The hierarchical restricted three-body problem
Lidov-Kozai mechanism and evection resonance

Taught at University of Rochester on November 27th, 2023

Jérémy COUTURIER

Contents

Abstract

The three-body problem is a configuration where the trajectories of three celestial objects moving under their mutual gravity are to be determined. The hierarchical restricted three-body problem is a simplification of the more general three-body problem. The term *restricted* means that one of the three bodies is a massless particle, whereas *hierarchical* refers to the fact that the particle is much closer to one body than the other. The latter body then acts as a perturbation to the motion of the particle around the former. Since the massive bodies are not affected by the particle, their motion is given by the well-known two-body problem. The aim of the hierarchical restricted three-body problem is therefore to constrain the trajectory of the massless particle. Provided additional assumptions, this problem is integrable and has an analytical solution.

1 Framework

Let m_0 be the mass of the perturbing body and m_1 the mass of the body being orbited by the massless particle. The setting of the hierarchical restricted three-body problem is adapted, for example, to these cases

- The Moon orbiting the Earth and perturbed but the Sun.
- A spacecraft orbiting a satellite and perturbed by the planet.
- Mercury orbiting the Sun and perturbed by Jupiter.
- A planet orbiting a star and perturbed by a binary companion star.

We make no assumption on m_0 and m_1. In cases 1 and 2, m_0 is much larger than m_1. In case 3, m_1 is much larger than m_0. In case 4, they could be of similar values. I denote vectors with a bold font and their norms with the same unbolded letter. The derivation with respect to time is denoted by an upper dot. The gravitational constant is written $\mathcal{G}$.

1.1 Hamiltonian of the hierarchical restricted 3-body problem

I start by obtaining the Hamiltonian of the hierarchical restricted 3-body problem. Let $\boldsymbol{u}$, $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ be the positions of the particle, the perturbing body and the orbited body in an inertial reference frame, respectively. The equations of motion are given by

$$\begin{aligned}
\ddot{\boldsymbol{u}} &= -\frac{\mathcal{G}m_0}{|\boldsymbol{u}-\boldsymbol{u}_0|^3}\left(\boldsymbol{u}-\boldsymbol{u}_0\right) - \frac{\mathcal{G}m_1}{|\boldsymbol{u}-\boldsymbol{u}_1|^3}\left(\boldsymbol{u}-\boldsymbol{u}_1\right),\\
\ddot{\boldsymbol{u}}_0 &= -\frac{\mathcal{G}m_1}{|\boldsymbol{u}_0-\boldsymbol{u}_1|^3}\left(\boldsymbol{u}_0-\boldsymbol{u}_1\right),\\
\ddot{\boldsymbol{u}}_1 &= -\frac{\mathcal{G}m_0}{|\boldsymbol{u}_1-\boldsymbol{u}_0|^3}\left(\boldsymbol{u}_1-\boldsymbol{u}_0\right).
\end{aligned} \tag{1}$$

We are interested in knowing the position of the particle and of the perturbing body with respect to the orbited body. Hence I define $\boldsymbol{r}=\boldsymbol{u}-\boldsymbol{u}_1$, $\boldsymbol{r}_0=\boldsymbol{u}_0-\boldsymbol{u}_1$ and $\boldsymbol{r}_1=\boldsymbol{u}_1$. The equation of the motion of the particle in these new variables reads

$$\ddot{\boldsymbol{r}} = -\frac{\mathcal{G}m_1}{r^3}\boldsymbol{r} - \mathcal{G}m_0\left(\frac{\boldsymbol{r}_0}{r_0^3}+\frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}\right). \tag{2}$$

Let $\tilde{\boldsymbol{r}}=\dot{\boldsymbol{r}}$. The Hamiltonian $\mathcal{H}$ of the hierarchical restricted 3-body problem, where $(\tilde{\boldsymbol{r}},\boldsymbol{r})$ is a pair of conjugated variables, must verify the Hamilton equations

$$\dot{\tilde{\boldsymbol{r}}} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{r}} = \ddot{\boldsymbol{r}} \quad \text{and} \quad \dot{\boldsymbol{r}} = \frac{\partial\mathcal{H}}{\partial\tilde{\boldsymbol{r}}} = \tilde{\boldsymbol{r}}. \tag{3}$$

This yields

$$\mathcal{H} = \mathcal{H}_K + \mathcal{H}_P, \tag{4}$$

where the Keplerian part $\mathcal{H}_K$ governs the orbit of the particle around m_1 and the perturbative part $\mathcal{H}_P \ll \mathcal{H}_K$ is due to m_0. They read

$$\mathcal{H}_K = \frac{1}{2}\tilde{r}^2 - \frac{\mathcal{G}m_1}{r}, \tag{5}$$

and

$$\mathcal{H}_P = -\mathcal{G}m_0 \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}_0}{r_0^3} \right), \tag{6}$$

where $\boldsymbol{r}_0(t)$ is a known function of time, given by the two-body problem.

1.2 Quadrupolar expansion in the Legendre polynomials

The hardest term to deal with in the perturbative part $\mathcal{H}_P$ is clearly the inverse distance between the particle and the perturbing body. We can greatly simplify its expression by using the assumption of hierarchy. In other words, I expand to second order the Hamiltonian of the hierarchical restricted 3-body problem in power series of the small quantity r/r_0. The expansion is given by

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} = \frac{1}{r_0} \sum_{n=0}^{+\infty} \left(\frac{r}{r_0} \right)^n P_n(\cos\theta), \tag{7}$$

where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_0$ and P_n is the n^{th} Legendre polynomial. Even if the Legendre polynomials are not known, the expansion is straightforward to compute. I simply write

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} = [(\boldsymbol{r} - \boldsymbol{r}_0) \cdot (\boldsymbol{r} - \boldsymbol{r}_0)]^{-1/2} = \frac{1}{r_0} \left[1 + \frac{r^2}{r_0^2} - 2\frac{\boldsymbol{r} \cdot \boldsymbol{r}_0}{r_0^2} \right]^{-1/2}, \tag{8}$$

and I use $(1+x)^{-1/2} = 1 - x/2 + 3x^2/8 + \mathcal{O}(x^3)$ to end up with

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} = \frac{1}{r_0} \left[1 + \frac{\boldsymbol{r} \cdot \boldsymbol{r}_0}{r_0^2} - \frac{1}{2}\frac{r^2}{r_0^2} + \frac{3}{2}\frac{(\boldsymbol{r} \cdot \boldsymbol{r}_0)^2}{r_0^4} \right] + \mathcal{O}\left(\frac{r^3}{r_0^4} \right). \tag{9}$$

The term of order 1 cancels itself[1] with the second term of $\mathcal{H}_P$ (Eq. (6)). As for the term of order 0, it only contributes to add to the Hamiltonian a quantity independent on both $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$. Therefore, it can be removed from the expansion without modifying the equations of motion. Only terms of second order remain and the quadrupolar expansion of the Hamiltonian of the hierarchical restricted 3-body problem reads

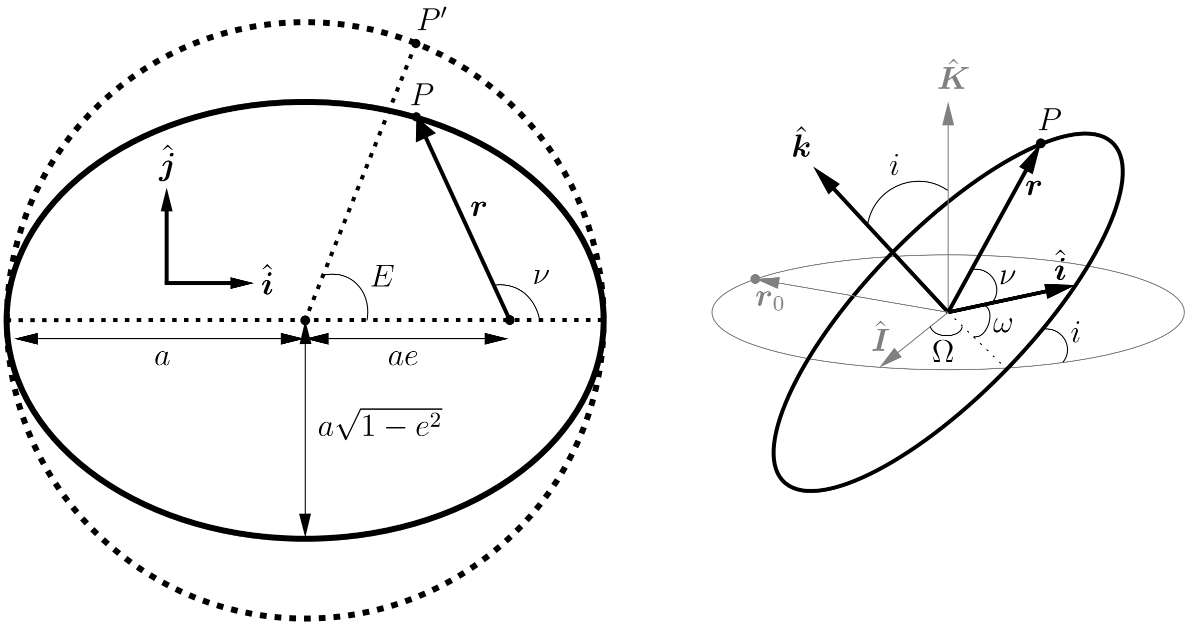
$$\mathcal{H} = \mathcal{H}_K + \frac{\mathcal{G}m_0}{2r_0^3} \left[r^2 - 3\frac{(\boldsymbol{r} \cdot \boldsymbol{r}_0)^2}{r_0^2} \right]. \tag{10}$$

[1] Which is why a second order expansion is needed

1.3 Reminders about the two-body problem

If the particle was affected only by m_1, its trajectory would be planar and along an ellipse of semi-major axis a. I present here the main results of the two-body problem, that will be useful for the rest of this course. The reader interested in a complete overview can read Sect. 2.1 of Antoine Petit's PhD manuscript.

Let a be the semi-major axis of the particle's orbit, e its eccentricity and i its inclination with respect to the orbital plane of the perturbing body (taken as reference). Let ν, E and M be the true anomaly, the eccentric anomaly and the mean anomaly of the particle, respectively. These quantities are shown in the schema, where P is the particle and P' is the vertical projection of P on the dashed circle.



The longitude of the ascending node and the argument of the periapsis of the particle's orbit are respectively denoted Ω and ω, and are also shown on the schema. Let $\left(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\right)$ be an orthonormal reference frame such that the particle's orbits lies in the plane $\left(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}\right)$, with $\hat{\boldsymbol{i}}$ towards the periapsis of the orbit. I define $\left(\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}, \hat{\boldsymbol{K}}\right)$ similarly for the perturbing body and I write $\boldsymbol{r} = \mathcal{X}\hat{\boldsymbol{i}} + \mathcal{Y}\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_0 = \mathcal{X}_0\hat{\boldsymbol{I}} + \mathcal{Y}_0\hat{\boldsymbol{J}}$. The mean anomaly is not shown in the schema. It is defined as

$$M = \frac{2\pi}{T}t := nt, \tag{11}$$

where T is the particle's orbital period, the time t is 0 when the particle passes at periapsis and n is the particle's mean motion. Since the area swept by $\boldsymbol{r}$ is proportional to time, M is proportional both to time and to the area swept by $\boldsymbol{r}$ since the particle last reached its periapsis.

The vectors $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are uniquely determined by the six variables $(a, e, i, M, \omega, \Omega)$, called the elliptic elements of the orbit. In the two-body problem, all the elliptic elements are constant except M, which evolves linearly with time. In the hierarchical restricted 3-body problem, the variables $(a, e, i, \omega, \Omega)$ are a priori not constant, but they evolve secularly[2] with time. Looking at the schema, it is easy to verify that we have

$$\begin{aligned}\mathcal{X} &= r\cos\nu = a\left(\cos E - e\right),\\ \mathcal{Y} &= r\sin\nu = a\sqrt{1-e^2}\sin E,\\ r &= \frac{a\left(1-e^2\right)}{1+e\cos\nu} = a\left(1-e\cos E\right).\end{aligned} \tag{12}$$

These relations allow the Keplerian part of the Hamiltonian to be rewritten

$$\mathcal{H}_K = -\frac{\mathcal{G}m_1}{2a} := -\frac{\mu}{2a}. \tag{13}$$

Finally, the solution of the two-body problem shows that the true, eccentric and mean anomalies are related by

$$dM = \frac{r}{a}dE = \frac{r^2}{a^2\sqrt{1-e^2}}d\nu, \tag{14}$$

a relation that will prove very useful in Sect. 1.4

1.4 Average over the mean anomaly of the particle

The quadrupolar expansion, although it simplified the Hamiltonian, did not remove any degree of freedom. The Hamiltonian (10) still has three degrees of freedom and is not adapted to an analytical work. For the problem to be integrable, I need to end up with only one degree of freedom. Using the conservation of the particle's angular momentum, one of them can be lost.

The Hamiltonian contains short-term variations due to the small orbital period of the particle around m_1. These short-term variations are not relevant since I am only interested in the secular[2] evolution of the particle's orbit. Therefore, the idea is to smooth the trajectory by removing the short-term variations. More precisely, one additional degree of freedom can be lost by averaging the Hamiltonian along the orbit of the particle.

I define the new averaged Hamiltonian as

$$\bar{\mathcal{H}} = \frac{1}{T}\int_0^T \mathcal{H}\,dt. \tag{15}$$

This operation on the Hamiltonian is mathematically justified in the sense that it is just a first-order Lie serie expansion. The reader interested in perturbation theory and Lie series expansion can find an overview in Sect. 2.2.2 of my PhD manuscript[3]. What is interesting is that the results of the two-body problem allow for $\bar{\mathcal{H}}$ to be computed analytically using Eqs. (12) and (14). Since the mean longitude M is proportional to the time, averaging with respect to the time is equivalent to averaging with respect to M. The computation of $\bar{\mathcal{H}}$ requires to compute the average of r^2 and $(\boldsymbol{r}\cdot\boldsymbol{r}_0)^2$ with respect to M. We have

$$\overline{r^2} = \frac{1}{T}\int_0^T r^2\,dt = \frac{1}{2\pi}\int_0^{2\pi} r^2\,dM = \frac{1}{2\pi}\int_0^{2\pi} a^2\left(1-e\cos E\right)^3\,dE = a^2\left(1+\frac{3}{2}e^2\right). \tag{16}$$

[2]Secular comes from the French *siècle*, meaning century. It means much slower than the orbital period.
[3]https://jeremycouturier.com/img/PhD_manuscript.pdf

The case of $(\boldsymbol{r}\cdot\boldsymbol{r}_0)^2$ is trickier. The idea is to write[4] $(\boldsymbol{r}\cdot\boldsymbol{r}_0)^2 = {}^t\boldsymbol{r}_0\,(\boldsymbol{r}\,{}^t\boldsymbol{r})\,\boldsymbol{r}_0$ and to compute the average of the matrix $\boldsymbol{r}\,{}^t\boldsymbol{r}$ with respect to M. We have

$$\overline{\boldsymbol{r}\,{}^t\boldsymbol{r}} = \frac{1}{2}a^2\left[\left(1-e^2\right)\left(\mathbb{I}-\hat{\boldsymbol{k}}\,{}^t\hat{\boldsymbol{k}}\right)+5e^2\hat{\boldsymbol{\imath}}\,{}^t\hat{\boldsymbol{\imath}}\right], \tag{17}$$

and therefore

$$\overline{(\boldsymbol{r}\cdot\boldsymbol{r}_0)^2} = {}^t\boldsymbol{r}_0\,\overline{\boldsymbol{r}\,{}^t\boldsymbol{r}}\,\boldsymbol{r}_0 = \frac{1}{2}a^2\left[\left(1-e^2\right)\left(r_0^2-\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}_0\right)^2\right)+5e^2\left(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{r}_0\right)^2\right]. \tag{18}$$

From now on, I drop the upper bar from the Hamiltonian. Finally, the common Hamiltonian to study both the Lidov-Kozai mechanism and the evection resonance is

$$\mathcal{H} = -\frac{\mu}{2a}-\frac{\mathcal{G}m_0a^2}{4r_0^3}\left[1-6e^2-3\left(1-e^2\right)\frac{\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}_0\right)^2}{r_0^2}+15e^2\frac{\left(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{r}_0\right)^2}{r_0^2}\right]. \tag{19}$$

2 Lidov-Kozai mechanism

This mechanism triggers when the orbital plane of the particle if very inclined with respect to the orbital plane of the perturbing body. It leads to very large eccentricities for the particle's orbit.

2.1 Average over the mean anomaly of the perturbing body

In order to simplify even further the Hamiltonian of the problem, I average it over the mean motion M_0 of the perturbing body. This is done in the same manner as the average over M. Using

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r_0^3}dM_0 &= \frac{1}{a_0^3\left(1-e_0^2\right)^{3/2}}, \quad \text{and}\\ \frac{1}{2\pi}\int_0^{2\pi}\frac{\boldsymbol{r}_0\,{}^t\boldsymbol{r}_0}{r_0^5}dM_0 &= \frac{1}{2a_0^3\left(1-e_0^2\right)^{3/2}}\left(\mathbb{I}-\hat{\boldsymbol{K}}\,{}^t\hat{\boldsymbol{K}}\right),\end{aligned} \tag{20}$$

I find

$$\mathcal{H} = -\frac{\mu}{2a}+\frac{\mathcal{G}m_0a^2}{8a_0^3\left(1-e_0^2\right)^{3/2}}\left[1-6e^2-3\left(1-e^2\right)\left(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{K}}\right)^2+15e^2\left(\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{K}}\right)^2\right], \tag{21}$$

where a_0 and e_0 are the semi-major axis and eccentricity of the perturbing body's orbit. The right panel of the schema in Sect. 1.3 gives $\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{K}}=\cos i$ and $\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{K}}=\sin i\sin\omega$. Therefore, I write the Hamiltonian of the Lidov-Kozai mechanism as

$$\mathcal{H} = -\frac{\mu}{2a}+\frac{\mathcal{G}m_0a^2}{8a_0^3\left(1-e_0^2\right)^{3/2}}\left[1-6e^2-3\left(1-e^2\right)\cos^2 i+\frac{15}{2}e^2\sin^2 i\left(1-\cos 2\omega\right)\right]. \tag{22}$$

Note that averaging over M_0 did not remove any degree of freedom, it merely simplified the Hamiltonian.

[4]where ${}^t\cdot$ denotes the transpose operator.

2.2 Transformation to adapted canonical variables

In the Hamiltonian (22), the canonical variables $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ do not appear explicitly. Instead, this Hamiltonian is written in term of the elliptic elements $(a, e, i, M, \omega, \Omega)$. The problem is that the transformation from $(\tilde{\boldsymbol{r}}; \boldsymbol{r})$ to the elliptic elements is not canonical. Therefore. the Hamiltonian (22) is currently not written in a set of canonical variables. To get around this issue, I use the variables of Delaunay, defined as

$$\begin{array}{l|l} \Lambda = \sqrt{\mu a} & M, \\ G = \Lambda\sqrt{(1-e^2)} & \omega, \\ H = G\cos i & \Omega. \end{array} \tag{23}$$

The transformation $(\tilde{\boldsymbol{r}}; \boldsymbol{r}) \to (\Lambda, G, H; M, \omega, \Omega)$, where the semicolon separates the actions from their respective conjugated variables, is canonical (*e.g.* Laskar, 2017).

Due to the averaging process, the Hamiltonian no longer depends on M. Therefore, its conjugated action Λ, and hence a, is constant and the degree of freedom $(\Lambda; M)$ has been lost by the average. The semi-major axis a is a first-integral of the averaged hierarchical restricted 3-body problem. Terms of the Hamiltonian (22) depending only on a can be removed without changing the equations of motions.

Similarly, the Hamiltonian does not depend on the longitude of the ascending node Ω. Its conjugated action $H = G\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{K}}$, which is the projection on $\hat{\boldsymbol{K}}$ of the particle's angular momentum, is conserved. The degree of freedom $(H; \Omega)$ does not exist and the Hamiltonian is reduced to the unique degree of freedom $(G; \omega)$. I obtain the integrable Hamiltonian

$$\mathcal{H} = \frac{3\mathcal{G}m_0 a^2}{16 a_0^3 \left(1-e_0^2\right)^{3/2}} \left[e^2 - 2\cos^2 i \left(1 + \frac{3}{2}e^2\right) - 5e^2 \left(1 - \cos^2 i\right)\cos 2\omega\right], \tag{24}$$

where $e^2 = 1 - G^2/\Lambda^2$ and $\cos^2 i = H^2/G^2$. The problem with Delaunay's coordinates is that they are singular at small eccentricity and not dimensionless. I will rather use the rectangular canonical coordinates

$$x = \sqrt{2\left(\Lambda - G\right)}\cos\omega \quad \text{and} \quad y = \sqrt{2\left(\Lambda - G\right)}\sin\omega. \tag{25}$$

While the transformation $(G; \omega) \to (x; y)$ is canonical, the variables $(x; y)$ are still not dimensionless. I finally perform the transformation

$$X = \frac{x}{\sqrt{\Lambda}} = \sqrt{\frac{2\left(\Lambda - G\right)}{\Lambda}}\cos\omega \quad \text{and} \quad Y = \frac{y}{\sqrt{\Lambda}} = \sqrt{\frac{2\left(\Lambda - G\right)}{\Lambda}}\sin\omega. \tag{26}$$

The transformation $(x; y) \to (X; Y)$ is not canonical. However, the equations of motion stay in Hamiltonian form as long as the Hamiltonian is divided by Λ. The dimensionless variables $(X; Y)$ can thus be considered canonical.

2.3 Consequences of the Lidov-Kozai mechanism

The expression of the Hamiltonian (24) in the coordinates $(X; Y)$ is quite gruesome and does not allow for much analytical insight on the dynamics of the Lidov-Kozai mechanism. Instead, I expand it to second order in $(X; Y)$. Since

$$X = e\cos\omega + \mathcal{O}(e^3) \quad \text{and} \quad Y = e\sin\omega + \mathcal{O}(e^3), \tag{27}$$

the variables X and Y carry a lot of physical meaning at small eccentricity. After removing constant terms and dividing by Λ, the Hamiltonian of the Lidov-Kozai mechanism at second order in eccentricity takes the form

$$\mathcal{H} = -\frac{3\sigma}{8\left(1-e_0^2\right)^{3/2}}\left[2X^2 - \left(3-5c^2\right)Y^2\right], \tag{28}$$

where $\sigma = \eta^2/n$ is homogeneous to a frequency, $n = \sqrt{\mathcal{G}m_1/a^3} = \dot{M}$ is the mean motion of the particle and $\eta = \sqrt{\mathcal{G}m_0/a_0^3}$. I also defined the quantity $c = H/\Lambda$. Since both H and Λ are constant, c is just a parameter of the Hamiltonian. Its value is

$$c = \sqrt{1-e_{\text{in}}^2}\cos i_{\text{in}}, \tag{29}$$

where e_{in} and i_{in} are the initial eccentricity and inclination of the particle, respectively. The Hamiltonian (28) defines a conic and has an equilibrium at zero eccentricity ($X = Y = 0$).

- If the equilibrium is elliptic (or stable), that is, if the conic is an ellipse, then a quasi-circular orbit for the particle is stable.
- On the contrary, if the equilibrium is hyperbolic (or unstable), that is, if the conic is a hyperbola, then a quasi-circular orbit for the particle is unstable.

The conic is a hyperbola if, and only if, the terms in front of X^2 and Y^2 have different signs. Therefore, quasi-circular orbits for the particle are unstable if $c < \sqrt{3/5}$. With $e_{\text{in}} = 0$, the condition for instability is

$$i_{\text{in}} > i_{\text{crit}} := \arccos\sqrt{\frac{3}{5}} \approx 39.23^\circ, \tag{30}$$

where i_{crit} is sometimes called Kozai inclination. The untruncated Hamiltonian (24) has more that one equilibrium, but the expansion to second-order removed some equilibria. In order to better understand the dynamics of the Lidov-Kozai mechanism, I expand the Hamiltonian to fourth order in eccentricity instead. I obtain

$$\mathcal{H} = \frac{-3\sigma}{8\left(1-e_0^2\right)^{3/2}}\left[2X^2 - \left(3-5c^2\right)Y^2 - \frac{1}{2}X^4 + \frac{1}{4}\left(3+15c^2\right)Y^4 + \frac{1}{4}\left(1+15c^2\right)X^2Y^2\right]. \tag{31}$$

When $c \leq \sqrt{3/5}$, this Hamiltonian has two more equilibria besides the equilibrium at zero eccentricity, located at $X = 0$ and

$$Y = \pm\frac{\sqrt{6\left(3-5c^2\right)\left(1+5c^2\right)}}{3+15c^2}. \tag{32}$$

I plot the phase space of the Hamiltonian (31) in Fig. 1. As long as $c \geq \sqrt{3/5} \approx 0.7746$, all trajectories evolve around $(X, Y) = 0$ and quasi-circular orbits for the particle are possible. However, when $c < \sqrt{3/5}$, the equilibrium $(X, Y) = 0$ becomes hyperbolic and two equilibria at higher eccentricity appear. When c is sufficiently small, or equivalently, when i_{in} is sufficiently large, all trajectories have a large eccentricity.

As a consequence, putting a spacecraft on a polar circular low-Moon-orbit orbit would very likely result in the spacecraft crashing on the Moon due to perturbations from the Earth. Indeed, the growing eccentricity of the orbit, coupled with a constant semi-major axis, would lead to the altitude $a\left(1-e\right)$ of the periapsis to become smaller than the Moon's radius.

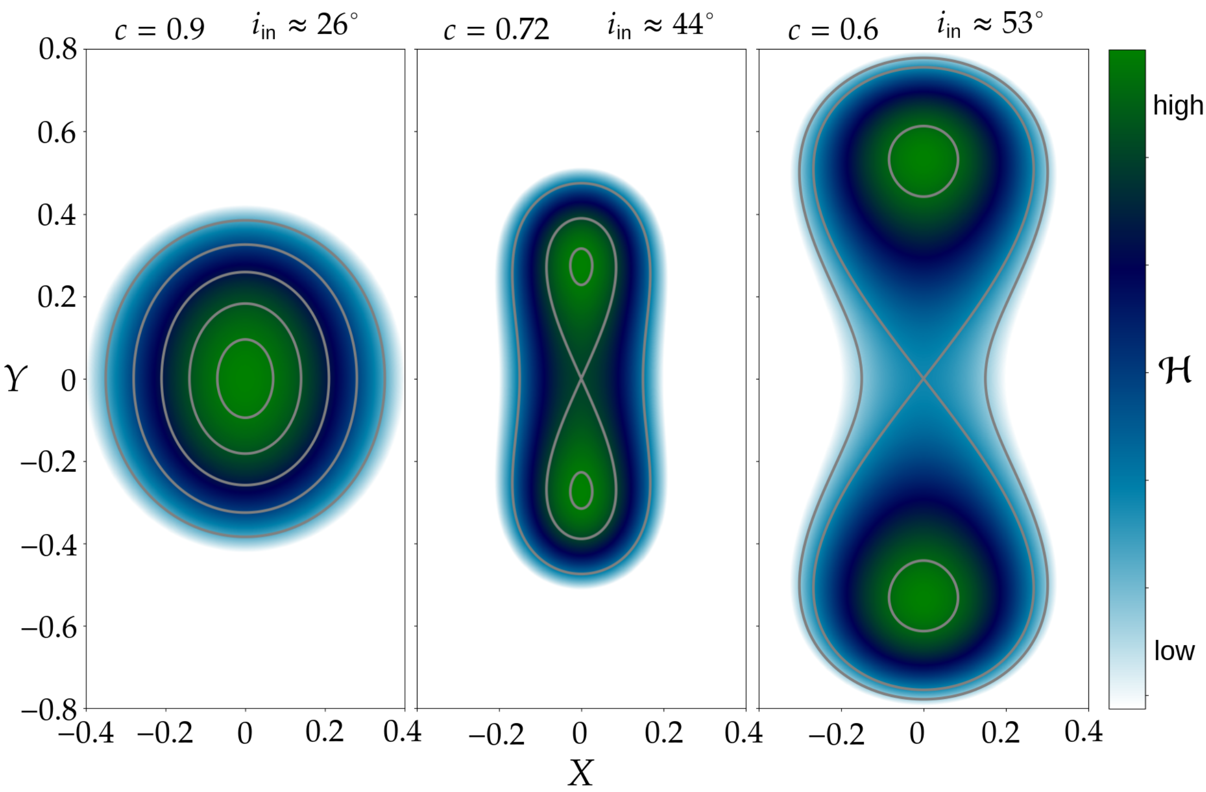


Fig. 1 — Phase space of the Hamiltonian (31) for different values of the parameter c. The corresponding value of $i_{\rm in}$ is given assuming $e_{\rm in} = 0$.

3 Evection resonance

The evection resonance corresponds to a commensurability between the precession frequency of the particle's periapsis and the mean motion of the perturbing body. In other words, it occurs when the ellipse of the particle's orbit precesses roughly at the same rate as m_0 and m_1 orbit each other. Like the Lidov-Kozai mechanism, it can lead to large eccentricities for the particle's orbit.

3.1 Hamiltonian of the evection resonance

Of course, averaging the Hamiltonian over the mean motion M_0 of the perturbing body is not possible, as this would remove dependency on an important variable of the evection resonance. Instead, I start from the Hamiltonian of the hierarchical restricted 3-body problem averaged over M, given by Eq. (19). Since a is constant in the averaged problem (Sect. 2.2), I remove from the Hamiltonian terms depending only on a, as this leaves the equations of motions unaffected. The Hamiltonian thus reads

$$\mathcal{H} = \frac{3\mathcal{G}m_0 a^2}{2r_0^3}\left[e^2 + \frac{1}{2}\left(1-e^2\right)\frac{\left(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}_0\right)^2}{r_0^2} - \frac{5}{2}e^2\frac{\left(\hat{\boldsymbol{\imath}}\cdot\boldsymbol{r}_0\right)^2}{r_0^2}\right]. \tag{33}$$

Since I cannot average over M_0, the Hamiltonian is still quite difficult to deal with. I simplify it by making the assumption that m_0 and m_1 orbit each other on a circular trajectory, and we have $r_0 = a_0$. Unlike the Lidov-Kozai mechanism, the evection resonance exists even when the particle and the perturbing body are coplanar. Therefore, I simplify even further by assuming coplanarity, or equivalently, that $\hat{\boldsymbol{k}} = \hat{\boldsymbol{K}}$ (schema in Sect. 1.3).

The coplanarity induces $i = 0$, and Delaunay's variables, introduced in Sect. 2.2, are singular in that case. I use instead the canonical[5] Poincaré's variables defined as

$$\begin{array}{l|l} \Lambda = \sqrt{\mu a} = \sqrt{\mathcal{G} m_1 a} & \lambda = M + \varpi, \\ D = \Lambda - G = \Lambda\left(1 - \sqrt{1-e^2}\right) & -\varpi = -\omega - \Omega, \\ \mho = G - H = G\left(1 - \cos i\right) & -\Omega. \end{array} \tag{34}$$

where λ is the mean longitude, ϖ is the longitude of the periapsis and D is called the angular momentum deficit. Using the coplanar and circular hypothesis, the vector $\boldsymbol{r}_0$ takes a simple expression

$$\boldsymbol{r}_0 = a_0\left(\cos\lambda_0 \hat{\boldsymbol{I}} + \sin\lambda_0 \hat{\boldsymbol{J}}\right). \tag{35}$$

According to the two-body problem, the mean longitude of the perturbing body around m_1 reads $\lambda_0(t) = n_0 t$ and is proportional to the time, where the mean motion of the perturbing body is $n_0 = \sqrt{\mathcal{G}\left(m_0 + m_1\right)/a_0^3}$. Both the argument of the periapsis ω and the longitude of the ascending node Ω evolve secularly[2] due to the perturbing body. Therefore, the particle's ellipse precesses at a rate $\dot{\varpi} = \dot{\omega} + \dot{\Omega}$. The commensurability that leads to the evection resonance can formally be written

$$\dot{\varpi} \sim n_0. \tag{36}$$

Due to this precession, the frame $\left(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}\right)$ of the particle's orbit slowly rotates at frequency $\dot{\varpi}$, whereas the frame $\left(\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}}\right)$ of the perturbing body's orbit does not. Both frames are thus related by

$$\begin{pmatrix} \hat{\boldsymbol{I}} \\ \hat{\boldsymbol{J}} \end{pmatrix} = \begin{pmatrix} \cos\varpi & -\sin\varpi \\ \sin\varpi & \cos\varpi \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \end{pmatrix}, \tag{37}$$

which yields

$$\boldsymbol{r}_0 = a_0\left[\cos\left(\lambda_0 - \varpi\right)\hat{\boldsymbol{i}} + \sin\left(\lambda_0 - \varpi\right)\hat{\boldsymbol{j}}\right], \tag{38}$$

from which I obtain $\hat{\boldsymbol{k}} \cdot \boldsymbol{r}_0 = 0$ and $\hat{\boldsymbol{i}} \cdot \boldsymbol{r}_0 = a_0 \cos\left(\lambda_0 - \varpi\right)$. The Hamiltonian for the planar and circular evection resonance hence takes the simple form

$$\mathcal{H} = -\frac{3\mathcal{G} m_0 a^2 e^2}{8 a_0^3}\left[1 + 5\cos\left(2\lambda_0 - 2\varpi\right)\right]. \tag{39}$$

3.2 Simple model of second order resonance

The Hamiltonian (39) depends on $(D; -\varpi, t)$ (through e and λ_0) and has more than one degree of freedom[6]. I remove the time-dependency by going into the extended phase space. That is, I introduce the action Λ_0, conjugated to λ_0. Since $\lambda_0 = n_0 t$, the time-dependency can be removed from the Hamiltonian by rewriting it as

$$\mathcal{H} = n_0\Lambda_0 - \frac{3\mathcal{G} m_0 a^2 e^2}{8 a_0^3}\left[1 + 5\cos\left(2\lambda_0 - 2\varpi\right)\right]. \tag{40}$$

[5]The canonicity of the transformation from Delaunay's to Poincaré's variables is easy to prove by verifying that the Jacobian is a symplectic matrix.

[6]It has 3/2 degrees of freedom.

This Hamiltonian has the two degrees of freedom $(D, \Lambda_0; -\varpi, \lambda_0)$. One degree of freedom can nevertheless be lost by performing the linear canonical transformation $(D, \Lambda_0; -\varpi, \lambda_0) \to (\Sigma', \Sigma'_2; \sigma, \sigma_2)$ where $\sigma = \lambda_0 - \varpi$, $\sigma_2 = \lambda_0$, $\Sigma' = D$ and $\Sigma'_2 = \Lambda_2 - D$. Using $n^2 = \mathcal{G}m_1/a^3$, I end up with the one-degree-of-freedom Hamiltonian

$$\mathcal{H} = n_0\Sigma' - \frac{3\eta^2 e^2 \Lambda}{8n}\left(1 + 5\cos 2\sigma\right), \tag{41}$$

where $\eta^2 = \mathcal{G}m_0/a_0^3 = n_0^2/(1 + m_1/m_0)$ and where terms depending only on Σ'_2 have been removed since they are constant. The only remaining degree of freedom is $(\Sigma'; \sigma)$ and this Hamiltonian is integrable. Note that a dependency on Σ' is hidden in $e^2 = (2 - \Sigma'/\Lambda)\,\Sigma'/\Lambda$. In order to work with dimensionless variables, I use the variable

$$\Sigma = \frac{\Sigma'}{\Lambda}. \tag{42}$$

The transformation $(\Sigma'; \sigma) \to (\Sigma; \sigma)$ is not canonical, but the equations of motion stay in Hamiltonian form if the Hamiltonian is divided by Λ. Therefore, I consider the variables $(\Sigma; \sigma)$ to be canonical. The general form for the Hamiltonian of a resonance of order k in polar coordinates reads (*e.g.* Delisle *et al.*, 2014)

$$\mathcal{H} = \mathfrak{a}\Sigma + \mathfrak{b}\Sigma^2 + \mathfrak{c}\Sigma^{k/2}\cos k\sigma, \tag{43}$$

where $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are parameters. I can retrieve from the Hamiltonian (41) the general form of a second-order resonance by expanding it to second-order in Σ for the non-resonant terms and to first-order for the resonant terms. After dividing the Hamiltonian by Λ and expanding it, I find (Touma and Wisdom, 1998)

$$\mathcal{H} = \left(n_0 - \frac{3\eta^2}{4n}\right)\Sigma + \frac{3\eta^2}{8n}\Sigma^2 - \frac{15\eta^2}{4n}\Sigma\cos 2\sigma, \tag{44}$$

which is exactly the Hamiltonian of a second order resonance[7]. Like in Sect. 2.3, I use the canonical rectangular coordinates

$$X = \sqrt{2\Sigma}\cos\sigma = e\cos\sigma + \mathcal{O}(e^3) \quad \text{and} \quad Y = \sqrt{2\Sigma}\sin\sigma = e\sin\sigma + \mathcal{O}(e^3). \tag{45}$$

In the coordinates $(X; Y)$, the Hamiltonian reads

$$\mathcal{H} = \left(n_0 - \frac{3\eta^2}{4n}\right)\frac{X^2 + Y^2}{2} + \frac{3\eta^2}{32n}\left(X^2 + Y^2\right)^2 - \frac{15\eta^2}{8n}\left(X^2 - Y^2\right). \tag{46}$$

3.3 The unstable circular orbit of the Moon

The Moon is inclined on the ecliptic by only 5.1° and the eccentricity of Earth's orbit around the Sun is only $e_0 \approx 0.0167$. Therefore, the circular and coplanar assumption is valid for the Sun−Earth−Moon system and the Hamiltonian (40) is adapted to study the influence of the Sun (the perturbing body) on the trajectory of the Moon (the massless particle) around the Earth (the orbited body).

[7] Note that the Lidov-Kozai Hamiltonian (24) can also be written as a second-order resonance.

Because of the perturbations from the Sun, the currently quasi-circular orbit of the Moon could become unstable if the Moon orbits too far away from Earth. I now consider the conic part of Hamiltonian (46) by truncating it to second-order in eccentricity. Furthermore, since the mass of the Earth is negligible with respect to that of the Sun, I consider that $n_0 = \eta$. I obtain

$$\mathcal{H} = \frac{1}{2}\eta \left[\left(1 - \frac{9\eta}{2n}\right) X^2 + \left(1 + \frac{3\eta}{n}\right) Y^2 \right]. \tag{47}$$

Using the same arguments as in Sect. 2.3, I know that a quasi-circular orbit for the Moon is only possible if the term in front of X^2 has the same sign as the term in front of Y^2. The instability will occur if

$$\frac{9\eta}{2n} > 1 \quad \Leftrightarrow \quad a > \left(\frac{4m_1}{81m_0}\right)^{1/3} a_0. \tag{48}$$

In the case of the Sun−Earth−Moon system, this gives

$$a_{\mathrm{crit}} \approx 794\,000 \text{ km}. \tag{49}$$

Currently, the Moon is orbiting the Earth at a distance $a = 386\,000$ km and is safe from destabilization. However, tidal forces are responsible for the Moon to recess away from Earth at a rate of 3.8 centimeters per year, and eventually, the quasi-circular orbit of the Moon will become unstable and the Moon will get captured by the Sun. Note that this destabilization will occur much before the Moon reaches the edge of Earth's sphere of influence[8], located 1.5 million kilometers away from Earth.

A Problem

The framework of the evection resonance generally involves an additional term to the Hamiltonian due to the orbited body being non spherical. For example, the Earth is flattened at the poles and bulges at the equator due to its own rotation. When a body of mass m_1 is not spherical or point-mass, its gravitational potential is not $-\mathcal{G}m_1/r$ anymore.

The aim of this problem is to study the evection resonance when the orbited body has an equatorial bulge. At the quadrupolar order, the equatorial bulge of the orbited body can be taken into account by adding to the Hamiltonian the term (*e.g.* Couturier *et al.*, 2023, Appendix B)

$$V_{\mathrm{bulge}} = -\frac{\mathcal{G}m_1 R_1^2}{2r^5} J_2 \left[r^2 - 3\left(\hat{\boldsymbol{p}} \cdot \boldsymbol{r}\right)^2 \right], \tag{50}$$

where R_1 is the mean radius of the orbited body, $\hat{\boldsymbol{p}}$ is a unit vector towards its North pole, and J_2 is a small dimensionless parameter measuring the equatorial bulge.

1 - Assume that the massless particle orbits in the equatorial plane of the orbited body[9] and rewrite V_{bulge} in a simpler way.

2 - Show that

$$\frac{1}{2\pi}\int_0^{2\pi} V_{\mathrm{bulge}}\, dM = -\frac{n\Lambda J_2 R_1^2}{2a^2\left(1 - e^2\right)^{3/2}}, \tag{51}$$

[8] Also called Hill sphere.

[9] That is, assume $\hat{\boldsymbol{p}} = \hat{\boldsymbol{k}}$.

where M is the mean anomaly of the particle and $n = \sqrt{\mathcal{G}m_1/a^3}$ is its mean motion. According to Eq. (39), the Hamiltonian of the evection resonance with an equatorial bulge in the coplanar and circular case, averaged over M, can be written

$$\mathcal{H} = -\frac{3\mathcal{G}m_0 a^2 e^2}{8a_0^3}\left[1 + 5\cos\left(2\lambda_0 - 2\varpi\right)\right] - \frac{n\Lambda J_2 R_1^2}{2a^2\left(1-e^2\right)^{3/2}}. \tag{52}$$

3 - Reproduce the steps of Sect. 3.2 and write the Hamiltonian under the form of Eq. (43) with $k = 2$. To this aim, expand the Hamiltonian to second order in Σ, and get rid of terms proportional to $\Sigma^2 \cos 2\sigma$. Give the expression of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$.

4 - Show that the equilibria of the Hamiltonian, when they exist, verify

$$(X, Y) \in \left\{(0,0)\,;\left(\pm\sqrt{\frac{\mathfrak{c}-\mathfrak{a}}{\mathfrak{b}}}, 0\right);\left(0, \pm\sqrt{-\frac{\mathfrak{a}+\mathfrak{c}}{\mathfrak{b}}}\right)\right\}, \tag{53}$$

where X and Y are defined in Eq. (45).

5 - Draw the phase space of the Hamiltonian, in the case where it has one, three and five equilibria. You can either linearize the equations of motion in the vicinity of each equilibria in order to deduce which are elliptic and which are hyperbolic (by computing the eigenvalues), or you can ask this python script to draw the phase space for you.

The Moon was likely formed ~ 4.5 Gy ago by the accretion of material from a debris disk caused by a giant impact between the Earth and a Mars-sized object. My simulations conclude that two sub-Moons generally form from the debris disk, one with semi-major axis typically $\lesssim 4R_\oplus$ and another one with semi-major axis typically $\gtrsim 6R_\oplus$, where $R_\oplus$ is the radius of the Earth. From there, these sub-Moons generally fail to quickly collide to give a complete Moon, but rather see their semi-major axis start to slowly increase due to tidal forces with the Earth.

6 - Using $J_2 = 0.043$ and results from question 5, explain how the evection resonance could have helped make the sub-Moons collide and complete the formation of the Moon. In particular, explain why it is important that the increase in semi-major axis of the sub-Moons be very slow, for the completion of the Moon to be successful.

References

Couturier, J., Quillen, A. C. and Nakajima, M. (2023). *NcorpiON : A O(N) Software for N-body Integration in Collisional and Fragmenting Systems.*

Delisle, J.-B., Laskar, J. and Correia, A. C. M. (2014). Resonance Breaking Due to Dissipation in Planar Planetary Systems. *Astronomy & Astrophysics, Volume 566, id.A137*, 566, A137.

Laskar, J. (2017). Andoyer Construction for Hill and Delaunay Variables. *Celestial Mechanics and Dynamical Astronomy*, 128, pp. 475–482.

Touma, J. and Wisdom, J. (1998). Resonances in the Early Evolution of the Earth-Moon System. *The Astronomical Journal*, 115, pp. 1653–1663.