PHY411 Lecture notes Part 11

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Contents

1 Geodesics

Within the context of the variational Lagrangian approach to describing dynamics we can ask if there is a geometric description.

1.1 Minimum of an action along a path and geodesics

Recall that given a Lagrangian $\mathcal{L}(q, \dot{q}, t)$, and paths described as $q(\tau)$, the action or integral on a path

$$S(q(\tau)) = \int \mathcal{L}(q, \dot{q}, \tau) d\tau \tag{1}$$

is minimized on a path that satisfies the Euler-Lagrange's equations at each position in the path

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

Let us compare this to a *geodesic* which is a path on a surface that has minimum *length*. To measure length we need something called a *metric*. A metric allows us to measure distances. Given a local coordinate system with coordinates $\mathbf{x}$ the square of the distance along a path with elements $d\mathbf{x}$ is

$$ds^2 = g_{ij} dx^i dx^j$$

using summation notation convention. Here g_{ij} is a symmetric invertible matrix that can depend on the local position in a manifold. We require that

$$g^{ij} g_{jk} = \delta^i_k$$

For a nice coordinate system $g_{ij} = \delta_{ij}$. Upper and lower indices refer to wether the item is in the cotangent or tangent space and the metric allows one to compare one and another.

It may be convenient to write

$$\frac{ds}{d\tau} = \frac{\sqrt{g_{ij} dx^i dx^j}}{d\tau} = \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}}$$

The length of a path is the integral of ds along the path or

$$l = \int ds = \int \sqrt{g_{ij} dx^i dx^j} = \int \sqrt{g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}} d\tau \tag{2}$$

Minimizing the path length gives a geodesic. If the Lagrangian can be written as a path then, then Lagrange's equations are equivalent to the geodesic equations.

1.2 Action comprised of square of path lengths

Kinetic energy is often written as a sum of velocities and so involves a square of velocities rather than a square root of velocities. Consider the action

$$S = \int \frac{1}{2} \left(\frac{ds}{d\tau} \right)^2 d\tau = \int \frac{1}{2} g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau \tag{3}$$

We use a Lagrange multiplier λ that is a function of t on the path. Consider

$$S(\lambda) = \int \frac{1}{2} \left[g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \lambda^{-1}(\tau) + \lambda(\tau) \right] d\tau \tag{4}$$

Taking the functional derivative with respect to the function λ

$$\begin{aligned}\frac{\delta S}{\delta \lambda(\beta)} &= \lim_{\epsilon \to 0} \frac{1}{2} \int \left[g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} (\lambda + \epsilon\delta(\beta - \tau))^{-1} + \lambda + \epsilon\delta(\beta - \tau) \right] d\tau \\ &= \frac{1}{2} \int \left[g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \lambda^{-2} \delta(\beta - \tau) + \delta(\beta - \tau) \right] d\tau \\ &= -\lambda(\beta)^{-2} g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau}(\beta) + 1 \\ &= -|\dot{x}(\beta)|^2 \lambda(\beta)^{-2} + 1\end{aligned}$$

This is minimized with

$$\lambda(\beta) = |\dot{x}| = \sqrt{g_{ij}\dot{x}^i\dot{x}^j} = \sqrt{g_{ij}\frac{x^i}{d\tau}\frac{x^j}{d\tau}} \tag{5}$$

at any time $t = \beta$. This is a constant velocity condition! Inserting this back into $S(\lambda)$, equation 4 gives

$$\begin{aligned}S(\lambda) &= \int \frac{1}{2} \left[g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} (\sqrt{g_{ij}\dot{x}^i\dot{x}^j})^{-1} + \sqrt{g_{ij}\dot{x}^i\dot{x}^j} \right] d\tau \\ &= \int \sqrt{g_{ij} \frac{x^i}{d\tau} \frac{x^j}{d\tau}} d\tau\end{aligned}$$

Our action S (equation 3) involving squares of velocities is minimized w.r.t to the function $\lambda(t)$ if and only if we chose $\lambda(t)$ so as to give a constant velocity. With this condition our action then gives the path length. In this sense $\lambda(t)$ can be considered a transformation of the way the path is parametrized by time. Recall that the length of a path is independent how we describe time

$$\int_{\tau_1}^{\tau_2} \sqrt{g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}} d\tau = \int_{t_1}^{t_2} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} dt$$

for $\tau = f(t)$ with $\tau_1 = f(t_1)$ and $\tau_2 = f(t_2)$ and the function f is continuous and never has zero or infinite slope.

Let us look again at the first part of equation 3 and change time with $t = f(\tau)$ and

$$\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \frac{dx^i}{dt} f'(\tau)$$

and $d\tau = dt/f'$.

$$
\begin{aligned}
S(\lambda) &= \int_{\tau_1}^{\tau_2} \frac{1}{2}\left[g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\lambda^{-1}\right] d\tau \\
&= \int_{\tau_1}^{\tau_2} \frac{1}{2} d\tau g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} f'(\tau)^2 \lambda^{-1} \\
&= \int_{t_1}^{t_2} \frac{1}{2} dt g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} f' \lambda^{-1}
\end{aligned}
$$

If we chose $f(\tau)$ such that $f'(\tau) = \lambda(\tau)$ then we can remove λ with a parametrization of time and this action does not change (except possibly by a dilating factor involving the boundary). Since we are only constrained by the derivative of f we can choose that $t_1 = \tau_1$.

What about the second half of equation 3

$$
\begin{aligned}
\int_{\tau_1}^{\tau_2} d\tau \frac{1}{2}\lambda(\tau) &= \int_{\tau_1}^{\tau_2} \frac{1}{2} dt \lambda [f'(\tau(t))]^{-1} \\
&= \frac{t_2 - t_1}{2}
\end{aligned}
$$

This only depends on the time interval. This can be written as a time derivative of something and we show below that if two Lagrangian's differ by the time derivative of something then they are minimized on the same paths. Altogether (except for some uncertainty about the time on the boundary) putting this together we have shown that this action

$$
\int g_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} d\tau \tag{6}
$$

is minimized on the same paths that minimize path length.

Comparing the action (equation 3) with the action for a Lagrangian (equation 1) we can associate

$$
\mathcal{L}(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\frac{dq^i}{d\tau}\frac{dq^j}{d\tau}
$$

If we can write a Lagrangian in the above form, then we can equivalently think about the dynamics in terms of paths being geodesics of the metric g. Systems that have Lagrangian equal to a kinetic energy function, such as rigid bodies and hydrodynamics, can be written in this form!

1.3 The geodesic equation

Let us just choose a Lagrangian

$$
\mathcal{L}(x,\dot{x}) = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j
$$

and look at the Euler-Lagrange equations. First let us compute some derivatives

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij}\dot{x}^j = p_i$$

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2}(\partial_k g_{ij})\dot{x}^i\dot{x}^j$$

The Euler-Lagrange equations are

$$\frac{d}{d\tau}\left[g_{ij}\frac{dx^j}{d\tau}\right] = \frac{1}{2}(\partial_i g_{jk})\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}$$

These can also be written as

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

with Γ^i_{jk} known as Christoffel symbols that depend on derivatives of the metric coefficients. Here the equations of motion are also called geodesic equations. The above form is commonly used in general relativity to compute trajectories of particles in space that is curved due to gravity.

1.4 An example with a holonomic constraint

From Arnold's book.

Consider a particle in a three-dimensional space in cylindrical coordinate that is constrained to a surface defined by a function of radius $R(z)$.

The kinetic energy per unit mass of the particle

$$\begin{aligned}\frac{T}{m} &= \frac{1}{2}\dot{\mathbf{r}}^2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2 + r^2\dot{\theta}^2\right)\end{aligned}$$

Using our surface constraint function $R(z)$

$$\dot{r} = R'(z)\dot{z}$$

and

$$\frac{T}{m} = \frac{1}{2}\left((1 + R'(z)^2)\dot{z}^2 + R(z)^2\dot{\theta}^2\right)$$

is now a two-dimensional system. An integral

$$S = \int \mathcal{L}dt = \int T dt$$

can be written as

$$S = \int g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau$$

with diagonal metric in the two dimensional space z, θ

$$g_{zz} = (1 + R'(z)^2) \qquad g_{\theta\theta} = R(z)^2$$

Thus the trajectories of free particles constrained to the surface defined with $R(z)$ are geodesics with the above metric.

Because there is no potential energy $H = \mathcal{L} = T$ is conserved. Also as H does not depend on θ the associated momentum

$$p_\theta = r^2 \dot{\theta}$$

is conserved. This conserved quantity is equivalent to the z component of the angular momentum. The kinetic energy $T/m = \frac{1}{2}|v|^2$ is conserved so the magnitude of the velocity (in 3d) $|v|$ is conserved. If we define an angle for the velocity vector α such that

$$r\dot{\theta} = |v| \sin\alpha$$

then conservation of $p_\theta = r|v|\sin\theta$ and conservation of $|v|$ implies that

$$r \sin\alpha = \text{constant}$$

This implies that the orbit can be reflected if it reaches a small enough radius that depends upon its initial energy. Orbits are meridians, closed orbits and dense rings covering a cylindrical region of the surface.

1.5 Maupertuis' principle

Hamilton's equation states that the actions $S = \int \mathcal{L} d\tau$ is minimized along a path. $q(t)$ is specified at the end points we must specify q_1, t_1 and q_2, t_2. Recall that

$$H(p, q) = p\dot{q} - \mathcal{L}$$

with $\dot{q}(p) = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. If energy is conserved then

$$\int p\dot{q} d\tau - \int \mathcal{L} d\tau = \text{constant}$$

Taking an extremum of the above we find that

$$\int p\dot{q} d\tau = \int p dq$$

is also an extremum. That $S = \int p dq$ is minimized when energy is conserved is called Maupertuis' principle. It is convenient that the integral can be specified with only the coordinates of the end points (and it is not necessary to specify the times).

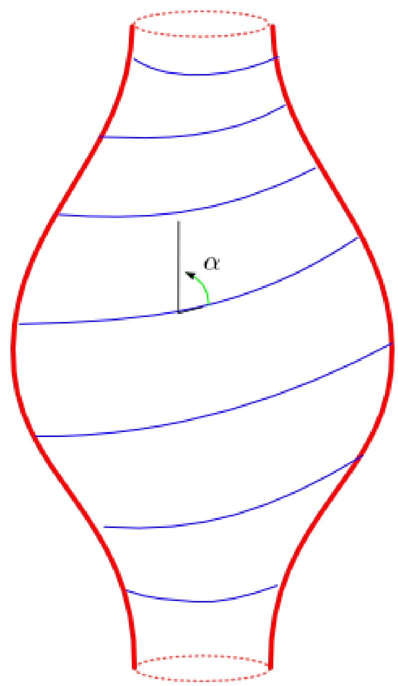

Figure 1: Orbits on a surface of revolution

1.6 Geodesics in Newtonian gravity

Using a kinetic energy in the form

$$T = \frac{1}{2} m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau}$$

and a Lagrangian

$$\mathcal{L} = T - V(q)$$

where V is a potential energy that includes two-body interactions. The matrix m_{ij} we would expect is diagonal in a Cartesian coordinate system where multiple bodies each have their own separate coordinates. The Hamiltonian

$$H = T + V$$

is conserved so

$$E - V = \frac{1}{2} m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau}$$

We can define a momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = m_{ij} \frac{dq^j}{d\tau}$$

so

$$p_i \dot{q}^i = m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} = 2(E - V(q))$$

With energy conserved and using Maupertuis' principle

$$S = \int p_i \dot{q}^i d\tau = \int m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau} d\tau = \int 2(E - V(q)) d\tau$$

is minimized along a trajectory with $q(\tau)$ that satisfies the equations of motion. It is convenient to split the term inside the integral so we can write it as

$$S = \int \sqrt{m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau}} \sqrt{2(E - V(q))} d\tau$$

This action looks like a path length

$$S = \int ds$$

with

$$\begin{aligned} ds &= \sqrt{2(E - V(q)) m_{ij} \frac{dq^i}{d\tau} \frac{dq^j}{d\tau}} d\tau = \sqrt{2(E - V(q)) m_{ij} dq^i dq^j} \\ &= \sqrt{g_{ij} dq^i dq^j} \end{aligned}$$

with a metric

$$g_{ij} = 2(E - V(q))m_{ij}$$

To rephrase, the equations of motion are consistent with minimizing the path length using a metric such that distance is measured with

$$ds^2 = 2(E - V(q))m_{ij}dq^i dq^j$$

As an example consider a Keplerian system

$$H(p_r, r, p_\theta, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \frac{k}{r}$$

and this gives a metric with

$$ds^2 = 2\left(E + \frac{k}{r}\right)(dr^2 + r^2 d\theta^2)$$

The equations of motion also minimize distances measured with the above metric.

Remark This should be equivalent to the non-relativistic limit of the Schwarzschild metric.

1.7 Hamiltonian formulation of geodesics

If we take

$$\mathcal{L} = \frac{1}{2} g_{ij} \frac{dq^i}{d\tau}\frac{dq^j}{d\tau}$$

then the momentum

$$p_i = \frac{\partial \mathcal{L}}{\dot{q}_i} = g_{ij}\frac{dq^j}{d\tau}$$

We can invert this with the metric

$$g^{ki}p_i = g^{ki}g_{ij}\frac{dq^j}{d\tau} = \delta^k_j \frac{dq^j}{d\tau} = \frac{dq^k}{d\tau}$$

the Hamiltonian is

$$H = p_i \frac{dq^i}{d\tau} - \mathcal{L} = \frac{1}{2}p_i\frac{dq^i}{d\tau} = \frac{1}{2}g^{ij}p_i p_j$$

Hamilton's equations give

$$\frac{dp_i}{d\tau} = \frac{1}{2}(\partial_i g^{ij})p_j p_k$$

$$p_i = g_{ij}\frac{dq^j}{d\tau}$$

If the metric is independent of a coordinate, its conjugate momentum is conserved. If the metric can be transformed into a coordinate system that is independent of certain coordinates then conserved quantities can be found. Another way to thing about this, is if the metric space can be described as having some symmetries (like a sphere), then it can be easier to find geodesics. This is similar to using the Hamilton-Jacobi equation to solve for conserved variables (or canonical momenta).

2 Equivalent actions

Question How many ways are there to create actions that are minimized on the same trajectories?

We will show that if two Lagrangians, $\mathcal{L}_1, \mathcal{L}_2$ give the same equations of motion then there is a function $\Phi(q,t)$ such that

$$\mathcal{L}_1 - \mathcal{L}_2 = \frac{d\Phi}{dt}$$

Consider two Lagrangians that give the same equations of motion.

$$\frac{d}{dt}\frac{\partial \mathcal{L}_i}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}_i}{\partial q_i} = \Lambda_i(q, \dot{q}, \ddot{q}, t)$$

for each Lagrangian $i = 1, 2$. If the equations of motion are the same then

$$\Lambda_1(q, \dot{q}, \ddot{q}, t) = \Lambda_2(q, \dot{q}, \ddot{q}, t)$$

$$\Lambda_1 - \Lambda_2 = 0$$

So we can define a function

$$\Psi(q, \dot{q}, t) = \mathcal{L}_1 - \mathcal{L}_2$$

$$\begin{aligned}
\Lambda_1 - \Lambda_2 &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}}(\mathcal{L}_1 - \mathcal{L}_2) - \frac{\partial}{\partial q}(\mathcal{L}_1 - \mathcal{L}_2) = 0 \\
&= \left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right)\Psi \\
&= \frac{\partial^2 \Psi}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 \Psi}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial^2 \Psi}{\partial \dot{q}\partial t} - \frac{\partial \Psi}{\partial \dot{q}} = 0
\end{aligned}$$

This must be true for all values of $\ddot{q}, \dot{q}, q, t$. But Ψ is only a function of $\dot{q}, q, t$. This implies that the coefficient with $\ddot{q}$ must be zero so $\frac{\partial^2 \Psi}{\partial \dot{q}^2} = 0$. Consequently Ψ is linear in $\dot{q}$.

$$\Psi = F(q,t)\dot{q} + G(q,t)$$

Inserting this back into the previous equation gives

$$\frac{\partial F}{\partial t} - \frac{\partial G}{\partial q} = 0 \tag{7}$$

This can be called an integrability condition as we can find a function $\Phi(q,t)$ that satisfies

$$F = \frac{\partial \Phi}{\partial q} \qquad G = \frac{\partial \Phi}{\partial t}$$

and equation 7 is automatically satisfied. It follows that

$$\Psi(\dot{q}, q, t) = \dot{q}\frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial t} = \frac{d\Phi}{dt} = \mathcal{L}_1 - \mathcal{L}_2$$

Question Is there a metric description for the KdV? There is a way to think about the operators within context of Lax operators as a zero curvature equation with connection for vector fibers over the space of x, t.

3 Some notes on Celestial mechanics

With a Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^2 + \frac{k}{|\mathbf{q}|}$$

transferring into cylindrical coordinates we obtain a Hamiltonian with $L = p_\theta$

$$H(p_r, L; r, \theta) = \frac{1}{2}p_r^2 + \frac{L^2}{2r^2} - \frac{k}{r}$$

with $k = GM$ and potential energy $V(r) = -k/r$, (and I have neglected motion out of the plane but we could include it by adding $p_z^2/2$). The radial degree of freedom gives an equation of motion

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{L^2}{r^3} + \frac{k}{r^2} = -\ddot{r}$$

Because the Hamiltonian is independent of θ, angular momentum (in the z direction), $L = p_\theta$ is conserved. Using Hamilton's equations we find that

$$L = r^2\dot{\theta}$$

It is convenient to use a variable $u = 1/r$ with

$$\dot{u} = -\frac{\dot{r}}{r^2}$$

$$\frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt}$$

using $L = r^2\dot{\theta}$
$$\frac{du}{dt} = \frac{du}{d\theta}\frac{L}{r^2}$$
Putting this together with our previous expression for $\dot{u}$
$$-\dot{r} = \frac{du}{d\theta}L$$
Taking the time derivative of this
$$\begin{aligned} -\ddot{r} &= \frac{d}{dt}\frac{du}{d\theta}L = \dot{\theta}\frac{d}{d\theta}\frac{du}{d\theta}L \\ &= \frac{d^2u}{d\theta^2}L^2u^2 \end{aligned}$$
The equations of motion are
$$\ddot{r} = \frac{L^2}{r^3} - \frac{k}{r^2} = L^2u^3 - ku^2$$
Putting these together
$$\begin{aligned} \frac{d^2u}{d\theta^2}L^2u^2 &= -L^2u^3 + ku^2 \\ u'' + u &= kL^{-2} \end{aligned}$$
This has solution
$$u = (1 + e\cos\theta)p^{-1}$$
with $p = L^2/k$ and free parameter e known as the eccentricity. Inverting this for radius
$$r = \frac{p}{1 + e\cos f}$$
and we have replaced θ with angle f called the true anomaly. For $f = 0$ the orbit is a pericenter. The minimum and maximum radius are $r_{min} = p/(1+e)$ and $r_{max} = p/(1-e)$ giving a semi-major axis a
$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2}$$
so that
$$p = a(1 - e^2)$$

With some manipulation it is possible to show that
$$\begin{aligned} E &= -\frac{k}{2a} \\ L &= \sqrt{ka(1-e^2)} \end{aligned}$$
These are appropriate for elliptical orbits. With some generalization a similar description covers parabolic and hyperbolic orbits (e.g., hyperbolic orbits have e greater than 1, $a < 0$).

3.1 Eccentric anomaly

In a coordinate system defined from the ellipse focal point, a point on the orbit

$$\begin{aligned} x &= r\cos f \\ y &= r\sin f \end{aligned}$$

Here f is the true anomaly and r the radius. This coordinate system uses as origin an ellipse focal point which is also the location of the Sun for the orbit of a planet in motion around the Sun.

In a coordinate system with origin at the center of the ellipse, the orbit clearly defines an ellipse obeying

$$\left(\frac{\bar{x}}{a}\right)^2 + \left(\frac{\bar{y}}{b}\right)^2 = 1$$

with semi-major axis a and semi-minor axis $b = a\sqrt{1-e^2}$. The coordinates for a point on the orbit can be written in terms of an angle called the eccentric anomaly E

$$\begin{aligned} \bar{x} &= a\cos E \\ \bar{y} &= b\sin E = a\sqrt{1-e^2}\sin E = y \end{aligned}$$

And

$$x = a(\cos E - e)$$

These relations can be read off Figure 2 showing the orbit.

Also useful are relations

$$r = a(1 - e\cos E)$$

a relation between true and eccentric anomaly

$$\tan(f/2) = \tan(E/2)\sqrt{\frac{1+e}{1-e}}$$

$$\dot{r} = ae\sin E\dot{E}$$

$$(r\dot{f})^2 = \frac{L^2}{r^2}$$

$$v^2 = (r\dot{f})^2 + (\dot{r})^2$$

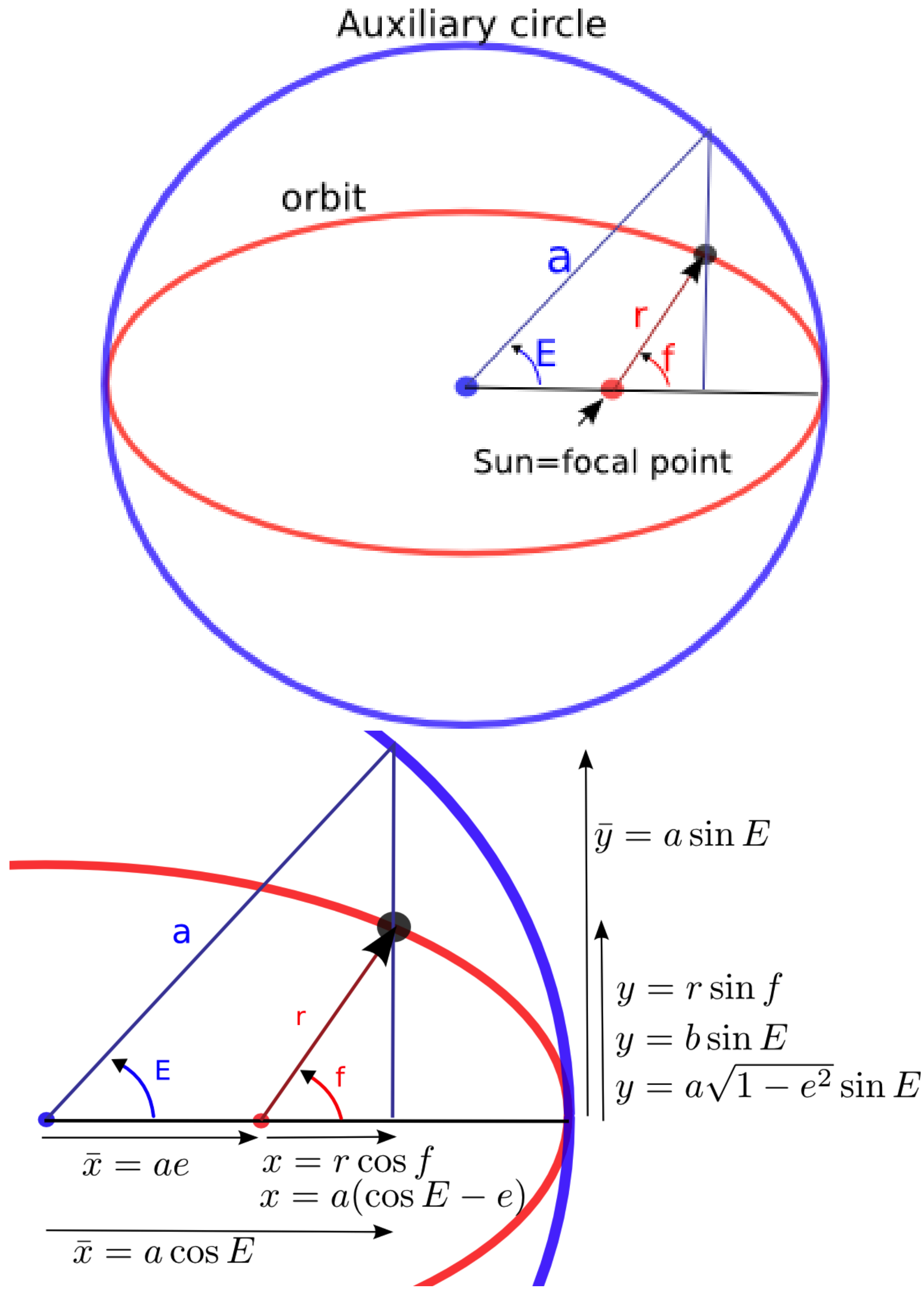

Figure 2: a) One focal point of an elliptic orbit is the location of the Sun. Also drawn is the auxiliary circle with radius a, the angle known as the true anomaly f, and the eccentric anomaly E. The true anomaly f is defined with respect to origin at the Sun and the focal point of the ellipse. The Eccentric anomaly is defined with origin at the center of the ellipse and the auxiliary circle. b) Coordinate relations. x, y are positions with origin at the Sun. $\bar{x}, \bar{y}$ are coordinates with origin at the center of the ellipse.

3.2 The mean anomaly and Kepler's equation

Using above relations for $\dot{r}$ in terms of the mean anomaly it can be shown than

$$\dot{E} = \frac{n}{1 - e\cos E}$$

where the frequency n is called the mean motion

$$n = \sqrt{\frac{k}{a^3}}$$

. We assume that there is an angle M, known as the mean anomaly, that advances with angular rotation rate given by n.

$$M = M_0 + nt$$

so that $\dot{M} = n$. The mean anomaly is not a physical angle on the sky. Now insert this into our equation for $\dot{E}$

$$\dot{E} = \frac{\dot{M}}{1 - e\cos E}$$

we can integrate this equation finding

$$M = E - e\sin E$$

This is known as Kepler's equation.

Kepler's equation cannot be solved analytically. In other words, given M, it is not possible to solve analytically for E. However extremely rapid numerical techniques that converge to third order are known (Laguerre's method). To converge to a solution to a precision of order 10^{-16} (double precision floating point) it takes less than 6 or 7 iterations of the method.

To advance an orbit it time, M is advanced, then E computed. From E, the position in the orbit can be computed. Then the orbit is rotated according to its longitude of perihelion ω, inclination, i, and longitude of the ascending node Ω. The reverse procedure is done to convert a cartesian positions and velocities to orbital elements. The orbital elements are a, e, i and angles M, ω, Ω. The ordering is important as canonical momenta depending primarily on a, e, i are conjugate to canonical angles either equal to or related to the angles M, ω, Ω, respectively.

3.3 Geometric view in velocity space

The equations of motion give

$$\frac{d\mathbf{v}}{dt} = -\frac{k\mathbf{r}}{r^3}$$

and this implies that

$$\frac{|d\mathbf{v}|}{dt} = \frac{k}{r^2}$$

Let us define a regularized distance

$$ds = \frac{dt}{r}$$

Multiplying this on with the relation for $d|v|$

$$\frac{d|v|}{ds} = \frac{k}{r}$$

Using $E = \frac{v^2}{2} - \frac{k}{r}$

$$\frac{d|v|}{ds} = \frac{v^2}{2} - E$$

$$ds = \frac{2dv}{v^2 - 2E}$$

$$ds^2 = (\mathbf{v}^2 - 2E)^{-2} 4\mathbf{dv}^2$$

In the space of velocity trajectories the path distance in ds is minimized. Also we can think of the above relation for ds^2 as giving a metric. In cartesian coordinates (for velocity)

$$g_{uv} = 4(\mathbf{v}^2 - 2E)^{-2} \delta_{ij}$$

Using this metric, trajectories in velocity space are minimized. It is nice to write this in terms of $\mathbf{w} = \mathbf{v}/v^2$ with $|d\mathbf{w}|^2 = |dv|^2 v^{-2}$. With this variable change

$$ds^2 = 4|dw|^2(1 - 2Ew^2)^{-2} = 4|dw|^2(1 + Kw^2)^{-2}$$

In the positive curvature case we can compare this to the standard metric on a 3-sphere with radius $1/\sqrt{K}$. The coefficient $K = -2E$ can be considered the curvature.

xxxx show how this is like curvature.

Remark The integral $\int ds = \int dt/r$ is called the Levi-Civita integral. The geometric interpretation is due to Osipov and Belbruno, but see a nice paper by Milnor on the geometry of Keplerian orbits. In this paper there is also a nice description of how trajectories in velocity space are circles. John Milnor, The geometry of the Kepler problem, AMS Notices 90 (June-July 1983), 353-365.

Also take a look at http://math.ucr.edu/home/baez/gravitational.html

3.4 Notes

Following Rajeev's lecture notes on geometry, and Ashok Das's book on integrable models. An example of a holonomic constraint taken from Arnold's book on Math methods of Classical mechanics. Also see John Milnor, The geometry of the Kepler problem, AMS Notices 90 (June-July 1983), 353-365.

Also see notes by John Baez `http://math.ucr.edu/home/baez/gravitational.html`