# PHY256 Lecture notes on the shift and Baker maps

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### 0.1 Introduction

Following 'A First Course in Discrete Dynamical Systems' by Richard A. Holmgren (1994). The Baker map discussion is from "Chaotic Dynamics" by Tel and Gruiz and currently has mistakes in the discussion of period three orbits.

We have seen *chaos* exhibited in different ways. Chaos is associated with a change in dimension of orbits in Hamiltonian systems and in maps created from them. In iterative maps on the real line we saw examples of sensitivity to initial conditions. Here we will look at maps more abstractly, specifically at maps on sets. In the realm of symbolic dynamics, we also find periodic orbits and maps that are strongly mixing. We are building here on the properties of maps to cause strong mixing. Even though these maps are deterministic we find behavior typical of ergodic or stochastic systems.

**Definition** An *ergodic* system is one that, given sufficient time, includes or impinges on all points in a given space and can be represented statistically by a reasonably large selection of points.

**Definition** A *stochastic* system is one that is randomly determined; having a random probability distribution or pattern that may be analyzed statistically but may not be predicted precisely.

## 1 Symbolic Dynamics

Symbolic dynamics is concerned with maps on sets. We focus here on an example using the set $\Sigma_2$.

We define $\Sigma_2$ as the set of infinite sequences of 0's and 1's. An element or point in $\Sigma_0$ is something like 0000000... or 010101010101..... repeating. We can refer to the point $s$ as $s_0s_1s_2s_3s_4$..... and the point $t$ as $t_0t_1t_2t_3$.... We can think of $s$ and $t$ as close to one another if their sequences are similar at the beginning of the sequence.

This space has a nice metric that we can use to tell if two points in $\Sigma_2$ are close to each other.

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Since $|s_i - t_i|$ is always either 0 or 1

$$0 \leq d[s,t] \leq \sum_{i=0}^{\infty} 2^{-i} = 2$$

**Definition** Recall that a metric on a set $X$ has the properties

- $d[x,y] \geq 0$
- $d[x,y] = d[y,x]$
- $d[x,y] \leq d[x,z] + d[z,y]$

for all $x, y, z$ in the set $X$.

A metric space is a set along with a metric allowing one to measure distances between points in the set.

A nice thing about this metric is that we can describe the distance between two points in terms of the number of digits in the beginning of the sequence that agree. For example, if $s, t$ agree in their first 2 digits (first 2 numbers of the sequence) then $d[s,t] \leq 2^{-1}$. If $s, t$ agree in their first $n$ digits then $d[s,t] \leq 2^{-n+1}$. The size of a neighborhood can be described in terms of the numbers of matching digits of points in the neighborhood.

### 1.1 The Shift map

The shift map on $\Sigma_2$ is defined as

$$\sigma(s_0 s_1 s_2 ...) = s_1 s_2 s_3$$

where $s \in \Sigma_2$. The shift map forgets the first digit of the sequence. For example $\sigma(10010101...) = 0010101....$

If two points are close together with respect to our metric $d$ then they remain close together after they are shifted by the shift map. But if the shift map is reapplied many times then we eventually expect the orbits to differ.

There are periodic points of the shift map. For example the repeating sequence $s =$ 1010101010.... is a fixed point of $\sigma^2$.

The shift map has $2^n$ periodic points of period $n$. This follows as there are $2^n$ ways to chose 1,0 for $n$ positions in a repeating sequence.

We can call a point *eventually* periodic if it starts with a sequence and then is periodic.

**Definition** It is useful to use the word *dense*. A set $A$ is *dense* in another set $X$ if for every point in $x \in X$, every neighborhood of $x$ contains at least a point in $A$.

In other words for each point $x \in X$ and each $\epsilon > 0$ there exists a $y \in A$ such that $|x - y| < \epsilon$. Another equivalent statement is that for every point $x \in X$ there is a sequence of points in $A$ that converge to $x$. A dense set is one that is present everywhere in $X$.

The set of periodic points of the shift map is *dense* in $\Sigma_2$. Suppose we choose an $x \in \Sigma_2$ that is not periodic. We will show that there are periodic points near it no matter what $x$ is. We can define distance in terms of the number of digits that agree. Suppose we choose a neighborhood with all points that are the same as $x$ up to $2^n$ digits. We can find a periodic point in this neighborhood by choosing a point that repeats with the same $2^n$ digits as $x$. Since for every size neighborhood (every possible $n$) and every $x$ we can find a periodic orbit in the neighborhood, we have shown that the set of periodic points is dense in $\Sigma_2$.

There is a point in $\Sigma_2$ whose *orbit* is dense in $\Sigma_2$. To show that this is true, all we have to do is construct such an orbit and show that it is dense. The Morse sequence

$$0\ 1\ 00\ 01\ 10\ 11...$$

and then has all permutations and choices for blocks of three elements and then all permutations and choices for blocks of four elements. As the shift map is applied this orbit will eventually cover all possible sized neighborhoods near any point in $\Sigma_2$.

**Definition** A function $f : D \to D$ is *topologically transitive* if for all open sets $U, V$ in $D$ there is an $x$ in $U$ and a natural number $n$ such that $f^n(x)$ is in $V$.

If a function is topologically transitive, then we can arbitrarily chose two neighborhoods and we can always find an orbit that contains points in both neighborhoods. A function that is topologically transitive *mixes* the domain. We saw mixing in the chaotic regions of the area preserving maps and for certain parameters in the logistic map, though we didn't determine whether the mixing was present everywhere. We did not show that there was an orbit connecting *any* two neighborhoods.

### 1.2 $\Sigma_2$ is well mixed by the shift map

We can show that the shift map is *topologically transitive* and so well mixes $\Sigma_2$. Chose any two points, $s, t$, in $\Sigma_2$ and any two sized neighborhoods around them. We can specify the size the neighborhood around $s$ with a number of digits $n$ for points in $\Sigma_2$ at the beginning of the sequence that are the same as those of $s$. A point $z$ in this neighborhood of $s$ has distance $d[s, z] \leq 2^{-n}$. We can similarly specify the size of the neighborhood around $t$ with $m$ digits. We can construct a point $x$ that has first $n$ numbers in its sequence that are the same as $s$ but the following $m$ numbers in its sequence that are the same as $t$. This $x$ has orbit such that it stars in the neighborhood of $s$ and $\sigma^n$ puts it in the neighborhood of $t$.

### 1.3 The shift map is sensitive to initial conditions

**Definition** For $D$ a metric space with metric $d$. The function $f : D \to D$ exhibits *sensitive dependence* on initial conditions if there exists a $\delta > 0$ such that for all $x \in D$ and all $\epsilon > 0$, there is a $y \in D$ and a natural number $n$ such that

$$d[x, y] < \epsilon \qquad \text{and} \qquad d[f^n(x), f^n(y)] > \delta$$

No matter how small a neighborhood we chose around any point $x$, we can always find a nearby point $y$ (in the neighborhood) that has an orbit that eventually sends it some distance $\delta$ away from the orbit of $x$. That means it is hard to predict orbits after many iterations if we have uncertainty in initial conditions.

The shift map is sensitive to initial conditions. Choosing an $\epsilon$ gives an $m$ the number of digits of $y$ that must be in common with $x$ to satisfy $d[x, y] < \epsilon$. We set $\delta$ to be $1/4$. We can chose $y$ to have $m$ digits in common with $x$ and then differ in the next digit. After $m$ applications of the shift map the orbits are $1/2$ apart. This means for any $\epsilon$ we can find a $y$ and an $n = m$ that has distance between the orbits greater than $\delta = 1/4$. Thus the shift map is 'sensitive to 'initial conditions', as defined above.

### 1.4 Symbolic Definition of Chaos (Devaney)

Definition due to Devaney 1989 in his book on chaotic dynamical systems.

**Definition** Let $D$ be a metric space. The function $f : D \to D$ is *chaotic* if

a) The periodic orbits of $F$ are dense in $D$.

b) The function $f$ is topologically transitive. In other words $f$ mixes the set really well.

c) The function $f$ exhibits extreme sensitivity to initial conditions.

It turns out that a,b imply c if $D$ is infinite (proved by Banks, Brooks Cairns, David and Stacey in 1992).

## 1.5 The shift map on the unit interval

A point in $\Sigma_2$ can be turned into a number on the unit interval by inserting a decimal point before the sequence and assuming that the number is in base 2. For example $s = 01011....$ is equivalent to the number 0.01011.. Recall our metric in $\Sigma_2$. This metric is equivalent to distance between two points in the unit interval.

What is the equivalent of the shift map but on the unit interval? If the number is less than 1/2 then the shift map simply multiplies by 2. If the number is greater than 1/2 then the shift map multiplies the number by 2 and then subtracts it by 1.

$$\sigma(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2x - 1 & 1/2 \leq x \leq 1 \end{cases}$$

Another way to write this map is

$$\sigma(x) = 2x \mod 1$$

It is easy for us to compute the Lyapunov exponent for this map from its derivative which is always 2.

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \ln |f'(x_i)| = \ln 2$$

How is this related to the Lyapunov exponent of the shift map? Consider two points $s, t$ that are the same for their first 100 digits but differ afterwards. At the beginning the distance between the two points is $d(s,t) \leq 2^{-100}$ but after a single application of the shift map $d(\sigma(s), \sigma(t)) \leq 2^{99}$ and so on. The ratio of these distances is $2^n$ where the map is applied $n$ times. The natural log of the ratio of these distances is $n \ln 2$.

Recall we can define a Lyapunov exponent for a map as

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \frac{|\delta_n|}{|\delta_0|}$$

Instead of using Euclidian distances if we use the metric computing

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{d(\sigma^n(s), \sigma^n(t))}{d(s,t)} \right| = \ln 2$$

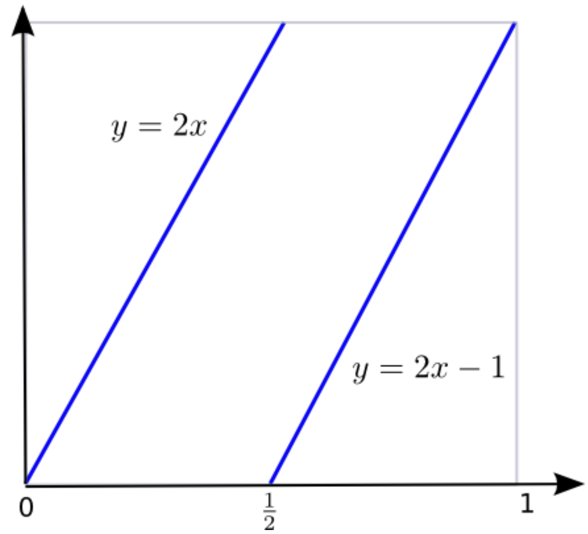


Figure 1: The shift map on $\Sigma_2$ is equivalent to this map on the unit interval. The map stretches the interval.

**Question** Do we have to be careful how we chose two initial points for computing trajectories? Periodic points are dense. What happens if we chose two periodic orbits?

**Remark** This map stretches the unit interval. It stretches both the region 0 to 1/2 and the region 1/2 to 1 back into the region 0 to 1. Stretching and folding is typical of chaotic maps and how they mix.

## 2 Baker map

The baker map is a prototype for 2D chaotic dynamical systems. See Figure 2. On $x \in [0, 1]$ and $y \in [0, 1]$

$$(x_{n+1}, y_{n+1}) = \begin{cases} (cx_n, 2y_n) & \text{for} \quad y_n \leq 1/2 \\ (1 + c(x_n - 1), 1 + 2(y_n - 1)) & \text{for} \quad y_n > 1/2 \end{cases} \tag{1}$$

Or

$$x_{n+1} = \begin{cases} cx_n & \text{for} \quad y_n \leq 1/2 \\ 1 + c(x_n - 1) & \text{for} \quad y_n > 1/2 \end{cases} \qquad y_{n+1} = \begin{cases} 2y_n & \text{for} \quad y_n \leq 1/2 \\ 1 + 2(y_n - 1) & \text{for} \quad y_n > 1/2 \end{cases}$$

The form of the map depends on the critical line $y = 1/2$. We assume that $0 < c \leq 1/2$. Looking at the $y$ part of the map alone. The map is a sawtooth and so reminds us of the shift map on the unit interval. We already know that this map is 'chaotic'. The $x$ part of the map is more complicated as it depends on the $y$ value.

**Remark** The Baker map, and the Smale horseshoe map are topologically conjugate. The area preserving Baker map is topologically conjugate to the Arnold cat map. The 1D tent map and the shift map are not topologically conjugate. All these statements should be checked!!!!

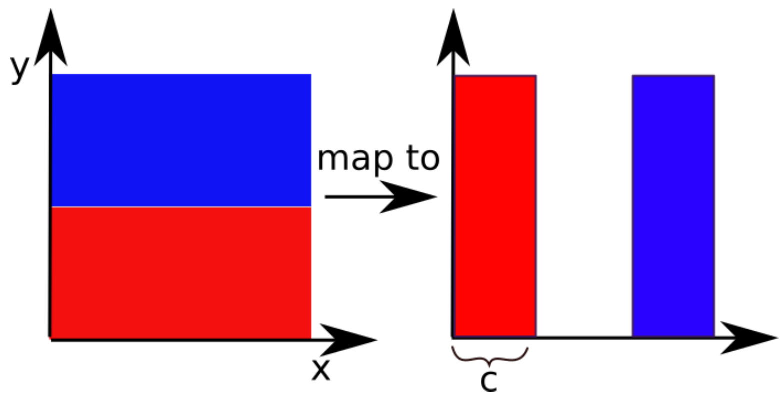


Figure 2: Illustration of the baker map. The $y$ direction is stretched while the $x$ direction is compressed. The Lyapunov exponent is determined from the stretching in the $y$ direction. The lower left and right most corners are hyperbolic fixed points of the map.

The origin $(0,0)$ and $(1,1)$ are fixed points. We will show that these two points are *hyperbolic* fixed points.

Linearizing the map about these two points Near the origin

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Near $(1,1)$

$$\begin{pmatrix} x_{n+1} - 1 \\ y_{n+1} - 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_n - 1 \\ y_n - 1 \end{pmatrix}$$

The Jacobian matrix for transformations near both fixed points

$$J = \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix}$$

The eigenvalues of this matrix are $2, c$. The determinant of the matrix is $J = 2c$ which is 1 if $c = 1/2$. The map is area preserving if the determinant of the matrix is 1 (and when $c = 1/2$). For $c < 1$ one direction shrinks volume and the other stretches volume. With $c < 1/2$ there is continual shrinking of volume.

## 2.1 Classification of fixed points in a linear two-dimensional map

Writing the map in this form

$$
\begin{aligned}
x_{n+1} &= f_x(x_n, y_n) \\
y_{n+1} &= f_y(x_n, y_n)
\end{aligned}
$$

Suppose we have a fixed point $x*, y*$ of a two-dimensional map Compute the Jacobian matrix

$$
J = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix}
$$

and evaluate it at the fixed point $x*, y*$. Near the fixed point the map looks like

$$
\begin{pmatrix} x_{n+1} - x_* \\ y_{n+1} - y_* \end{pmatrix} = \mathbf{J} \begin{pmatrix} x_n - x_* \\ y_n - y_* \end{pmatrix}
$$

Using a characteristic equation we can solve for eigenvalues of $J$, $\lambda_1, \lambda_2$. The size of the eigenvalues is important. If the eigenvalue is negative then $\Delta x$ will oscillate back and forth. The distance away from the fixed point will grow if the absolute value of an eigenvalue is greater than 1.

1. If the eigenvalues are both real and $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then the map is expanding. Orbits are exponentially diverging away from the fixed point with distance along directions determined by the eigenvectors. This type of fixed point is also called a *node repeller*.

2. If both eigenvalues are real and absolute value less than 1. The fixed point is called a *node attractor*. Orbits are exponentially decaying toward the fixed point.

3. If both eigenvalues are real but one has absolute value greater than 1 and the other has absolute value less than 1, then the point is *hyperbolic*. There is one eigenvector along which the orbit decays to the fixed point. Along the other direction the trajectory moves outwards.

4. If the determinant of $J$ is 1 then the map is area (or volume) preserving.

5. If both eigenvalues are complex then the fixed point is said to be *elliptic*. Orbits circulate about the fixed point.

6. If the fixed points have complex parts but their real part has absolute value less than 1 then the node is a *spiral attractor*.

7. If the fixed points have complex parts but their real part has absolute value greater than 1 then the node is a *spiral repeller*.

8. If the matrix is degenerate and there is only eigenvector then all trajectories are parallel to a single eigendirection.

**Definition** A **hyperbolic** fixed point has Jacobian matrix with two real eigenvalues and one of them has absolute value greater than 1.

**Remark** The classification of fixed points in a two-dimensional map is similar but not identical to that of a two-dimensional linear dynamical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. In that case the sign of the eigenvalue is important and determines whether trajectories exponential diverge from or converge to the fixed point.

## 2.2 Hyperbolic points in the Baker map

Recall for the Baker map, the eigenvalues for the fixed points at $0, 0$ and $1, 1$ were $2, c$. Because there is an eigenvalue greater than 1, there is a trajectory moving away from the fixed point exponentially fast. The fixed points at (0,0) and (1,1) are both hyperbolic.

## 2.3 Lyapunov exponent of the Baker map

The Baker map looks exactly like the shift map in the $y$ direction. In fact the map in the $y$ direction is independent of the $x$ value. The Lyapunov exponent is equivalent to that of the shift map and is $\lambda = \ln 2$.

## 2.4 Periodic orbits in the Baker map

How many points, $N_m$ are fixed points of the map $B^m$? There are two points that are fixed in the map (those at the corners). So $N_1 = 2$.

We can write the Baker map in terms of two linear transformations. From equation 1

$$
\begin{aligned}
B_-(\mathbf{x}) &= \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} \\
B_+(\mathbf{x}) &= \begin{pmatrix} 1-c \\ -1 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}
\end{aligned}
$$

The transformation $B_-$ is used for $y \leq 1/2$ and that for $B_+$ for $y > 1/2$. A period two point is a fixed point of either $B_-^2$ or $B_+^2$ or $B_+B_-$. If we evaluate $B_+^2$ we find that the only fixed point is at (1,1). Likewise the only fixed pint of $B_-^2$ is the origin.

$$B_+B_-(\mathbf{x}) = \begin{pmatrix} 1-c \\ -1 \end{pmatrix} + \begin{pmatrix} c^2 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x}$$

This has fixed point $1/(c+1), 1/3$.

There is one 2-cycle giving two more fixed points for $B^2$. Altogether including the two fixed points of $B$ there are four points that are fixed for $B^2$ so $N_2 = 4$.

We can similarly count the number of 3-cycles. There are two 3-cycles each giving 3 fixed points of $B^3$. Altogether giving $N_3 = 6 + 2 = 8$ fixed points of $B^3$ (we don't need the 2-cycles for $B^3$). And so on for other cycles. The number of fixed points of $B^m$ is $N_m = 2^m$.

## 2.5 Topological entropy

A quantity called *topological entropy* can be computed from the number of fixed points of the map. Take $\log N_m = \log 2^m = m \log 2$ for the Baker map. In general

$$h = \lim_{m\to\infty} \frac{1}{m} \ln N_m$$

And this implies that

$$N_m \sim e^{hm}$$

The topological entropy is a measure of complexity of the attractor. Thus the topological entropy is the average (per iteration) amount of information needed to describe long iterations of the map.

## 2.6 Chaotic attractor for the area contracting Baker map

When $c < 1/2$ the map is not area preserving. Each application of the map shrinks the volume. As each application of the map shrinks the volume where points can go, every orbit must converge eventually onto something that has no area (has measure zero).

Reapplication of the map many times converges to a set. The set has measure zero when $c < 1/2$. The set is called a *chaotic attractor*. Points at the boundaries are fixed or periodic points. The chaotic attractor looks like a Cantor set. There are cycles of all integers and these are all part of the chaotic attractor (see Figure 3).

I found that to compute more than a hundred iterations of the Baker map I had to increase the precision of the computations above that of float.

The attractor contains all periodic fixed points. Like the Cantor set it has measure zero. Most members of the Cantor set are not endpoints of deleted intervals. Since each step removes a finite number of intervals and the number of steps is countable, the set of endpoints is countable while the whole Cantor set is uncountable. Are there members of

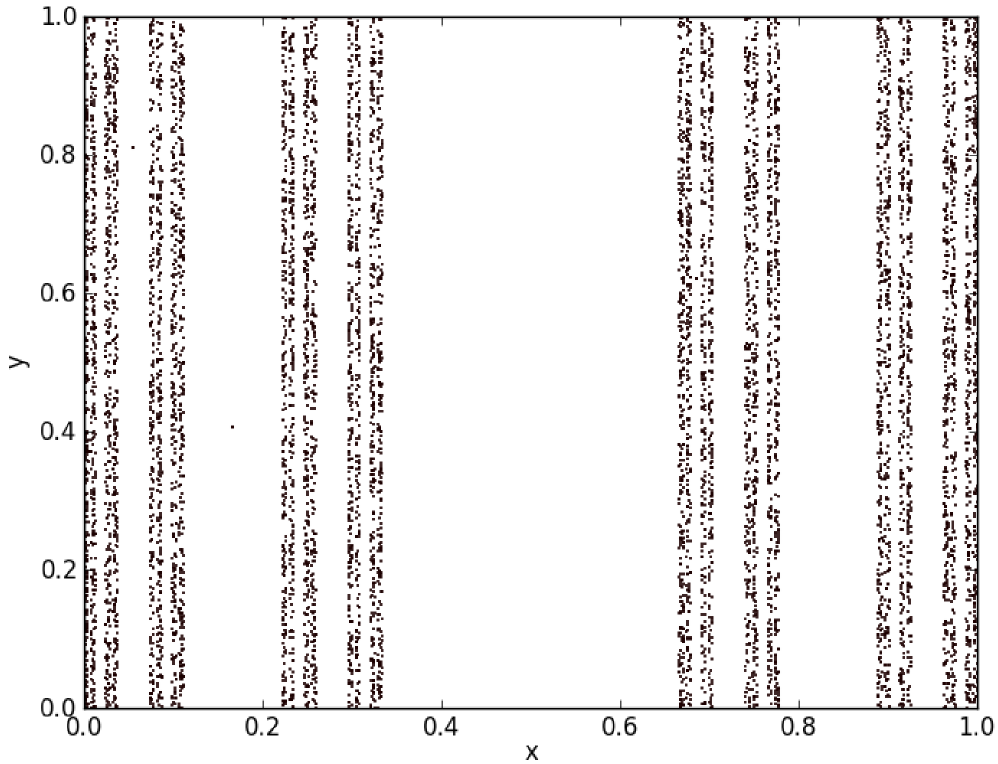


Figure 3: For $c = 1/3$ and initial conditions $x_0 = 0.5$, $y_0 = 2/\pi^2$, 8000 iterations of the Baker map are plotted. I found I had to use a high level of precision. With regular floats in python, the orbit was incorrectly converging on the origin. The self-similar structure of the chaotic attractor can be imagined from this illustration.

the attractor for the Baker map that are not periodic points? (Yes as they can start from irrational initial conditions). And if so does that mean that it is uncountable? (Yes). The number of periodic points is countable.

An *attractor* attracts all orbits starting within a sufficiently small open setting containing the attractor. If the map $c = 1/2$ is area preserving then all orbits shuffle things around and orbits never decay to anything. An attractor is incompatible with area preservation and can only be possible in physical systems that are dissipative. An attractor cannot contain any repelling fixed points.

The Baker map has provided a setting for calculating various types of dimensions, Hausdorff, information, pointwise, fractal and Lyapunov dimensions for the attractor. For example, the Hausdorff dimension is the exponent used to describe how the number of balls needed to cover the set scales with the size of the balls.

What is the Hausdorff dimension of the Baker map's attractor?