove theorem

It is tempting to assume that it is always possible to take the Poisson bracket and turn it into a commutator

$$\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]. \tag{83}$$

Here A, B are functions and $\hat{A}, \hat{B}$ are associated operators. To do this we need a map Q which takes a function f and returns an operator $\hat{f}$. Let $Q(f) = \hat{f}$. Is it possible to find a map Q such that

$$Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)] \qquad \text{for all functions } f, g? \tag{84}$$

The Groenewold-van Hove theorem dictates that the map Q does not exist. Specifically, if $\hat{q}$ and $\hat{p}$ are taken to be the usual position and momentum operators, then no quantization scheme can perfectly reproduce the Poisson bracket relations among all classical observables. It is possible to find such a map that reproduces the Poisson bracket relations among polynomials with degree less than 3.

It is possible to find a map Q that works for all polynomials with degree less or equal to two. There is precisely one such map, and it is the Weyl quantization. The map itself is the Weyl transform defined in equation 81.

Remark There is no unique way to quantize classical systems. There is an ordering ambiguity: classically, the position and momentum variables x and p commute, but their quantum mechanical operator counterparts do not.

The phase space formulation of quantum mechanics can be considered as describing noncommutative spaces in the setting of non-commutative geometry.

4.6 Phase space formulation of quantum mechanics and the Moyal product

The *phase-space formulation* of quantum mechanics is based on a quasiprobability distribution $f(q, p)$ in phase space which is not necessarily derived from a wave function or a density operator.

The time evolution of the phase space distribution is given by a quantum modification of Liouville flow. The time evolution of the phase space distribution $f(q, p)$ is given by

$$\frac{\partial f(q, p)}{\partial t} = -\frac{1}{i\hbar}(f \star H - H \star f) \tag{85}$$

Here $H(q, p)$ is the Hamiltonian, and a plain function of phase-space, and in many settings it is the same as the classical Hamiltonian.

The Moyal product, shown with $\star$, can be written in terms of left and right derivatives or in terms of a convolution. For any two functions f, g of phase space

$$(f \star g)(q,p) = \frac{1}{\pi^2 \hbar^2} \int f(q+q', p+p') g(q+q'', p+p'') e^{\frac{2i}{\hbar}(q'p''-q''p')} dq' dq'' dp' dp'' \qquad (86)$$

The Moyal product can also be written in terms of left and right derivatives. The operation $f \star g - g \star f$ is known as the *Moyal bracket* of functions f, g.

4.7 Husimi distribution function

The problem of the Wigner function is that is can take negative values. An alternative probability density which is always positive is the Husimi distribution function. The Husimi distribution function smooths the Wigner function W_ρ with a function f;

$$H_\rho(q,p) = \int d\mathbf{q}' d\mathbf{p}' \; W_\rho(\mathbf{q}', \mathbf{p}') f(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}'). \qquad (87)$$

The Husimi function is the convolution of the Wigner function by a coarse graining function f. The usual choice for the smooth function is

$$f(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}') = (\pi\hbar)^{-d} \exp\left[-\left(\frac{(\mathbf{q}-\mathbf{q}')^2}{\sigma_q^2} + \frac{\sigma_q^2 (\mathbf{p}-\mathbf{p}')^2}{\hbar^2} \right) \right] \qquad (88)$$

The parameter σ_q is known as the squeezing parameter. The above definition is equivalent to defining the Husimi function, also known as the Q-function in quantum optics, by projecting directly the wave vector $|\psi\rangle$ onto a coherent state and taking the absolute value. For a pure state with wave vector $|\psi\rangle$, the Hussimi function can also be written

$$H_{|\psi\rangle}(q,p) = |\,\langle \psi | \alpha \rangle\,|^2 \qquad (89)$$

where $|\alpha\rangle$ is a coherent state described with a complex number α. The arguments q, p, of H in equation 89 are the expectation values $\langle \alpha | \, \hat{q} \, | \alpha \rangle = \langle \hat{q} \rangle_\alpha$, $\langle \alpha | \, \hat{p} \, | \alpha \rangle = \langle \hat{p} \rangle_\alpha$ of the coherent state (and these are equal to $\sqrt{2}\,\mathrm{Re}\{\alpha\}, \sqrt{2}\,\mathrm{Im}\{\alpha\}$, respectively).

Using equation 124 (repeated here) the coherent state satisfies

$$\langle q | \alpha \rangle = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(q - \langle q \rangle)^2 + \frac{i}{\hbar} \langle p \rangle (q - \langle q \rangle)}, \qquad (90)$$

In equation 90, q is dimensionless, and p has units of $q/\hbar$. For a harmonic oscillator, units of q can be restored via equation 116. In equation 88, σ_q has dimensions of q. The mean values should be related to the squeezing parameter σ_q, (see problem 4.9 in the book by Wimberger which seems very tedious).

4.8 The Weyl Symbol

We approximately follow *Phase space representation of quantum dynamics. Lecture notes. Boulder Summer School*, 2013, by Anatoli Polkovnikov. `http://physics.bu.edu/~asp28/teaching/lecture_notes_wigner_boulder_2013.pdf` Also see: `https://arxiv.org/abs/0905.3384` Note our Fourier definitions differ by a factor of 2π and the notation is somewhat different from these two manuscripts,

The Wigner function, $W_{\hat{\rho}}(q,p)$ (equation 69) as we defined it in section 4.2, is a function of phase space that is generated from a density operator $\hat{\rho}$. The **Weyl Symbol**, is the same as the Wigner function, except it is a function of any operator and the factors of 2π are usually neglected. We denote the Weyl symbol of operator $\hat{f}$ as $(\hat{f})_W(q,p)$.

What is the Weyl symbol of $\hat{q}$? In one dimension,

$$\begin{aligned}
\hat{q}_W(q,p) &= \int dx \ \langle q+x/2|\, \hat{q}\, |q-x/2\rangle \, e^{-ipx/\hbar} \\
&= \int dx \ dq' \ \langle q+x/2|\, q' \, |q'\rangle \, \langle q'| \, |q-x/2\rangle \, e^{-ipx/\hbar} \\
&= \int dx \ dq' \ q'\delta(q+x/2-q')\delta(q'-(q-x/2))e^{-ipx/\hbar}
\end{aligned}$$

The expression inside the integral is only non-zero if $x=0$, where giving

$$(\hat{q})_W(q,p) = \int dq' \ q'\delta(q-q') = q \tag{91}$$

This implies that the Weyl symbol of any operator $f(\hat{q})$ that is a function of $\hat{q}$, and does not depend upon $\hat{p}$, is equal to

$$(f(\hat{q}))_W = f(q).$$

Likewise $(f(\hat{p}))_W = f(p)$ for any function $f(\hat{p})$.

Let's compute the Weyl symbol for $\hat{q}\hat{p}$.

$$\begin{aligned}
(\hat{q}\hat{p})_W(q,p) &= \int dx \ \langle q+x/2|\, \hat{q}\hat{p}\, |q-x/2\rangle \, e^{-ipx/\hbar} \\
&= \int dx \ (q+x/2) \, \langle q+x/2| \int dp' \ p' \, |p'\rangle \, \langle p'|q-x/2\rangle \, e^{-ipx/\hbar} \\
&= \int \frac{dxdp'}{2\pi\hbar}(q+x/2)p' e^{i(q+x/2)p'/\hbar} e^{-i(q-x/2)p'/\hbar} e^{-ipx/\hbar} \\
&= \int \frac{dxdp'}{2\pi\hbar}(q+x/2)p' e^{i(p'-p)x/\hbar} \\
&= q\int dp' \ p' \int \frac{dx}{2\pi\hbar} e^{ix(p'-p)/\hbar} + \frac{1}{2}\int dx \ x \int \frac{dp'}{2\pi\hbar} p' e^{ix(p'-p)/\hbar}
\end{aligned}$$

$$
\begin{aligned}
&= q\int dp'\ p'\delta(p'-p) + \frac{1}{2}\int dx\ x\int \frac{dp'}{2\pi\hbar}p'\partial_{p'}\frac{e^{ixp'/\hbar}}{ix/\hbar}e^{-ixp/\hbar} \qquad \text{by parts}\\
&= qp + \frac{i\hbar}{2}\int dx\ \int \frac{dp'}{2\pi\hbar}e^{ix(p'-p)\hbar}\\
&= qp + \frac{i\hbar}{2}\int dp'\delta(p'-p)\\
&= qp + \frac{i\hbar}{2}
\end{aligned}
$$

Flipping the order of the two operators, we can similarly compute

$$
(\hat{p}\hat{q})_W(q,p) = qp - \frac{i\hbar}{2}. \tag{92}
$$

These relations are consistent with what is called the Bopp representation where

$$
\hat{q} \to q + \frac{i\hbar}{2}\partial_p \tag{93}
$$

$$
\hat{p} \to p - \frac{i\hbar}{2}\partial_q. \tag{94}
$$

Using these, and being careful with the order of $\hat{p}, \hat{q}$ operators you can find the Weyl symbol for any polynomial of momentum and positions operators. Furthermore, one can show that for *any operator* that is a function of momentum and position operators $\hat{\Omega}(\hat{q},\hat{p})$, the Weyl symbol

$$
(\hat{\Omega})_W(q,p) = \Omega\left(q + \frac{i\hbar}{2}\partial_p, p - \frac{i\hbar}{2}\partial_q\right)1 \tag{95}
$$

where the 1 symbolizes that a derivative applied to a constant gives zero.

Suppose we order $\hat{\Omega}(\hat{q},\hat{p})$ so that coordinate operators are to the left the momentum operators. For convenience let's consider $\Omega(\hat{q},\hat{p}) = f(\hat{q})g(\hat{p})$

$$
\begin{aligned}
\Omega_W(q,p) &= \int dx\ \ \langle q+x/2|\, f(\hat{q})g(\hat{p})\, |q-x/2\rangle\, e^{-ixp/\hbar}\\
&= \int dx\ \ \langle q+x/2|\, f(q+x/2)\int dk\, |k\rangle\, \langle k|\, g(\hat{p})\, |q-x/2\rangle\, e^{-ixp/\hbar}\\
&= \int \frac{dx\ dk}{2\pi\hbar}\ f(q+x/2)g(k)e^{i(q+x/2)k/\hbar}e^{-i(q-x/2)k/\hbar}e^{-ixp/\hbar}\\
&= \int \frac{dx\ dk}{2\pi\hbar}\ f(q+x/2)g(k)e^{ix(k-p)/\hbar}
\end{aligned}
$$

Let $k = p - k'/2$, giving $dk = -dk'/2$, and $k - p = -k'/2$. The limits go from $\infty \to -\infty$ so if we flip the limits we correct the sign flip caused by $dk = -dk'/2$ giving,

$$
\Omega_W(q,p) == \int \frac{dx\ dk'}{4\pi\hbar}\ f\left(q+\frac{x}{2}\right)g\left(p-\frac{k}{2}\right)e^{-\frac{ixk'}{2\hbar}} \qquad \text{for } \Omega(\hat{q},\hat{p}) = f(\hat{q})g(\hat{p}). \tag{96}
$$

We computed the Weyl symbol for a function $\Omega(\hat{q},\hat{p}) = f(\hat{q})g(\hat{p})$, but equation 96 can be modified to hold for an *ordered operator*, where the operator is expanded in a polynomial of $\hat{q}$ and $\hat{p}$ and then each term is reordered using the commutator so that powers of $\hat{q}$ are before powers of $\hat{p}$, giving a polynomial in the form $\Omega(\hat{q},\hat{p}) = \sum_{mn} a_{mn}\hat{q}^m\hat{p}^n$. Let us define $\Omega(p,q) = \sum_{mn} a_{mn}\hat{q}^m\hat{p}^n$ be the ordered form for $\Omega(\hat{q},\hat{p})$. Then equation 96 implies that

$$\Omega_W(q,p) = \int \frac{dx\ dk}{4\pi\hbar}\ \Omega\left(q+\frac{x}{2}, p-\frac{k}{2}\right) e^{-\frac{ixk}{2\hbar}} \qquad \text{for ordered } \Omega(\hat{q},\hat{p}). \tag{97}$$

5 Coherent states

The simplest examples for the Wigner function seems to be based on coherent states! Coherent states also seem to be used as examples in phase space formulations for quantum mechanics.

Coherent states are eigenstates of the raising operator $\hat{a}$,

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle \tag{98}$$

with α a complex number.

Coherent states are described as a state which has dynamics most closely resembling the oscillatory behavior of a classical harmonic oscillator. Coherent states arise in the quantum theory of a wide range of physical systems. They are sometimes called *Gaussian* or Glauber states and are often used in optics. Because they have a Gaussian form, the Wigner function for coherent states is moderately straightforward to compute.

In terms of our momentum and position operators $\hat{p},\hat{q}$

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}\left(\hat{q}+\frac{i\hat{p}}{\hbar}\right) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\hat{q}-\frac{i\hat{p}}{\hbar}\right) \end{aligned} \tag{99}$$

and are consistent with

$$[\hat{a},\hat{a}^\dagger] = 1. \tag{100}$$

Let's write a coherent state in terms of the Foch basis. For

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle \tag{101}$$

$$\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\left|m\right\rangle. \tag{102}$$

You can show 102 by writing $|\alpha\rangle = \sum_n c_n |n\rangle$, operating on it with $\hat{a}^\dagger$, deriving a recursion relation for the coefficients, and then normalizing the state vector:

$$\begin{aligned}\hat{a}^\dagger |\alpha\rangle = a^\dagger \sum_n c_n |n\rangle &= \sum_{n=0} c_n \sqrt{n+1}\, |n+1\rangle \\ &= \hat{a}^\dagger \frac{\hat{a}}{\alpha} |\alpha\rangle = \frac{1}{\alpha} \sum_{n=0}^{\infty} c_n n\, |n\rangle \\ &= \frac{1}{\alpha} \sum_{n=0}^{\infty} c_{n+1}(n+1)\, |n+1\rangle \end{aligned} \tag{103}$$

Matching coefficients, this implies that

$$\frac{\alpha}{\sqrt{n+1}} c_n = c_{n+1} \qquad \rightarrow \qquad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0. \tag{104}$$

Lastly normalization (and with choosing $c_0 \in \mathbb{R}$) gives equation 102.

Coherent states are not orthogonal as

$$\langle \alpha | \beta \rangle = e^{-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta} \tag{105}$$

Coherent states satisfy $\langle \hat{n} \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^2$.

5.1 Expectation values of position and momentum

For coherent state $|\alpha\rangle$,

$$\begin{aligned}\langle \hat{q} \rangle_\alpha &= \frac{1}{\sqrt{2}} \left\langle \hat{a} + \hat{a}^\dagger \right\rangle = \frac{\alpha + \alpha^*}{\sqrt{2}} \\ &= \sqrt{2}\, \mathrm{Re}\{\alpha\} \end{aligned} \tag{106}$$

$$\begin{aligned}\langle \hat{p} \rangle_\alpha &= \frac{1}{\sqrt{2}} \frac{\hbar}{i} \left\langle \hat{a} - \hat{a}^\dagger \right\rangle = \hbar \frac{\alpha - \alpha^*}{\sqrt{2} i} \\ &= \hbar \sqrt{2}\, \mathrm{Im}\{\alpha\} \end{aligned} \tag{107}$$

$$\begin{aligned}\left\langle \hat{q}^2 \right\rangle_\alpha &= \frac{1}{2} \left\langle \hat{a}^2 + (\hat{a}^\dagger)^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} \right\rangle \\ &= \frac{1}{2} \left(\alpha^2 + (\alpha^*)^2 + \left\langle 1 + 2\hat{a}^\dagger \hat{a} \right\rangle \right) \\ &= \frac{1}{2} \left(\alpha^2 + (\alpha^*)^2 + 1 + 2\alpha^* \alpha \right) \\ &= \frac{1}{2}((\alpha + \alpha^*)^2 + 1) = 2\, \mathrm{Re}\{\alpha\}^2 + \frac{1}{2}. \end{aligned} \tag{108}$$

Likewise for $|\alpha\rangle$,

$$\begin{aligned}\left\langle \hat{p}^2\right\rangle_\alpha &= \frac{1}{2}\frac{\hbar^2}{i^2}\left\langle \hat{a}^2 + (\hat{a}^\dagger)^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right\rangle \\ &= \frac{1}{2}\frac{\hbar^2}{i^2}\left(\alpha^2 + (\alpha^*)^2 - 1 - 2\alpha\alpha^*\right) \\ &= \frac{1}{2}\frac{\hbar^2}{i^2}((\alpha-\alpha^*)^2 - 1) \\ &= \hbar^2\left(2\,\mathrm{Im}\{\alpha\}^2 + \frac{1}{2}\right). \end{aligned} \tag{109}$$

Computing the variances for $|\alpha\rangle$,

$$\Delta\hat{q} = \sqrt{\langle\hat{q}^2\rangle_\alpha - (\langle\hat{q}\rangle_\alpha)^2} = \sqrt{2\,\mathrm{Re}\{\alpha\}^2 + \frac{1}{2} - 2\,\mathrm{Re}\{\alpha\}^2} = \frac{1}{\sqrt{2}} \tag{110}$$

$$\Delta\hat{p} = \sqrt{\langle\hat{p}^2\rangle_\alpha - (\langle\hat{p}\rangle_\alpha)^2} = \hbar\sqrt{2\,\mathrm{Im}\{\alpha\}^2 + \frac{1}{2} - 2\,\mathrm{Im}\{\alpha\}^2} = \frac{\hbar}{\sqrt{2}} \tag{111}$$

$$\Delta\hat{q}\Delta\hat{p} = \frac{\hbar}{2}. \tag{112}$$

Coherent states have the minimum possible value for the uncertainty product.

With Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{113}$$

the time evolution of a coherent state

$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}\,|\alpha(0)\rangle\,. \tag{114}$$

It is convenient that the Foch basis states are eigenvalues of the Hamiltonian. With $\alpha(0) = \alpha_0$,

$$\begin{aligned}|\alpha(t)\rangle &= e^{-i\hat{H}t/\hbar}\,|\alpha_0\rangle \\ &= e^{-i(a^\dagger a + \frac{1}{2})\omega t}e^{-\frac{|\alpha_0|^2}{2}}\sum_m \frac{\alpha_0^m}{m!}\,|m\rangle \\ &= e^{-\frac{i\omega t}{2}}e^{-\frac{|\alpha_0|^2}{2}}\sum_m \frac{\alpha_0^m}{m!}e^{-i\omega t m}\,|m\rangle \\ &= e^{-\frac{i\omega t}{2}}e^{-\frac{|\alpha_0|^2}{2}}\sum_m \frac{(e^{-i\omega t}\alpha_0)^m}{m!}\,|m\rangle \\ &= e^{-\frac{i\omega t}{2}}\left|e^{-i\omega t}\alpha_0\right\rangle. \end{aligned}$$

Except for a global phase, the coherent state evolves with constant amplitude and

$$\alpha(t) = \alpha_0 e^{-i\omega t}. \tag{115}$$

The mean position and the mean momentum oscillate, but they are not in phase; exactly like the classical, deterministic momentum and position of a harmonic oscillator. Coherent states are the quantum states most similar to classical states.

Our position operator $\hat{q}$ is unitless, and $\hat{p}$ has units of $\hat{q}/\hbar$. If the system is a harmonic oscillator then we can restore units so that the momentum and position operators $\hat{Q}, \hat{P}$ are

$$\sqrt{m\omega/\hbar}\ \hat{Q} = \hat{q} \text{ and } \sqrt{\hbar m\omega}\ \hat{P} = \hat{p}. \tag{116}$$

The ground state of the Harmonic oscillator Hamiltonian $\hat{H}$ is a coherent state, with $\alpha = 0$

$$|0\rangle = \pi^{-\frac{1}{4}} \int dq\ e^{-\frac{q^2}{2}} |x\rangle . \tag{117}$$

We can use the same symbol, $|0\rangle$, to denote both the ground state (with $n = 0$) and the coherent state with $\alpha = 0$.

5.2 The displacement operator

Consider the operator

$$\hat{D}_\alpha = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}. \tag{118}$$

What does this do in terms of $\hat{q}, \hat{p}$? Inserting equations 99 for $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{q}, \hat{p}$,

$$\begin{aligned} \hat{D}_\alpha &= e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \\ &= e^{\frac{\hat{q}}{\sqrt{2}}(\alpha - \alpha^*) - \frac{i\hat{p}}{\sqrt{2}\hbar}(\alpha + \alpha^*)} \\ &= e^{i\sqrt{2}\,\mathrm{Im}\{\alpha\}\hat{q} - \frac{i}{\hbar}\sqrt{2}\,\mathrm{Re}\{\alpha\}\hat{p}} \end{aligned} \tag{119}$$

If we think of $\hat{q}$ as the infinitesimal generator for translations in momentum space and $\hat{p}$ as the generator for translations in position space, then it makes sense to call $\hat{D}_\alpha$ a **displacement operator**.

A coherent state can be created via the displacement operator. We will show that

$$\hat{D}_\alpha |0\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle = |\alpha\rangle \tag{120}$$

where $|0\rangle$ is the ground state; $\hat{a}|0\rangle = 0$.

To show 120, we use the Baker-Campbell-Hausdorff formula for the exponential of two operators that commute with their commutator. If $[\hat{A}, \hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$, then

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}.$$

Because $[\hat{a}, \hat{a}^\dagger] = 1$, the operators $\hat{a}, \hat{a}^\dagger$ commute with their commutator. Applying the Baker-Campbell-Hausdorff formula

$$\begin{aligned} e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} |0\rangle &= e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-\frac{\alpha\alpha^*}{2}} |0\rangle \qquad (121) \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} |0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} (\hat{a}^\dagger)^m |0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = |\alpha\rangle . \qquad (122) \end{aligned}$$

Using expressions for the expectation values $\bar{q} = \langle\alpha| q |\alpha\rangle$ equation 119 can also be written

$$\hat{D}_\alpha = e^{\frac{i}{\hbar}\bar{p}\hat{q} - \frac{i}{\hbar}\bar{q}\hat{p}}. \qquad (123)$$

It is moderately tedious to show that

$$\begin{aligned} \langle q|\alpha\rangle &= \pi^{-\frac{1}{4}} e^{\frac{i}{\hbar}\langle\hat{p}\rangle(q - \frac{\langle\hat{q}\rangle}{2}) - \frac{1}{2}(q - \langle\hat{q}\rangle)^2} \\ &= \pi^{-\frac{1}{4}} e^{i\sqrt{2}\,\mathrm{Im}\{\alpha\}(q - \mathrm{Re}\{\alpha\}/\sqrt{2}) - \frac{1}{2}(q - \sqrt{2}\,\mathrm{Re}\{\alpha\})^2} \qquad (124) \end{aligned}$$

I use shorthand $\langle\hat{q}\rangle_\alpha = \bar{q}$. We can show this using the displacement operator

$$\begin{aligned}
\langle q|\alpha\rangle &= \langle q|\, D_\alpha\,|0\rangle \\
&= \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}+\frac{i}{\hbar}\bar{p}\hat{q}}\,|0\rangle \qquad \text{use Baker} - \text{Campbell} - \text{Hausdorff formula} \\
&= \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}} e^{\frac{i}{\hbar}\bar{p}\hat{q}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}}\,|0\rangle \\
&= \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}} e^{\frac{i}{\hbar}\bar{p}\hat{q}}\,|0\rangle\, e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \qquad \text{use ground state (equation 117)} \\
&= \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}} e^{\frac{i}{\hbar}\bar{p}\hat{q}} \int dq'\ \pi^{-\frac{1}{4}} e^{-\frac{q'^2}{2}}\,\left|q'\right\rangle e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \int dq'\ \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}}\,\left|q'\right\rangle e^{\frac{i}{\hbar}\bar{p}q'} e^{-\frac{q'^2}{2}}\,\pi^{-\frac{1}{4}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \int dq'\ dp\ \ \langle q|\, e^{-\frac{i}{\hbar}\bar{q}\hat{p}}\,|p\rangle \left\langle p\middle|q'\right\rangle e^{\frac{i}{\hbar}\bar{p}q'} e^{-\frac{q'^2}{2}}\,\pi^{-\frac{1}{4}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \int dq'\ dp\ e^{-\frac{i}{\hbar}\bar{q}p}\,\langle q|p\rangle \left\langle p\middle|q'\right\rangle e^{\frac{i}{\hbar}\bar{p}q'} e^{-\frac{q'^2}{2}}\,\pi^{-\frac{1}{4}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \int dq'\ dp\ e^{-\frac{i}{\hbar}\bar{q}p} e^{\frac{ip}{\hbar}(q-q')}\,\frac{1}{2\pi} e^{\frac{i}{\hbar}\bar{p}q'} e^{-\frac{q'^2}{2}}\,\pi^{-\frac{1}{4}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \int dq'\ \delta(-\bar{q}+q-q') e^{\frac{i}{\hbar}\bar{p}q'} e^{-\frac{q'^2}{2}}\,\pi^{-\frac{1}{4}} e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \pi^{-\frac{1}{4}} e^{\frac{i}{\hbar}\bar{p}(q-\bar{q})} e^{-\frac{(q-\bar{q})^2}{2}}\, e^{\frac{i}{2\hbar}\bar{p}\bar{q}} \\
&= \pi^{-\frac{1}{4}} e^{\frac{i}{\hbar}\bar{p}(q-\bar{q}/2)} e^{-\frac{(q-\bar{q})^2}{2}}
\end{aligned}$$

We have succeeded in showing equation 124. (This is what I meant by moderately tedious).

Notice that equation 124 is a Gaussian that is displaced in phase space from the origin. Sometimes coherent states are labelled according to their expectation value of p and q as $\langle\hat{q}\rangle$ depends only on the real part of α and $\langle\hat{p}\rangle$ depends only the imaginary part of α. You might see the notation $|q,p\rangle$ to denote a coherent state for which $\langle q,p|\,\hat{q}\,|q,p\rangle = q$ and $\langle q,p|\,\hat{p}\,|q,p\rangle = p$.

5.3 The identity operator

Even though coherent states are not orthogonal, when written in terms of coherent states, the identity operator is not complicated;

$$\mathbf{I} = \int |\alpha\rangle\,\langle\alpha|\,\frac{d^2\alpha}{\pi} \tag{125}$$

where $d^2\alpha = d\,\mathrm{Re}\{\alpha\} d\,\mathrm{Im}\{\alpha\}$. With the goal of making it clearer what is meant by d^α, we check the identity.

$$\begin{aligned} \langle m' | \mathbf{I} | m \rangle &= \int \langle m' | \alpha \rangle \langle \alpha | m \rangle \frac{d^2\alpha}{\pi} \\ &= \int e^{-|\alpha|^2} \frac{\alpha^{m'}}{\sqrt{m'!}} \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{d^2\alpha}{\pi}. \end{aligned} \tag{126}$$

We transfer the integral into polar coordinates r, ϕ with

$$\mathrm{Re}\{\alpha\} = r\cos\phi \qquad \mathrm{Im}\{\alpha\} = r\sin\phi \qquad r = |\alpha|. \tag{127}$$

This gives

$$d^2\alpha = r dr d\phi, \tag{128}$$

and $\alpha^{m'} = r^{m'} e^{im'\phi}$. Equation 126 becomes

$$\langle m' | \mathbf{I} | m \rangle = \frac{1}{\pi} \int r dr d\phi \; e^{-r^2} \frac{r^{m'm}}{\sqrt{m!m'!}} e^{i(m'-m)\phi}. \tag{129}$$

The integral is zero if $m \neq m'$. Integrating over ϕ we gain a factor of 2π because the exponential is part of a delta function.

$$\langle m' | \mathbf{I} | m \rangle = 2 \int r dr \; e^{-r^2} \frac{r^{2m}}{m!} \delta_{mm'} \tag{130}$$

It is helpful to look up the moments of a Gaussian `https://en.wikipedia.org/wiki/Gaussian_integral`

$$\int_0^\infty x^{2n+1} e^{-x^2} dx = \frac{n!}{2} \tag{131}$$

This inserted into equation 130 gives

$$\langle m' | \mathbf{I} | m \rangle = \delta_{mm'} \tag{132}$$

and giving us some feeling for what is meant in equation 125 for the identity operator.

5.4 The Wigner function of a coherent state

I think we are finally read to compute the Wigner function for a coherent state! Using equation 124

$$\begin{aligned}
W_{|\alpha\rangle\langle\alpha|}(q,p) &= \frac{1}{2\pi\hbar}\int dx \,\langle q+x/2|\alpha\rangle\,\langle\alpha|q-x/2\rangle\, e^{-ixp/\hbar} \\
&= \frac{1}{2\pi\hbar}\int dx\ \pi^{-\frac{1}{4}} e^{i\sqrt{2}\,\mathrm{Im}\{\alpha\}(q+x/2-\mathrm{Re}\{\alpha\}/\sqrt{2})-\frac{1}{2}(q+x/2-\sqrt{2}\,\mathrm{Re}\{\alpha\})^2} \\
&\quad\times \pi^{-\frac{1}{4}} e^{-i\sqrt{2}\,\mathrm{Im}\{\alpha\}(q-x/2-\mathrm{Re}\{\alpha\}/\sqrt{2})-\frac{1}{2}(q-x/2-\sqrt{2}\,\mathrm{Re}\{\alpha\})^2} e^{-ixp/\hbar} \\
&= \frac{1}{2\pi\hbar}\int dx\ \pi^{-\frac{1}{2}} e^{i\sqrt{2}\,\mathrm{Im}\{\alpha\}x} e^{-(q-\sqrt{2}\,\mathrm{Re}\{\alpha\})^2} e^{-x^2/4} e^{-ixp/\hbar} \\
&= \pi^{-\frac{1}{2}} e^{-(q-\sqrt{2}\,\mathrm{Re}\{\alpha\})^2} \frac{1}{2\pi\hbar}\int dx\ e^{ix(\sqrt{2}\hbar\,\mathrm{Im}\{\alpha\}-p)/\hbar} e^{-x^2/4}
\end{aligned}$$

Let's complete the square

$$\begin{aligned}
&\int dx\ e^{-\frac{1}{4}(x-2i(\sqrt{2}\,\mathrm{Im}\{\alpha\}-p/\hbar))^2} e^{-(\sqrt{2}\,\mathrm{Im}\{\alpha\}-p/\hbar)^2} \\
&= 2\sqrt{\pi} e^{-(\sqrt{2}\,\mathrm{Im}\{\alpha\}-p/\hbar)^2}
\end{aligned}$$

This gives

$$W_{|\alpha\rangle\langle\alpha|}(q,p) = \frac{1}{\pi\hbar} e^{-(q-\langle\hat{q}\rangle)^2} e^{-(p-\langle\hat{p}\rangle)^2/\hbar^2} \tag{133}$$

We have succeeded in computing the Wigner function for the coherent state and unsurprisingly it looks like a Gaussian function in phase space.

We can transfer the Wigner function itself to a coherent state representation

$$\begin{aligned}
W_\rho(q,p) &= \int dx\ \ \langle q+x/2|\,\rho\,|q-x/2\rangle\, e^{-ixp/\hbar} \\
&= \int dx d^2\alpha d^2\alpha'\, \langle q+x/2|\alpha\rangle\,\langle\alpha|\,\rho\,|\alpha'\rangle\,\langle\alpha'|q-x/2\rangle\, e^{-ixp/\hbar} \\
&= \int dx d^2\alpha d^2\alpha' \pi^{-\frac{1}{2}} e^{\frac{i}{\hbar}\langle\hat{p}\rangle_\alpha(q+x/2-\langle\hat{q}\rangle_\alpha/2)-\frac{1}{2}(q+x/2-\langle\hat{q}\rangle_\alpha)^2} \\
&\quad\times e^{-\frac{i}{\hbar}\langle\hat{p}\rangle'_\alpha(q-x/2-\langle\hat{q}\rangle'_\alpha/2)-\frac{1}{2}(q-x/2-\langle\hat{q}\rangle'_\alpha)^2}\,\langle\alpha|\,\rho\,|\alpha'\rangle\, e^{-ixp/\hbar}
\end{aligned}$$

Let's use $u = \mathrm{Re}\{\alpha\} = \langle\hat{q}\rangle/\sqrt{2}$, $v = \mathrm{Im}\{\alpha\} = \langle\hat{p}\rangle/(\sqrt{2}\hbar)$.

$$\begin{aligned}
W_\rho(q,p) &= \int dx du dv du' dv' \alpha' \pi^{-\frac{1}{2}} e^{i\sqrt{2}q(v-v')+i\sqrt{2}(v+v')x/2+i(vu+v'u')} \\
&\quad\times e^{-u^2-u'^2-x^2/4-q^2+x(u-u')}\,\langle\alpha|\,\rho\,|\alpha'\rangle\, e^{-ixp/\hbar} \\
&\ldots\ldots \\
&=
\end{aligned}$$

With $\hat{a} \to a$ and $\hat{a}^\dagger \to a^*$, the above calculation should give Weyl symbol

$$\Omega_W(a, a*) = \int d\eta d\eta^* \langle a - \eta/2| \hat{\Omega}(\hat{a}, \hat{a}^\dagger) |a + \eta/2\rangle e^{\frac{1}{2}(\eta^* a - \eta a^*)}. \tag{134}$$

For a normal ordered function $\hat{\Omega}$ where $\hat{a}^\dagger$ is to the left of $\hat{a}$,

$$\Omega_W(a, a*) = \int d\eta d\eta^* \Omega(a^* - \eta^*/2, a + \eta/2) e^{-|\eta|^2/2} \qquad \text{normal ordered} \tag{135}$$

As in the coordinate representation, there is a Bopp representation for coherent states

$$\hat{a}^\dagger \to a^* - \frac{1}{2}\partial_a \qquad \hat{a} \to a + \frac{1}{2}\partial_{a^*} \tag{136}$$

with $u = \text{Re}\{a\}$ and $v = \text{Im}\{a\}$,

$$\partial_a = \frac{1}{2}(\partial_u - i\partial_v) \qquad \partial_{a^*} = \frac{1}{2}(\partial_u + i\partial_v) \tag{137}$$

Of interest: classical limit of subsystems when systems are linked. `https://arxiv.org/abs/2310.18271`

6 Discrete analogs

Consider an N dimensional Hilbert space, described with basis $|n\rangle$. A Fourier basis

$$|k\rangle_F = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i nk/N} |n\rangle \tag{138}$$

A Fourier basis satisfies $\langle n|k\rangle_F = \frac{1}{\sqrt{N}} e^{2\pi i nk/N}$. The subscript F lets us know which basis we mean. Both basis are labelled by integers in $\mathbb{Z}_N$. Addition for the indices is modulo N.

It is helpful to define a frequency

$$\omega \equiv e^{2\pi i/N}. \tag{139}$$

With this

$$|k\rangle_F = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \omega^{nk} |n\rangle \tag{140}$$

$$\langle n|k\rangle_F = \frac{1}{\sqrt{N}} \omega^{nk} \tag{141}$$

It is convenient to recall that the sum of the N-th roots of unity is either 0 or N.

$$\sum_{j=0}^{N-1} \omega^{jl} = N\delta_{l,0}. \tag{142}$$

It is helpful to define a Fourier transform operator

$$\hat{Q}_{FT} = \frac{1}{\sqrt{N}} \sum_{mn} \omega^{mn} |m\rangle \langle n| \tag{143}$$

This unitary operator relates the $|k\rangle_F$ basis to the $|n\rangle$ basis. To change basis, if you have a vector in the $|n\rangle$ basis, to put it into the Fourier basis, operate on it with $\hat{Q}_{FT}^\dagger$.

Let's first repeat our prior definition for the Wigner function (in 1 d)

$$W_\rho(q,p) = \frac{1}{2\pi\hbar} \int dx \, \langle q + x/2| \, \hat{\rho} \, |q - x/2\rangle \, e^{-ixp/\hbar}. \tag{144}$$

This is equivalent to the following

$$W_\rho(q,p) = \frac{1}{\pi\hbar} \int dx \, \langle q - x| \, \hat{\rho} \, |q + x\rangle \, e^{2ixp/\hbar}$$
$$W_\rho(q,p) = \frac{1}{\pi\hbar} \int dx \, \langle q + x| \, \hat{\rho} \, |q - x\rangle \, e^{-2ixp/\hbar}.$$

To create an analogous function, we would like to refer to states by integers n, k in $\mathbb{Z}_N$, with n analogous to q and k analogous to p. We would like to have $\sum_n W_\rho(n,k) = \langle k|_F \, \rho \, |k\rangle_F$ and $\sum_k W_\rho(n,k) = \langle n| \, \rho \, |n\rangle$, so that space and Fourier space probability distributions are recovered.

6.1 For the Wigner function

A (non-unique) possible choice for the discrete analog of the Wigner function is

$$\begin{aligned} W_{\hat{\rho}}(n,k) &= \frac{1}{2N} \sum_{x=0}^{N-1} \langle n - x| \, \hat{\rho} \, |n + x\rangle \, e^{4\pi i x k/N} \\ &= \frac{1}{2N} \sum_{x=0}^{N-1} \langle n - x| \, \hat{\rho} \, |n + x\rangle \, \omega^{2xk} \end{aligned} \tag{145}$$

where $\omega = e^{2\pi i/N}$. Summing over k gives $\sum_k W_\rho(n,k) = \langle n| \, \rho \, |n\rangle$ as expected. By inserting identities in the form $\sum_{k'} |k'\rangle_F \langle k'|_F$, we can show that $\sum_n W_\rho(n,k) =_F \langle k| \, \rho \, |k\rangle_F$. With a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, equation 145 can be written

$$
\begin{aligned}
W_{\hat{\rho}}(n,k) &= \frac{1}{2N}\sum_{x=0}^{N-1} \langle n-x|\psi\rangle \langle\psi|n+x\rangle\, \omega^{2xk} \\
&= \langle\psi|\, \frac{1}{2N}\sum_{x} |n+x\rangle\langle n-x|\, \omega^{2xk}\, |\psi\rangle \\
&= \langle\psi|\, \hat{A}_{nk}\, |\psi\rangle \qquad (146)
\end{aligned}
$$

with point operator

$$
\hat{A}_{nk} = \frac{1}{2N}\sum_{x=0}^{N-1} |n+x\rangle\langle n-x|\, \omega^{2xk}. \qquad (147)
$$

For our periodic space, addition within the bras and kets is modulo N.

A discrete analog for the Weyl quantization can be constructed using the point operator. An operator can be created from a function $a(n,k)$ in phase space with $n, k \in \mathbb{Z}_N$ via

$$
\hat{A} = \sum_{nk} a(n,k)\hat{A}_{nk}. \qquad (148)
$$

Note that the choice of point operator and associated Wigner function in a discrete system is not unique. Sometimes people use a lopsided point operator like this one,

$$
\hat{B}_{nk} = \frac{1}{2N}\sum_{x} |x\rangle\langle n-x|\, \omega^{xk}. \qquad (149)
$$

6.2 Displacement operators in discrete analogs

In discrete systems there are no two unique operators (or equivalently, no unique two sets of complete, mutually unbiased bases) that will correspond to position and momentum operators $\hat{q}$, $\hat{p}$ for arbitrary N. What is meant by **mutually unbiased**? Consider eigenstates $|n\rangle$ and $|k\rangle_F$ of operators $\hat{n}, \hat{k}$. They are mutually unbiased, (Woutters 86) if every outcome is equally probable when $|n\rangle$, an eigenstate of $\hat{m}$ is measured via $\hat{k}$.

For a discrete system, it is important decide is whether the system is behaving as it it were periodic (analogous to a loop) or not. For N states, labelled $|n\rangle$ with $n \in \{0, 1, ..., N-1\}$, if the system is periodic then $|0\rangle$ is next to $|N-1\rangle$ and we would do addition modulo N. If the system is analogous to a loop, then a good choice for displacement operators would be the $\hat{X}, \hat{Z}$ operators described below. These have eigenvalues on the unit circle and so all states are weighted equally. This is a popular choice when studying quantum chaos of maps of the unit interval (in phase space) with periodic boundary conditions.

If the system is not analogous to a loop then it might be better to construct displacement operators that are like the angular momentum operators J_+, J_-. For a spin system, We label states by J_z quantum number, $|-N/2\rangle$, $|-N/2+1\rangle$, $|-N/2+2\rangle$ ..... $|-1+N/2\rangle$, $|N/2\rangle$ and N can be odd. In this setting zero has a specific location, so we might want coherent states to be with respect to the $|0\rangle$ or $|\pm 1/2\rangle$ states.

It is always possible to create Fourier basis, whatever the value N of the dimension of the Hilbert space. However N a prime or a product of primes might be convenient as you get a finite field. I was hoping that there would be a unique choice for coherent states in a discrete system but many examples confine themselves to systems that are described by finite fields.

We might guess that α for a coherent state would be related to mean values of $\hat{n} = \sum_n n\,|n\rangle\,\langle n|$ and $\hat{k} = \sum_k k\,|k\rangle_F\,\langle k|_F$, so that $\langle\alpha|\,\hat{n}\,|\alpha\rangle = n_0$, and $\langle\alpha|\,\hat{k}\,|\alpha\rangle = k_0$. However, the eigenvalues of $\hat{n}$ and $\hat{k}$ are positive, and that's a problem as mean values for $\hat{p},\hat{q}$ for coherent state depend on a ground state that has a mean of zero. A way to get around this problem is to use operators that give eigenvalues on the unit circle in the complex plane and then construct the coherent states from those operators. I think that's the idea of the $\hat{Z},\hat{X}$ operators (also used by Balazs and Voros '89 on the quantized Baker map and by Bjork et al. 08).

$$\hat{Z} = \sum_n \omega^n\,|n\rangle\,\langle n| = \sum_k |k+1\rangle_F\,\langle k|_F \tag{150}$$

$$\hat{X} = \sum_k \omega^{-k}\,|k\rangle_F\,\langle k|_F = \sum_n |n+1\rangle\,\langle n| \tag{151}$$

$$\omega = e^{2\pi i/N}. \tag{152}$$

Here $\hat{Z}$ is a shift operator in Fourier space that is diagonal in the $|n\rangle$ basis. The operator $\hat{X}$ is a shift operator in $|n\rangle$ basis but diagonal in the Fourier basis. Eigenvalues for both operators are complex numbers on the unit circle. The operators satisfy

$$\hat{Z}\hat{X} = \omega\hat{X}\hat{Z}. \tag{153}$$

See *The discrete Wigner function*, Bjork et al. 2008, `https://www.sciencedirect.com/science/article/pii/S0079663807510073` With this notation the Fourier transform is

$$\hat{Q}_{FT} = \frac{1}{\sqrt{N}}\sum_{mn}\omega^{mn}\,|m\rangle\,\langle n| \tag{154}$$

and

$$\hat{Q}_{FT}\hat{X}Q^\dagger_{FT} = \hat{Z} \qquad \hat{Q}^\dagger_{FT}\hat{Z}Q_{FT} = \hat{X}. \tag{155}$$

A displacement operator can be defined as

$$\hat{D}(k,l) = \omega^{-kl/2}\hat{Z}^k\hat{X}^l \tag{156}$$

Because $\hat{X}$ acts like a raising operator in the $|n\rangle$ basis and $\hat{Z}$ acts like a raising operator in the $|k\rangle_F$ basis, $D(k,l)$ shifts between states. The displacement operator satisfies a composition relation

$$\hat{D}(k_1,l_1)\hat{D}(k_2,l_2) = \omega^{(k_1l_2-k_2l_1)/2} D(k_1+k_2, l_1+l_2) \tag{157}$$

and a parity relation

$$\frac{1}{N}\sum_{kl} \hat{D}(k,l) = \hat{P} \tag{158}$$

where $\hat{P}$ is a parity operator that satisfies $\hat{P}\,|n\rangle = |-n \bmod N\rangle$ or equivalently

$$\hat{P} = \sum_{j=0}^{N-1} |j\rangle\,\langle N-j|\,.$$

The displacement operator of equation 156 is similar to that of equation 118, for the continuum system, which I repeat here

$$\hat{D}_\alpha = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-\frac{1}{2}|\alpha|^2} \tag{159}$$

where on the right I have used the Baker-Campbell-Hausdorf relation. Both formulas have a similar phase. We could think of the operators $\hat{Z}$ and $\hat{X}$ as if they were exponentials of infinitesimal generators. However, for the continuous system, the raising and lower operators gave us a ground state which has $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$ and is equivalent to the coherent state with $\alpha = 0$. It is less obvious how to construct a replacement for a ground state in a discrete setting.

The displacement operator can used as a point operator which then can be used to construct a Weyl symbol. Recall equation 147 for a discrete version of the point operator which I repeat here

$$\hat{A}_{nk} = \frac{1}{2N}\sum_{x=0}^{N-1} |n+x\rangle\,\langle n-x|\,\omega^{2xk} \tag{160}$$

Following equations 152

$$\begin{aligned}
\hat{X}^n &= \sum_m |m+n\rangle\,\langle m| \\
\hat{X}^n\,|x\rangle &= \sum_m |m+n\rangle\,\langle m|\,|x\rangle = |x+n\rangle \\
\langle x|\,\hat{X}^n &= \langle x|\sum_m |m+n\rangle\,\langle m| = |x-n\rangle \\
\langle -x|\,\hat{X}^{-n} &= \langle -x|\sum_m |m-n\rangle\,\langle m| = |n-x\rangle\,.
\end{aligned}$$

With these we write the point operator of equation 147

$$\begin{aligned}\hat{A}_{nk} &= \frac{1}{2N}\sum_{x=0}^{N-1} |n+x\rangle \langle n-x| \,\omega^{2xk} \\ &= \frac{1}{2N}\sum_{x=0}^{N-1} \hat{X}^n |x\rangle \langle -x| \,\hat{X}^{-n}\omega^{2xk} \\ &= \frac{1}{2N}\hat{X}^n \sum_x \hat{Z}^k |x\rangle \langle -x| \,\hat{Z}^{-k}\hat{X}^{-n} \\ &= \frac{1}{2N}\hat{X}^n \hat{Z}^k \hat{P} \hat{Z}^{-k}\hat{X}^{-n},\end{aligned}$$

with parity operator

$$\hat{P} = \sum_x |x\rangle \langle -x \mod N| \,.$$

This is pretty ugly, even after using the commutator in equation 153 to rearrange the order of $\hat{Z}$ and $\hat{X}$. Hence the alternative choice of using the displacement operator of equation 156 as a point operator;

$$\begin{aligned}\hat{D}(k,l) &= \omega^{-kl/2}\hat{Z}^k\hat{X}^l \\ &= \omega^{-kl/2}\sum_n \omega^{nk} |n\rangle \langle n| \sum_{n'} \left|n'+l\right\rangle \left\langle n'\right| \\ &= \omega^{-kl/2}\sum_{nn'} \omega^{nk} |n\rangle \,\delta_{n,n'+l} \left\langle n'\right| \\ &= \omega^{-kl/2}\sum_n \omega^{nk} |n\rangle \langle n-l| \,.\end{aligned}$$

Except for a phase, this resembles equation 149 which we gave as another possible choice for a point operator. The phase arises in analogy to that acquired when using the Baker-Campbell-Hausdoff formula to expand an exponential of operators (see equation 159).

A discrete analog of the Weyl symbol can be defined via equation 148 using the displacement operator $\hat{D}(k,l)$. Given a function $a(n,k)$ in phase space with $n,k \in \mathbb{Z}_N$, we can construct an operator

$$\hat{A} = \sum_{nl} a(n,k)\hat{D}(n,k). \tag{161}$$

In summary: there are different choices for the Wigner function and Weyl symbol in discrete systems. One way to approach the problem is simply to turn q into an integer and insert that choice into the Wigner function. Alternatively one can start with a displacement operator and use that as a point operator to generate both Wigner function and Weyl symbol.

6.3 Coherent states

A possible choice for a coherent state in a discrete system is

$$|n_0, k_0\rangle = b \sum_{n=0}^{N-1} e^{-\frac{(n-n_0)^2}{4a^2}} \omega^{-k_0(n-n_0)} |n\rangle \tag{162}$$

where b is a normalization constant and a would define a width. If the space is periodic, then we could assert that $(n - n_0)$ is computed modulo N. In this case I find that $\hat{X} |n_0, k_0\rangle = |n_0 + 1, k_0\rangle$ and $\hat{Z} |n_0, k_0\rangle = |n_0, k_0 - 1\rangle$. This is promising because other coherent states in this form can be generated using a displacement operator that depends upon powers of $\hat{X}$ and $\hat{Z}$.

However, the choice of equation 162 for a coherent state has some problems. If the system is periodic, then unfortunately, a Fourier transform of this state does not exactly yield another Gaussian. Another problem with equation 162 is evident if we look at the ground state

$$|0, 0\rangle = b \sum_n e^{-n^2/4a} |n\rangle \tag{163}$$

It would make more sense to have $n \in -N/2$ to $N/2$ rather than 0 to $N - 1$. A possible solution to this problem is to use a parity operator in the definition.

Another choice for a coherent state leverages a state that is an eigenstate of the Fourier transform. The state

$$|\tilde{\eta}\rangle = b \sum_{m=0}^{N} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi(m-kN)^2}{N}} |m\rangle \tag{164}$$

is an eigenstate of the discrete Fourier transform with eigenvalue 1. In other words

$$Q_{FT} |\tilde{\eta}\rangle = |\tilde{\eta}\rangle$$

The state can be normalized by adjusting the coefficient b. Note that the expression inside the exponential is with conventional arithmetic (it is not modulo N). The state is periodic and unimodal (it has a single peak, it looks U shaped, see Figure 12). The infinite series can be written in terms of a special function known as the Jacobi theta function.

Let's normalize the state. We need to compute

$$\begin{aligned} b^{-2} &= \sum_{m=0}^{N} \sum_{k,k'=-\infty}^{\infty} e^{-\frac{\pi(m-kN)^2}{N}} e^{-\frac{\pi(m-k'N)^2}{N}} \\ &\approx \sum_{m=0}^{N} e^{2\pi m^2/N} \approx \sqrt{\frac{N}{2}}. \end{aligned} \tag{165}$$

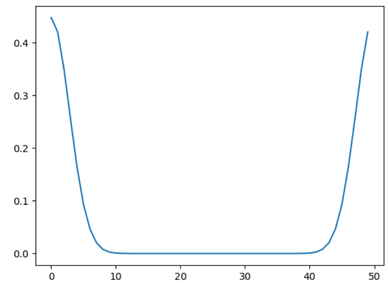

Figure 12: A state vector that is invariant under discrete Fourier transform. This plot shows $|\tilde{\eta}\rangle$ (equation 164) for $N = 50$.

We have used the integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. The normalization constant

$$b \approx \left(\frac{2}{N}\right)^{\frac{1}{4}} \tag{166}$$

Because the state is equal to its Fourier transform, it acts like a state of optimal uncertainty. This follows as

$$\langle\tilde{\eta}| f(\hat{X}) |\tilde{\eta}\rangle = \langle\tilde{\eta}| f(\hat{Z}) |\tilde{\eta}\rangle$$

for any polynomial function $f()$, because $\hat{Z} = \hat{Q}_{FT}\hat{X}\hat{Q}^{\dagger}_{FT}$ and $\hat{Q}_{FT} |\tilde{\eta}\rangle = |\tilde{\eta}\rangle$. So $\left\langle \hat{X}^2 \right\rangle = \left\langle \hat{Z}^2 \right\rangle$ for the state $|\tilde{\eta}\rangle$. Also $\left\langle \hat{X} - \hat{X}^{\dagger} \right\rangle = \left\langle \hat{Z} - \hat{Z}^{\dagger} \right\rangle = 0$ based on symmetry.

With this handy eigenstate of the discrete Fourier transform, we can define a discrete version of coherent states with a displacement operator. Using equation 156 which I repeat here

$$\hat{D}(k, l) = \omega^{-kl/2} \hat{Z}^k \hat{X}^l \tag{167}$$

giving the discrete analog of a coherent state

$$|n_0, k_0\rangle = D(k_0, n_0) |\tilde{\eta}\rangle . \tag{168}$$

Let's see what this looks like in the conventional basis,

$$
\begin{aligned}
|n_0, k_0\rangle &= \left(\frac{2}{N}\right)^{\frac{1}{4}} \omega^{-\frac{n_0 k_0}{2}} \hat{Z}^{k_0} \hat{X}^{n_0} b \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi(m-kN)^2}{N}} |m\rangle \\
&= \left(\frac{2}{N}\right)^{\frac{1}{4}} \omega^{-\frac{n_0 k_0}{2}} \hat{Z}^{k_0} b \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi(m-kN)^2}{N}} |m + n_0\rangle \\
&= \left(\frac{2}{N}\right)^{\frac{1}{4}} \omega^{-\frac{n_0 k_0}{2}} \hat{Z}^{k_0} b \sum_{m=0}^{N-1} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi(m-n_0-kN)^2}{N}} |m\rangle \\
&= \left(\frac{2}{N}\right)^{\frac{1}{4}} \omega^{-\frac{n_0 k_0}{2}} b \sum_{m=0}^{N-1} \omega^{k_0 m} \sum_{k=-\infty}^{\infty} e^{-\frac{\pi(m-n_0-kN)^2}{N}} |m\rangle
\end{aligned}
$$

We have used equations 152 for $\hat{X}, \hat{Z}$ and the result looks reasonable. Note the normalization used here is only approximate.

In summary. The state $|\tilde{\eta}\rangle$ which is invariant under the Fourier transform seems a good choice upon which to build coherent states in the discrete periodic setting. A displacement operator that is build from powers of $\hat{X}, \hat{Z}$ also seem like a good choice. There does not seem to be a consensus on the order of operators or the phase when constructing the displacement operator.

6.3.1 Some Notes:

Balazs and Voros '89 define coherent states via

$$|a, b\rangle = e^{\pi ab/N} \hat{X}^a \hat{Z}^b |\chi_0\rangle \qquad a, b \in \mathbb{Z}_N \tag{169}$$

$$|\chi_0\rangle : \qquad [\hat{X} - \hat{X}^\dagger + i(\hat{Z} - \hat{Z}^\dagger)] |\chi_0\rangle = 0 \tag{170}$$

Note that their condition on $|\chi_0\rangle$ is indeed satisfied by our choice of $|\tilde{\eta}\rangle$ in equation 164. Balazs and Voros's choice is similar to but not the same as ours!

On the circle (which is discrete but infinite in Fourier space, but continuous in θ) there are also analogies for the Wigner function (which seem straightforward) and coherent states (less straightforward) by Kowalski+21 https://iopscience.iop.org/article/10.1088/1751-8121/ac019d.

See for example https://www.researchgate.net/publication/7670213_Quantum_computation_and_analysis_of_Wigner_and_Husimi_functions_Toward_a_quantum_image_treatment The form of equation 162 is related to the probability density function of the wrapped normal distribution https://en.wikipedia.org/wiki/Wrapped_normal_distribution.

What if our quantum system is not that of a torus, but rather more like a spin system? The conventional basis would be generated from eigenstates of J_z. The states are

$|-N/2\rangle \ldots |N/2\rangle$ where indices differ by 1 and half values are allowed if N is odd. The indices are the eigenvalues of the J_z operator. We can construct J_x, J_y operators, analogous to Pauli operators. The operators $J_+ = J_x + iJ_y$ and $J_- = J_x - iJ_y$ would be alternatives to our $\hat{X}$ and $\hat{Z}$ operators which have eigenvalues on the unit circle. We can always define a discrete Fourier transform, however here there is no reason to associate the $|N/2\rangle$ state with the $|-N/2\rangle$ state, so the system is not periodic. Instead we would generate a displacement operator from J_+, J_-? In that setting a coherent state analog would be centered near spin 0 (or $\pm 1/2$). I am not certain what type of state would replace $|\tilde{\eta}\rangle$.

I don't understand why the displacement function has a phase in the definition. Since I don't understand why there is a phase included in the definition, I don't know how to check its sign.

6.4 Discrete analogs for the Husimi distribution

The Husimi distribution can be directly derived from the definition of a coherent state via $H_\psi(\alpha) = |\langle\psi|\alpha\rangle|^2$. Once you succeed at choosing a coherent state representation, then you can generate a Husimi distribution.

However, in the conventional setting, the Husimi distribution is a quasi probability distribution and it integrates to 1. Is that true for our coherent states generated via equation 168? In the conventional setting, while coherent states were not perpendicular, we could integrate over them. Here they are again not perpendicular, but we don't know if we can sum over them!

Let's compute some quantities:

$$
\begin{aligned}
\langle n_1, k_1 | n_2, k_2 \rangle &= \langle \tilde{\eta} | \hat{X}^{-n_1} \hat{Z}^{-k_1} \hat{Z}^{k_2} \hat{X}^{n_2} | \tilde{\eta} \rangle \\
&= \langle \tilde{\eta} | \hat{X}^{-n_1} \hat{Z}^{-k_1+k_2} \hat{X}^{n_2} | \tilde{\eta} \rangle \\
&= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_{kk' \in \mathbb{Z}} \sum_{mm' \in \mathbb{Z}_N} \langle m | e^{-\frac{\pi(m-n_1+Nk)^2}{N}} \omega^{(k_2-k_1)m'} e^{-\frac{\pi(m'-n_2+Nk')^2}{N}} |m'\rangle \\
&\approx \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_m e^{-\frac{\pi(m-n_1)^2}{N}} e^{-\frac{\pi(m-n_2)^2}{N}} \omega^{(k_2-k_1)m} \\
&= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_m e^{-\frac{2\pi}{N}(m-n_1/2-n_2/2)^2} e^{-\frac{\pi}{2N}(n_1-n_2)^2} \omega^{(k_2-k_1)m} \\
&= \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_m e^{-\frac{2\pi}{N}(m-(n_1+n_2)/2+i(k_2-k_1)/2)^2} e^{-\frac{\pi}{2N}(n_1-n_2)^2} e^{-\frac{\pi}{2N}(k_2-k_1)^2}
\end{aligned}
$$

I am not finding a simple closed form, though this peaks when $n_1 = n_2$ and $k_1 = k_2$ as expected.

Using the coherent states of equation 168, the Husimi function is

$$H_{|\psi\rangle}(n_0, k_0) = \frac{1}{N} |\langle\psi|n_0, k_0\rangle|^2 \tag{171}$$

My numerical attempts suggest that this is approximately normalized so that it integrates to 1.

Summary: Using discrete analogs of coherent states, it is possible to construct an analog of a Husimi distribution which can be used to look at phase space distributions for wave functions. This function is an alternative to a Wigner function for examining properties of wave functions in phase space for discrete systems. Classical maps of the unit interval, when quantized give an operator which can be approximated on a finite dimensional system, creating a unitary propagator. The eigenstates of this operator can be examined in phase space to see if they resemble orbits of the classical dynamical map. An example where this has been done is with the quantized Baker map (see below).

7 The semi-classical limit

7.1 Relationship between Feynman's path integral and Shrödinger's equation

Feynman's path integral formulation for quantum mechanics is based on a propagator

$$K(\mathbf{r}, \mathbf{r}', t) = \int_{\text{paths}} D\mathbf{q}\ e^{iS(\mathbf{q})/\hbar} \qquad \text{where paths start at } \mathbf{q}(0) = \mathbf{r} \text{ and end at } \mathbf{q}(t) = \mathbf{r}'. \tag{172}$$

Here the action S depends on a Lagrangian

$$S(\mathbf{q}_{\text{path}}) = \int_0^t \mathcal{L}(\mathbf{q}(t'), \dot{\mathbf{q}}(t'))dt' \qquad \text{on a path } \mathbf{q}(t). \tag{173}$$

The integral in equation 172 is over all paths that begin at $q(0) = \mathbf{r}$ and end at $\mathbf{q}(t) = \mathbf{r}'$.

If the Lagrangian is

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}(\dot{\mathbf{q}})^2 - V(\mathbf{q}), \tag{174}$$

with potential function V, then the propagator is equivalent to

$$K(\mathbf{r}, \mathbf{r}', t) = \langle\mathbf{r}|\, e^{-i\hat{H}t/\hbar} \left|\mathbf{r}'\right\rangle, \tag{175}$$

with Hamiltonian operator

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{q}}).$$

The probability that a particle goes from $\mathbf{r}$ to $\mathbf{r}'$ is equal to $KK\dagger$. The resulting evolution is also consistent with Shrödinger's equation and with $\hat{p} = -i\hbar\partial_q$.

Before we start, it is handy to have the following integral ready!

$$\int_{-\infty}^{\infty} dx \ e^{i\alpha x^2} = \sqrt{\frac{\pi}{|\alpha|}} e^{i\frac{\pi}{4}\text{sgn}(\alpha)}. \tag{176}$$

which comes from the integral $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and $\frac{1}{\sqrt{-i}} = \sqrt{i} = \pm e^{i\pi/4}$ and choosing the positive root.

Let us show that equation 175 for the propagator in terms of the Hamiltonian is equivalent to equation 172 written as a path integral. With a separable Hamiltonian and for a small time δt

$$\begin{aligned} e^{-\frac{i}{\hbar}\hat{H}\delta t} &= e^{-\frac{i\delta t}{\hbar}\left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{q}})\right)} \\ &\sim e^{-\frac{i\delta t}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}} e^{-\frac{i\delta t}{\hbar}V(\hat{\mathbf{q}})} + O(\delta t^2) \end{aligned}$$

$$\begin{aligned} K(\mathbf{r}, \mathbf{r}', \delta t) &= \langle \mathbf{r}| e^{-\frac{i}{\hbar}\hat{H}\delta t} |\mathbf{r}'\rangle \\ &\sim \langle \mathbf{r}| e^{-\frac{i\delta t}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}} |\mathbf{r}'\rangle e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \\ &= \langle \mathbf{r}| e^{-\frac{i\delta t}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}} \int d\mathbf{p} \,|\mathbf{p}\rangle \langle \mathbf{p}| \,|\mathbf{r}'\rangle e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \qquad \text{insert identity} \\ &= \int d\mathbf{p} \ e^{-\frac{i\delta t}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \langle \mathbf{r}|\mathbf{p}\rangle \langle \mathbf{p}|\mathbf{r}'\rangle \\ &= \int d\mathbf{p} \ e^{-\frac{i\delta t}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} e^{\frac{i(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}}{\hbar}} \frac{1}{(2\pi\hbar)^d} \end{aligned}$$

Complete the square

$$\begin{aligned} K(\mathbf{r}, \mathbf{r}', \delta t) &\sim \prod_{j=1}^{d} \left[\int \frac{dp_j}{2\pi\hbar} e^{-\frac{i\delta t}{\hbar 2m}(p_j - \frac{m}{\delta t}(r_j - r_j'))^2} e^{\frac{im}{\hbar 2\delta t}(r_j - r_j')^2}\right] e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \\ &= \prod_{j=1}^{d} \left[\int \frac{dp_j}{2\pi\hbar} e^{-\frac{i\delta t}{\hbar 2m}p_j^2}\right] e^{\frac{im}{\hbar 2\delta t}(\mathbf{r}-\mathbf{r}')^2} e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \\ &= \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{d}{2}} e^{\frac{im}{\hbar 2\delta t}(\mathbf{r}-\mathbf{r}')^2} e^{-\frac{i\delta t}{\hbar}V(\mathbf{r}')} \qquad \text{using equation 176} \\ &= \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{d}{2}} e^{\frac{i\delta t}{\hbar}\left(\frac{1}{2m}\frac{\mathbf{r}-\mathbf{r}'}{(\delta t)^2} - V(\mathbf{r}')\right)}. \end{aligned}$$

$$K(\mathbf{r}, \mathbf{r}', \delta t) = \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{d}{2}} e^{\frac{i\delta t}{\hbar}\mathcal{L}(\mathbf{r},\dot{\mathbf{r}})} \qquad \text{with } \dot{\mathbf{r}} = \frac{\mathbf{r}-\mathbf{r}'}{\delta t} \text{ and } \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{\dot{\mathbf{r}}^2}{2m} - V(\mathbf{r}) \tag{177}$$

One need not be real careful whether the potential is evaluated at $\mathbf{r}$ or $\mathbf{r}'$ if δt is small.

We use this to compute the propagator over a longer time interval t. We use N intermediate positions $\mathbf{r}_1, \mathbf{r}_2....\mathbf{r}_{N-1}$ with initial position $\mathbf{r}_0 = \mathbf{r}$ and final position $\mathbf{r}_N = \mathbf{r}'$. Because we don't know what the intermediate positions are, we integrate over all possible values for them. The propagator becomes

$$K(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 ... d\mathbf{r}_{N-1} K(\mathbf{r}, \mathbf{r}_1, \delta t) K(\mathbf{r}_1, \mathbf{r}_2, \delta t) ... K(\mathbf{r}_{N-1}, \mathbf{r}', \delta t), \tag{178}$$

with $t = N\delta t$. We then insert equation 177 into the right hand side of equation 178 to go across a longer time interval giving

$$K(\mathbf{r}, \mathbf{r}', t) = \left(\frac{m}{2\pi i \hbar \delta t}\right)^{\frac{Nd}{2}} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 ... d\mathbf{r}_{N-1} \exp\left\{\frac{i}{\hbar} \sum_{j=0}^{N-1} \mathcal{L}\left(\mathbf{r}_j, \dot{\mathbf{r}}_j\right) \delta t\right\}. \tag{179}$$

This is an example of what is meant by the path integral in equation 172.

7.2 Semi-classical limit using the constant phase approximation

We will use what is known as the **constant phase approximation** to derive a semi-classical approximation for the propagator in terms of the Lagrangian. We will use the resulting propagator to quantize the kicked rotator. In yet another way!

If $S/\hbar$ varies quickly then $e^{iS/\hbar}$ in equation 172 oscillates rapidly. The paths that contribute to the propagator are mostly those that have constant or nearly constant phase.

Consider the integral

$$A = \int_a^b g(x) e^{\frac{i}{\hbar} f(x)} dx \tag{180}$$

where g, f are real valued and well behaved functions. We make a list of all points where the phase is constant. We call these points x_ν where ν simply serves as an index to label them. Near x_ν we expand f

$$f(x - x_\nu) = f(x_\nu) + f'(x_\nu)(x - x_\nu) + \frac{1}{2} f''(x_\nu)(x - x_\nu)^2 \tag{181}$$

If x_ν is a point with stationary phase then $f'(x_\nu) = 0$. In the classical limit, we approximate equation 180 as

$$A \underset{\hbar \to 0}{\longrightarrow} \sum_\nu \int_a^b g(x_\nu) e^{\frac{i}{\hbar}\left[f(x_\nu) + \frac{1}{2} f''(x_\nu)(x - x_\nu)^2\right]} dx + O(\hbar). \tag{182}$$

Using the integral of equation 176 to carry out the Gaussian integral, equation 182 gives

$$A \underset{\hbar \to 0}{\longrightarrow} \sum_\nu g(x_\mu) e^{\frac{i}{\hbar} f(x_\nu)} \sqrt{\frac{2\pi\hbar}{|f''(x_\nu)|}} e^{\frac{i\pi}{4} \text{sign}(f''(x_\nu))}. \tag{183}$$

We generalize to a high dimensional system

$$A = \int g(\mathbf{x}) e^{\frac{i}{\hbar} f(\mathbf{x})} d\mathbf{x}. \tag{184}$$

Instead of zeros of $f'(x)$ we need to consider all zeros of the gradient ∇f which has components $\frac{\partial f}{\partial x_j}$. Denote $\mathbf{x}_\nu$ a root of the gradient of f, satisfying $\nabla f(\mathbf{x}_\nu = 0)$. We denote H the Hessian matrix with components $H_{jk}(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_k}$. Expanding about a root $\mathbf{x}_\nu$

$$f(\mathbf{x}) \approx f(\mathbf{x}_\nu) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_\nu) H(\mathbf{x}_\nu)(\mathbf{x} - \mathbf{x}_\nu)^T \tag{185}$$

where the Hessian matrix is evaluated at $\mathbf{x}_\nu$. Using an orthogonal coordinate system $\mathbf{y}$ where the Hessian matrix is diagonal

$$f(\mathbf{x}) \approx \frac{1}{2} \sum_j \mathbf{y}_j^2 \lambda_{\nu,j} \tag{186}$$

where $\lambda_{\nu,j}$ are the eigenvalues of $H(\mathbf{x}_\nu)$. The integral in equation 184 can be approximated with a sum for each root $\mathbf{x}_\nu$,

$$A \sim \sum_\nu g(\mathbf{x}_\nu) e^{\frac{i}{\hbar} f(\mathbf{x}_\nu)} \frac{(2\pi\hbar)^{\frac{d}{2}}}{\sqrt{|\det H(\mathbf{x}_\nu)|}} e^{\frac{i\pi}{4}\alpha_\nu} + O(\hbar). \tag{187}$$

Here the coefficient $\alpha_\nu = \sum_j \text{sign}(\lambda_{\nu,j})$ and $\det H(\mathbf{x}_\nu) = \prod_j \lambda_{\nu,j}$. The dimension d is that of $\mathbf{x}$.

7.3 The Van-Vleck propagator

The Van-Vleck propagator (or Van-Vleck-Gutzwiller propagator, also in work by Morette-DeWitt) is the propagator estimated by using a stationary phase approximation on the Feynman path integral.

However here $\mathbf{x}$ corresponds to the vector $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2 .... \mathbf{r}_{N-1}, \mathbf{r}'$ describing a discrete path from $\mathbf{r}$ to $\mathbf{r}'$. The roots $\mathbf{x}_\nu$ give a path that minimizes the integral of the Lagrangian. As $\mathbf{x}_\nu$ describes a path that minimizes the action, it locally satisfies Lagrange's equations (giving classical equations of motion). The Hessian matrix is computed on this classical path.

The van Vleck propagator replaces $\mathbf{x}$ the vector of different positions at different times with a function that depends only on $\mathbf{r}, \mathbf{r}'$. Something like this?

$$\langle q | U | q' \rangle = \left(\frac{\partial^2 S}{\partial q \partial q'} \right)^{\frac{1}{2}} e^{i2\pi i S} \tag{188}$$

where S is a generating function for a map?

I think it is not trivial to derive the van-Vleck propagator from the stationary phase approximation.

With the van-Vleck propagator we would have another way to create quantized systems of maps using a semi-classical approximation. The van-Vleck propagator is relevant for the Gutzwiller trace formula which gives the spectrum of a quantum system using closed orbits of a classical system, in the semi-classical limit.

Approaches: Carefully derive for a particular system (as in Linda Reichl's book). Refer to other work (as in Wimberger's book). Present as obvious (Backer's review).

7.4 The relation between WKB and Bohr Sommerfeld quantization

We adopt Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ and the Shrödinger equation $\hat{H}\psi(x) = (\frac{\hbar^2}{2m}\partial_{xx} + V(x))\psi(x)$.

We define two functions

$$\begin{aligned} p(x) &\equiv \sqrt{2m(E - V(x))} \quad \text{where } V(x) < E \\ \tilde{p}(x) &\equiv \sqrt{-2m(E - V(x))} \quad \text{where } V(x) > E \end{aligned} \tag{189}$$

The second case is the classically forbidden region. In the classically allowed region, the function $p(x)$ is consistent with the classical equation of motion and $p(x)$ equal to momentum, $p = m\dot{x}$.

The WKB approximation assumes that the wavefunction

$$\psi(x) = e^{ig(x)/\hbar}. \tag{190}$$

Inserting this wavefunction into the Shrödinger equation and we find (to first order in $\hbar$) that the wavefunction oscillates in the classically allowed region and decays exponentially in the forbidden region;

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{p(x)}}\left[\int Ae^{\frac{i}{\hbar}\int^x p(x)dx} + Be^{-\frac{i}{\hbar}\int^x p(x)dx}\right] \qquad \text{where } E > V(x) \\ &= \frac{1}{\sqrt{\tilde{p}(x)}}\left[\int Ce^{\frac{1}{\hbar}\int^x \tilde{p}(x)dx} + De^{-\frac{1}{\hbar}\int^x \tilde{p}(x)dx}\right] \qquad \text{where } E < V(x) \end{aligned} \tag{191}$$

for real constants A, B, C, D and using the functions defined in equations 189. A problem occurs where $E = V(x)$. Let us define x_0 as a location where $E = V(x_0)$. Near x_0, the ansatz of equation 190 is no longer a good approximation for a solution to Shrödinger's equation. Near x_0, the solution to the linearized Shrödinger equation is an Airy function. Near x_0, we expand the potential $V(x) = V(x_0) + V'(x_0)(x - x_0)$.... With $\alpha = V'(x_0)$, the solution to the Shrodinger equation is

$$\psi(x) = a\text{Ai}\left[\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}(x - x_0)\right] + b\text{Bi}\left[\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}(x - x_0)\right] \tag{192}$$

where Ai, Bi are Airy functions. Using the asymptotic limits of the Airy functions, the exponential and oscillating solutions are connected together. The result is

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{p(x)}}\left[\int A\cos\left(\frac{1}{\hbar}\int^x p(x)dx - \frac{\pi}{4}\right) + B\sin\left(\frac{1}{\hbar}\int^x p(x)dx - \frac{\pi}{4}\right)\right] \qquad \text{where } E > V(x) \\ &= \frac{1}{\sqrt{\tilde{p}(x)}}\left[\int \frac{A}{C}e^{\frac{1}{\hbar}\int^x \tilde{p}(x)dx} - Be^{-\frac{1}{\hbar}\int^x \tilde{p}(x)dx}\right] \qquad \text{where } E < V(x) \end{aligned} \tag{193}$$

Notice that in equation 193 (top) there is the integral $\int p(x)dx$ which is similar to the action variable in classical mechanics!

If a classical integrable system has a trajectory that is a loop then it can be quantized via $\int p(x)dx = 2\pi\hbar(n + \beta_1 + \beta_2)$ where the integral is over the loop. The real numbers β_1, β_2 take into account what to do where the is a sign ambiguity in $p(x)$ at either side of the loop. The β values are typically either 1/2 or 1/4, depending upon whether there is a hard or soft boundary.

Left to do is the quantization based on semi-classical limit for the kicked rotator

8 Quantizing maps of the square and torus

8.1 The space

Consider a rectangle of width and length L_q, L_p. If we think of this as phase space and use the uncertainty relation for p, q then N states can fit within the space, with

$$L_q L_p = 2\pi\hbar N. \tag{194}$$

Why 2π? We can divide L_q into N states via $L_q/N = 2\pi\hbar/L_p$ or we can divide L_p into N states via $L_p/N = 2\pi\hbar/L_q$. A basis for N states in position space is $|n\rangle$ with $n \in \mathbb{Z}_N$ and position operator

$$\hat{q}\,|n\rangle = \frac{2\pi\hbar}{L_p} n\,|n\rangle\,. \tag{195}$$

Likewise a basis for N states in momentum space is $|k\rangle_F$ with $k \in \mathbb{Z}_N$ and

$$\hat{p}\,|k\rangle_F = \frac{2\pi\hbar}{L_q} k\,|k\rangle_F \tag{196}$$

We can relate the two basis via a Fourier transform

$$|k\rangle_F = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{2\pi i n k/N}\,|n\rangle \tag{197}$$

This gives

$$\langle n|k\rangle_F = \frac{1}{\sqrt{N}} e^{2\pi i n k/N} \tag{198}$$

and

$$|n\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-2\pi i n k/N} \, |k\rangle_F \, . \tag{199}$$

Our choice for $\hat{q}, \hat{p}$ have only non-negative integer eigenvalues.

We rearrange the condition of equation 194

$$N = \frac{L_p L_q}{2\pi\hbar}. \tag{200}$$

For N to be an integer, only certain values of $\hbar$ are allowed. This problem is not the same as quantizing a square well, which can have any size and any value of $\hbar$. Nevertheless the limit $\hbar \to 0$ gives $N \to \infty$, as expected.

8.2 The Baker map

The classical area preserving Baker map (folded) is

$$(x, y) \to= \begin{cases} (2x, y/2) & \text{for } 0 \leq x < 1/2 \\ (2 - 2x, 1 - y/2) & \text{for } 1/2 \leq x < 1 \end{cases} \tag{201}$$

Equivalently

$$(x, y) \to (2x - \lfloor 2x \rfloor, \frac{y + \lfloor 2x \rfloor}{2}). \tag{202}$$

The floor function $\lfloor x \rfloor$ gives the greatest integer less than or equal to x. Both left and right halves of the unit interval are stretched out horizontally. Both left and right halves of the unit interval are squeezed vertically in half. The left half is mapped to the bottom half of the square. The right half is mapped to the top half of the square, as shown in Figure 13.

We assume that N is even, so that $N/2$ is an integer.

We create two pairs of projection operators. The top and bottom projection operators $\hat{P}_T$ and $\hat{P}_B$ project in the momentum or $|m\rangle_F$ basis.

$$\hat{P}_B = \sum_{k=0}^{N/2-1} |k\rangle_F \langle k|_F$$
$$\hat{P}_T = \sum_{k=N/2}^{N-1} |k\rangle_F \langle k|_F \, . \tag{203}$$

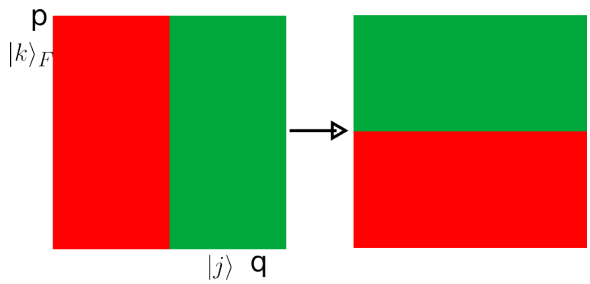

Figure 13: The Baker map.

They sum to the identity and so are a *complete* set $\hat{P}_T + \hat{P}_B = \mathbf{I}$.

Likewise also create two projection operators that project in the $|j\rangle$ basis. The left and right operators

$$\hat{P}_L = \sum_{n=0}^{N/2-1} |n\rangle \langle n|$$
$$\hat{P}_R = \sum_{n=N/2}^{N-1} |n\rangle \langle n| . \tag{204}$$

8.3 The Quantized Baker Map

To quantize the Baker map we want an operation that sends $|j\rangle \to |2j \mod N\rangle$. For $j < N/2$ the result should have a lower value of momentum and vice versa if $j > N/2$.

A version of the quantized Baker map by Balazs and Voror(1989)[4] can be described with the operator

$$\hat{B}_{QB} = \hat{Q}_N^\dagger \begin{pmatrix} \hat{Q}_{N/2} & 0 \\ 0 & \hat{Q}_{N/2} \end{pmatrix} \tag{205}$$

where $\hat{Q}_N$ is the $N \times N$ discrete quantum Fourier transform operator;

$$\hat{Q}_N = \sum_{j,k=0}^{N-1} \omega_N^{jk} |j\rangle \langle k| \tag{206}$$

and the phase $\omega_N = e^{2\pi i/N}$. This operator is unitary so we could call it a propagator.

[4]Balazs N.L, Voros A. (1989) Ann Phys 190:1

Suppose I start with a state vector $|\psi\rangle_L$ that is within the left side P_L projected space in the $|j\rangle$ basis (in the red rectangle on the left in Figure 13). That means that

$$|\psi\rangle_L = \sum_{j=0}^{N/2-1} a_j |j\rangle .$$

I operate on it with the operator $\hat{B}_{QB}$

$$\begin{aligned} \hat{B}_{QB} |\psi\rangle_L &= \hat{Q}_N^\dagger \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} |j'\rangle \langle k'| \sum_{j=0}^{N/2-1} a_j |j\rangle \\ &= \hat{Q}_N^\dagger \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} a_{k'} |j'\rangle \\ &= \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} a_{k'} |j'\rangle_F . \end{aligned} \tag{207}$$

There is no point in carrying out the inverse transform, as that is equivalent to changing into the Fourier basis. Note that the resulting states sum only range from 0 to $N/2-1$ so we are in the bottom half of Fourier space (and in the red rectangle on the right side of Figure 13). In other words we find that

$$\begin{aligned} \hat{P}_B \hat{B}_{QB} |\psi\rangle_L &= \hat{B}_{QB} |\psi\rangle_L \\ \hat{P}_T \hat{B}_{QB} |\psi\rangle_L &= 0. \end{aligned} \tag{208}$$

Now lets consider what happens to

$$|\psi\rangle_R = \sum_{j=N/2}^{N-1} b_j |j\rangle ,$$

within the green rectangle on the left in Figure 13.

$$\begin{aligned} \hat{B}_{QB} |\psi\rangle_R &= \hat{Q}_N^\dagger \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} |j'+N/2\rangle \langle k'+N/2| \sum_{j=N/2}^{N-1} b_j |j\rangle \\ &= \hat{Q}_N^\dagger \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} b_{k'} |j'+N/2\rangle \\ &= \sum_{j'k'=0}^{N/2-1} \omega_{N/2}^{j'k'} b_{k'} |j'+N/2\rangle_F . \end{aligned} \tag{209}$$

In this case the state is in the top half of phase space (in the green rectangle on the right side in Figure 13).

$$\hat{P}_B \hat{B}_{QB} |\psi\rangle_R = 0$$
$$\hat{P}_T \hat{B}_{QB} |\psi\rangle_R = \hat{B}_{QB} |\psi\rangle_R . \tag{210}$$

The propagator does project within momentum space, however, it is not as clear that it expands the state in the conventional basis, sending $j \to 2j$.

8.4 Egorov limits and quantum ergodicity

When the classical limit of a quantum dynamical system is a Hamiltonian dynamical system which is ergodic with respect to the Liouville measure, it is expected that in the classical limit (most of) the eigenfunctions of the quantum system become equidistributed with respect to the Liouville measure. In this is true, the quantum system is described as 'ergodic'.

The semi-classical Egorov Theorem states that quantization and evolution commute. Consider a quantum system described by unitary propagator $\hat{U}$ and an associated classical map $M(q,p)$. Consider an operator $\hat{O}(f)$ that is derived from a function of phase space $f(q,p)$. The operator could be constructed via Weyl quantization. Consider this function:

$$U^\dagger \hat{O}(f) U - \hat{O}(f \circ M). \tag{211}$$

The thing on the left is evolution of the observable $\hat{O}$ in the quantum system. The function $f \circ M$ represents the function f of phase space after the classical system has evolved due to the classical map M. The thing on the left should not be too different than the thing on the right if $\hat{U}$ is a quantum version of M. You can consider limits for $N \to \infty$ where N is the dimension of the Hilbert space for a discrete approximation. You can consider applying the propagator multiple times in which case you are computing

$$\hat{U}^{\dagger k} \hat{O}(f) \hat{U}^k - \hat{O}(f \circ M^k). \tag{212}$$

Limits on the sizes of these operators are constrained in a set of theorems that are sometimes called Erogov limits. When approximating a continuous classical map with a discrete quantum system, the validity of the approximation can be discussed in terms of 'Erogov' limits.

Any loop billiard problem can be made into an area preserving map of the torus. What is the relation between quantization via Laplacian vs quantization via maps of the unit square? Concept of Ergorov limit somewhere. Think about drifting systems?