PHY256 Lecture notes on Bifurcations and Maps

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Contents

0.1 Introduction

For bifurcations and maps we are following early and late chapters in the lovely book by Steven Strogatz (1994) called 'Non-linear Dynamics and Chaos' and the extremely clear book by Richard A. Holmgren (1994) called 'A First Course in Discrete Dynamical Systems'. On Lyapunov exponents, we include some notes from Allesandro Morbidelli's book 'Modern Celestial Mechanics, Aspects of Solar System Dynamics.'

In future I would like to add more examples from the book called Bifurcations in Hamiltonian systems (Broer et al.). Renormalization in logistic map is lacking.

1 Bifurcations of one-dimensional dynamical systems

We start by looking at dynamical systems that are functions of the real line. For example

$$\frac{dx}{dt} = f(x). \tag{1}$$

A heavily damped physical system could be in this category. For example, consider a mass m that is heavily damped. Newton's laws give

$$m\ddot{x} = F_{damp}(\dot{x}) + g(x) \tag{2}$$

where a damping force F_{damp} and coordinate dependent force $g(x)$ are applied to the mass. If $F_{damp} \propto \dot{x}$ and the mass is small then this equation can be written in the form of equation 1.

Here we look at one dimensional dynamical systems on the real line and maps from the real line back to itself.

1.1 Saddle-node bifurcation

Let us look at dynamical systems that are derived from maps of the real line back onto the real line

$$\dot{x} = f(x)$$

For example

$$\dot{x} = r + x^2 \tag{3}$$

- When $f(x) > 0$, then x will increase in time.
- When $f(x) < 0$, then x decreases in time.
- When $f(x) = 0$ then we have a fixed point.

It is straightforward to classify fixed points from a plot of the function $y = f(x)$. Fixed points are where the function crosses the x axis. On the plot x increases in time (goes to the right) where the function is above the x-axis and decreases in time (goes to the left) where the function is below the x-axis. It is also easy to see whether a fixed point is stable by looking at the direction of motion on either side of the fixed point.

For the dynamical system given by equation 3 (see Figure 1).

- When $r < 0$ there are two fixed points, one stable and one unstable. As r approaches zero, the two fixed points move together.
- When $r = 0$, we can say the fixed point is half stable because even a small change in r will make it disappear.
- For $r > 0$, there are no fixed points.

We can also look at the stability of a fixed point by expanding about the fixed point. We take x^* a fixed point and expand about it with $x = x^* + y$.

$$f(x^*) = 0 + f'(x^*)y + ...$$

The dynamical system near the fixed point

$$\dot{y} = f'(x^*)y$$

This has solution

$$y \propto \exp(f'(x^*)t)$$

so if $f'(x^*) > 0$ then the fixed point is unstable. If $f'(x^*) < 0$ then the fixed point is stable. If $f'(x^*) = 0$ then we need to expand to higher order to see how the dynamics behaves near the fixed point. If $f''(x^*) \neq 0$ then we know that $f(x^*)$ is a local maxima or minima. We can call this point either half unstable or half stable if a small change in a parameter like r changes its nature.

The dynamical system of equation 3 displays (as a function of r) a *bifurcation*. Going from positive r to negative r we say that a *saddle-node bifurcation* occurs at $r = 0$ with two fixed points appearing. The situation is illustrated in Figure 1.

The system

$$\dot{x} = r - x^2$$

also displays a saddle-node bifurcation but the bifurcation diagram is flipped both vertically and horizontally.

In a sense

$$\dot{x} = r + x^2 \qquad \dot{x} = r - x^2$$

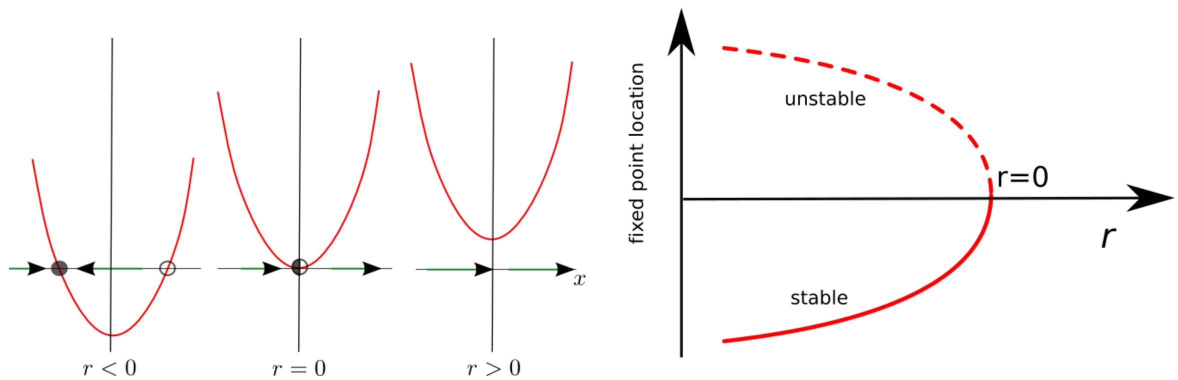

Figure 1: a) Fixed points for the dynamical system $\dot{x} = r + x^2$. When $r < 0$ the leftmost fixed point is stable and an attractor and the rightmost fixed point is unstable. When $r > 0$ there are no fixed points. The value of r determines whether there are stable or unstable fixed points. b) Bifurcation diagram for the same system. This is known as a *saddle-node* bifurcation.

are representative of *all* saddle-node bifurcations. Near a fixed point we can locally expand in a Taylor series. If the Taylor series contains a constant and a quadratic term, then by rescaling variables, we can transform the equations of motion near the fixed point to look like one of these two equations. These two models are sometimes call *normal forms* for the saddle-node bifurcation.

1.1.1 Example of a saddle-node bifurcation

Let us look at the system

$$\dot{x} = r - x - e^{-x}$$

The fixed points are given by $r - x - e^{-x} = 0$ but we can't solve this equation. We can think about it graphically by considering where a line with slope $-x$ crosses the function e^{-x} (see Figure 2). We expect that we will have a bifurcation when the line $y = r - x$ is tangent to the line e^{-x}. When the line $y = r - x$ intersects e^{-x} at two points then there are two fixed points. When the line $y = r - x$ does not intersect e^{-x} at all then there are no fixed points. When the line $y = r - x$ intersects e^{-x} at a single point, then the line is tangent to the curve and we have a bifurcation point. Let us label r_c the value of r at the bifurcation point. For $r > r_c$, there are two fixed points and for $r < r_c$ there are none.

Let us now solve for r_c. The bifurcation must take place when the two curves are just touching. At this point the tangent to the curve $y = e^{-x}$ is equal to that of the line $y = r - x$. The tangent to $y = e^{-x}$ is $-e^{-x}$. We set this equal to -1 and solve for x, finding that the bifurcation happens where $e^{-x} = 1$ or at $x_c = 0$. What r value is required for this to be a fixed point? $r - 0 - 1 = 0$, so $r_c = 1$. The bifurcation happens at $r_c = 1, x_c = 0$.

Let us expand about $x = 0$. We need to expand the exponential $e^{-x} \sim 1 - x + x^2/2..$ and our function becomes

$$\dot{x} = r - x - 1 + x - \frac{x^2}{2}.. = (r-1) - \frac{x^2}{2}..$$

which shows in another way that there is a saddle-node bifurcation at $r = 1$; see Figure 3.

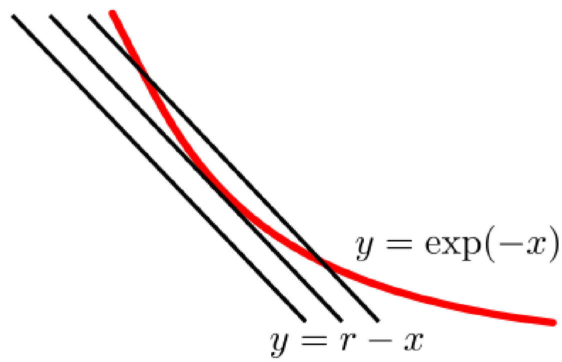

Figure 2: The system $\dot{x} = r - x - e^{-x}$ has a bifurcation where the curve $y = r - x$ is tangent to the curve $y = e^{-x}$.

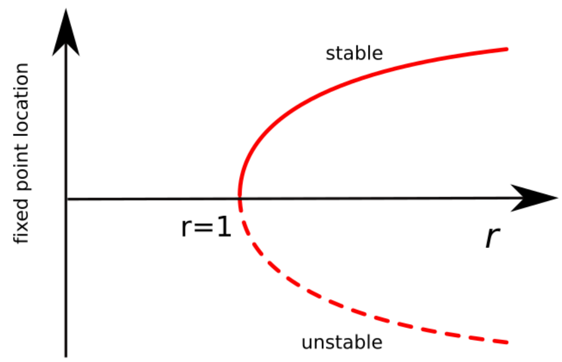

Figure 3: The system $\dot{x} = r - x - e^{-x}$ has a saddle node bifurcation with with two fixed points at $r > r_c$ with $r_c = 1$. This bifurcation diagram has the same shape as that for $\dot{x} = r - x^2$, except shifted in r by 1.

1.1.2 Another example

Consider the system

$$\dot{x} = rx - \sin x$$

with $rx - \sin x$ plotted in Figure 4. The system has a number of bifurcations but we will focus on the one that occurs with $r \sim 0.13$ at $x \sim 7.7$. We can see from the red line in the figure that when the function $rx - \sin(x)$ is above zero (near $x = 7.7$) there is no nearby fixed point. For larger r there are two fixed points and the leftmost one is stable and the rightmost one unstable. So it's a saddle-node bifurcation.

Let us define a function

$$f(x, r) = rx - \sin x$$

and consider where the new fixed point first appears. For a particular $r = r_c$ value the curve just touches the x axis, producing a fixed point at x^* with

$$f(x^*, r_c) = 0 \qquad \frac{\partial f}{\partial x}(x^*, r_c) = 0$$

Let us expand to first order in r about r_c and to second order in x about x^*.

$$f(x, r) = f(x^*, r_c) + \frac{\partial f}{\partial r}(x^*, r_c)(r - r_c) + \frac{\partial f}{\partial x}(x^*, r_c)(x - x^*) + \frac{\partial^2 f}{\partial x^2}(x^*, r_c)\frac{(x - x^*)^2}{2}$$

The first terms is zero (x^* is a fixed point for $r = r_c$). The third term is zero because the curve is tangent to the x axis at r_c, x^*. We evaluate the other two terms finding

$$f(x, r) = x^*(r - r_c) + \sin x^* \frac{(x - x^*)^2}{2}$$

If we define new variables $X = x - x^*$ and $R = x^*(r - r_c)$ we can rewrite this as

$$f(X, R) = R + \frac{\sin x^*}{2} X^2$$

and we recognize that this is in the form of a saddle-node bifurcation. A further rescaling of X can put it in the form $R + X^2$.

As any curve with a minimum dips below the x axis, locally it can be approximated by a quadratic function. We can shift r and x and rescale both. So any local minimum or maximum as it crosses the x axis can be made to look like a saddle-node bifurcation.

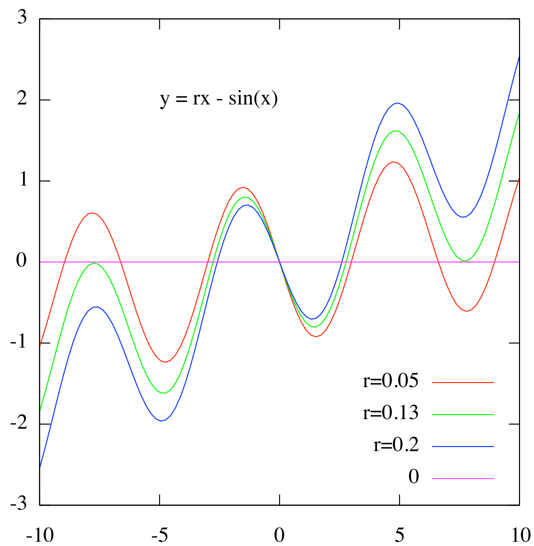

Figure 4: Saddle node bifurcation of $\dot{x} = rx - \sin x$. Plotted is $y = rx - \sin x$. There are saddle-node bifurcations at $r \sim 0.13$ and $x \sim \pm 7.5$. These can be seen with the green line touching the x axis at $x \sim \pm 7.5$.

1.2 Pitchfork bifurcations

Consider

$$\dot{x} = rx - x^3 \tag{4}$$

Depending on r there are either 3 roots or 1 root, so there are either 3 or 1 fixed points. The middle point is unstable (see figure 5). Solve $rx - x^3 = x(r - x^2) = 0$ finding fixed

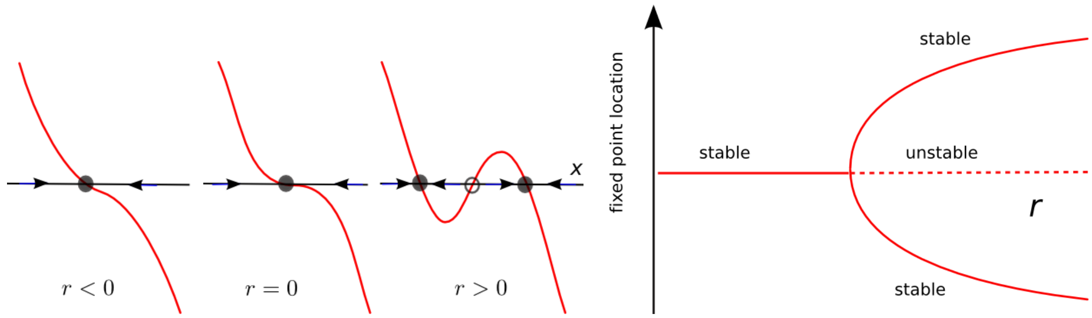

Figure 5: a) Fixed points for the dynamical system $\dot{x} = rx - x^3$. When $r < 0$ there is a single stable fixed point. When $r > 0$ there are three fixed points and the center one is unstable. The value of r determines the number of fixed points. b) Bifurcation diagram for the same system. This is known as a *supercritical pitch-fork bifurcation.*

points at $x^* = 0, \pm\sqrt{r}$.

When the sign of x^3 is flipped then the bifurcation diagram is inverted. For $r < 0$ there are three fixed points with the outer two unstable. For $r > 0$ there is a single unstable fixed point. The bifurcation is called a *subcritical pitchfork* bifurcation.

The system is vastly unstable, but can be made stable at large x with the addition of higher order terms (in x), and giving large amplitude stable branches and some hysteresis in its behavior as r is varied (see Figure 6b).

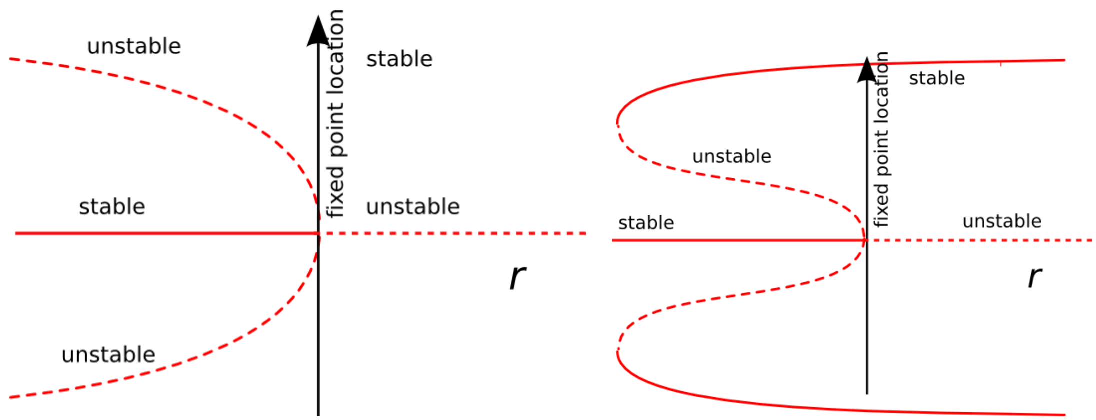

Figure 6: a) Bifurcation diagram for the dynamical system $\dot{x} = rx + x^3$. This is known as a *sub-critical pitch-fork bifurcation.* b) Bifurcation diagram for the dynamical system $\dot{x} = rx + x^3 - x^5$ that has stable fixed points at large x. This system exhibits hysteresis. Consider a system starting on the top right at high r. As r decreases the system follows the top stable curve and only jumps to the x-axis at large negative r. A system that starts on the left must reside on the stable curve on the x-axis. The system only jumps to high or low x when $r \sim 0$. The two trajectories are not the same. The system with decreasing r must reach a much lower r value before it makes the transition than the system with increasing r.

1.3 Trans-critical bifurcations

We consider the system

$$\dot{x} = rx - x^2$$

There is a fixed point at the origin in all cases.

- If $r < 0$ there is an unstable fixed point at $x = -r$ and the one at the origin is stable.
- If $r > 0$ there is an unstable fixed point at the origin and an unstable one at $x = r$.
- If $r = 0$ then there is a mixed type of fixed point at the origin. We can say that the fixed points swap locations at the origin.

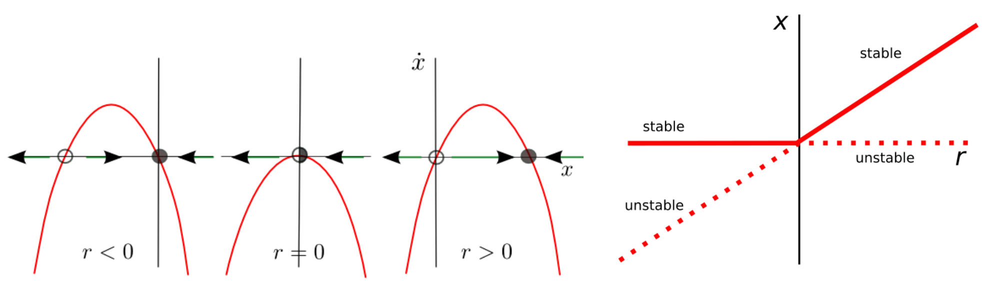

Figure 7: a) Fixed points for the dynamical system $\dot{x} = rx - x^2$. When $r < 0$ there are two fixed points. The leftmost is unstable and the rightmost is stable. When $r > 0$ there are also two fixed points but they have swapped locations. The bifurcation is known as a trans-critical bifurcation.

1.4 Imperfect bifurcations

Let us recall the models discussed so far

$\dot{x} = r + x^2$	$\dot{x} = r - x^2$	saddle-node
$\dot{x} = rx - x^2$	$\dot{x} = rx + x^2$	transcritical
$\dot{x} = rx - x^3$		supercritical pitchfork
$\dot{x} = rx + x^3$		subcritical pitchfork

The saddle and transcritical models are represented by quadratic polynomial systems. Additional free parameters in these systems will not change the nature of the bifurcation, it would only shift its location. However, the pitchfork bifurcation models have symmetric bifurcation diagrams and are represented by cubic equations. This implies that we can change the nature of the bifurcation diagram by adding a third parameter.

Consider

$$\dot{x} = h + rx - x^3$$

with imperfection parameter h. The critical points are solutions to a cubic equation. The cubic equation has local extrema at $r - 3x^2$ (taking the derivative of $h + rx - x^3$). This local max occurs at $x_{max} = \sqrt{r/3}$ and the local minimum at $x_{min} = -\sqrt{r/3}$. Insert these values into $f(x) = rx - x^3$ giving values at local maximum and minimum

$$f(x_{max}) = 2(r/3)^{3/2} \qquad f(x_{min}) = -2(r/3)^{3/2}$$

If $|h| < 2(r/3)^{3/2}$ and $r > 0$ then there can be three fixed points otherwise there will only be one. We can define a critical function

$$h_c(r) = 2(r/3)^{3/2}$$

When $|h| < h_c(r)$ allowing three fixed points but is not zero, then the bifurcation diagram the pitchfork disconnects into two pieces, with the lowest piece resembling a saddle bifurcation and the upper fixed point separate and remaining untouched as r varies (see Figure 8).

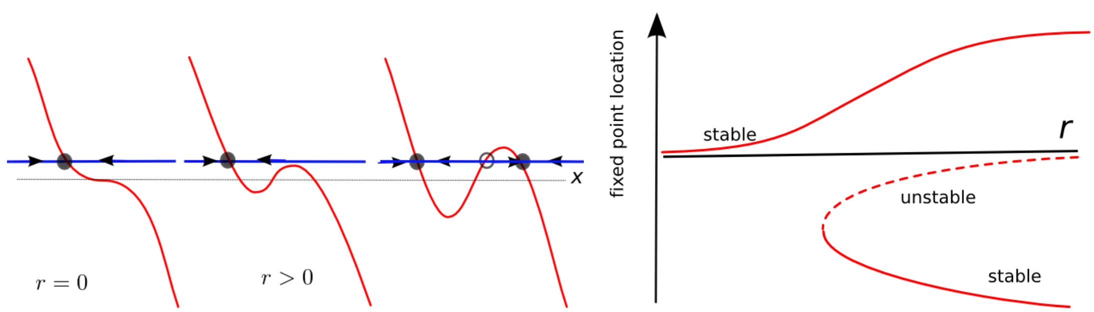

Figure 8: a) Fixed points for the dynamical system $\dot{x} = h + rx - x^3$. When $r < 0$ there is one fixed point. Not until r is equal to $3(|h|/2)^{2/3}$ are there two additional fixed points. When $h = 0$ there is a symmetric pitchfork bifurcation. When $h \neq 0$ the bifurcation is a saddle bifurcation. When $|h| < 2(r/3)^{3/2} = h_c(r)$, and $r > 0$, there are three fixed points, and the saddle bifurcation appears at $r = 3(h/2)^{2/3}$.

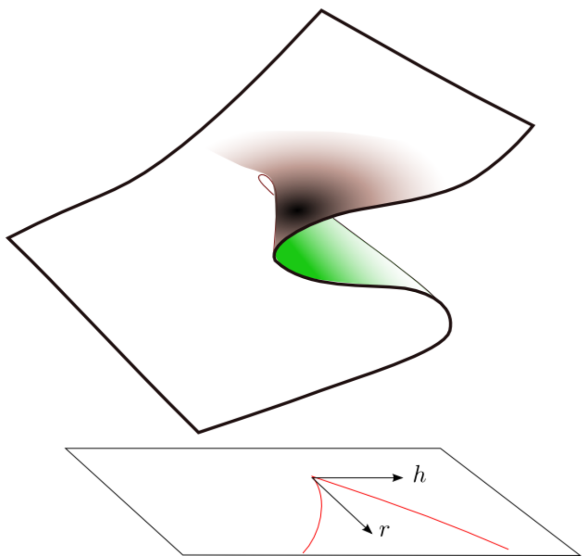

Figure 9: Another way to look at the imperfect bifurcations for the dynamical system $\dot{x} = h + rx - x^3$ as a function of two parameters h, r. Here the plane below is r vs h. Vertically we show x for the fixed points. When $h < h_c(r)$ and $r > 0$ there are three fixed points, otherwise there is only 1.

1.5 Relation to the fixed points of a simple Hamiltonian system

Recall the Hamiltonian system

$$H(J,\phi) = \frac{J^2}{2} + bJ + J^{1/2}\cos\phi$$

that is relevant for mean motion and Lindblad resonances. This system has a two dimensional phase space. It's actually a two-dimensional dynamical system. However the fixed points in this system satisfy a cubic equation. We can change variables to $x = \sqrt{2J}\cos\phi$, $y = \sqrt{2J}\sin\phi$ via cannonical transformation giving

$$H(x,y) = \frac{1}{2}(x^2+y^2)^2 + \frac{b}{2}(x^2+y^2) + \frac{x}{\sqrt{2}}$$

taking Hamilton's equations we find that fixed points occur at $y = 0$ and

$$\left.\frac{\partial H}{\partial x}\right|_{y=0} = \frac{x^3}{2} + bx + \frac{1}{\sqrt{2}} = 0$$

and satisfy a cubic equation with a linear term with free parameter b. We know we have either 1 or 3 fixed points. We would have to look at the linearized system about each fixed point to determine whether the bifurcation is sub or super critical pitchfork – recall that it was more similar to a supercritical pitchfork bifurcation because two fixed points were stable. The cubic system has a constant offset $1/\sqrt{2}$ (which we called an imperfection parameter). So the system really has a single fixed point that is separate and a saddle bifurcation appearing at negative b giving two additional fixed points.

This is a two dimensional Hamiltonian system and so is *not* like the one dimensional dynamical systems considered here. So why are we discussing it?

If we considered a damped version of our Hamiltonian system then we could imagine that trajectories would lose energy, decaying to a stable fixed point. So there can be a relation between end states (attracting orbits) of damped systems and fixed points of higher dimensional energy conserving systems.

The fixed points in this Hamiltonian system lie on a line, and so belong to a lower dimensional space and so could be similar to dynamical systems on a line. If the timescale for the orbit to decay is short compared to the timescale for r or b to vary then the orbit first decays to a fixed point and then the fixed point moves along the bifurcation diagram.

Here we can classify the type of *bifurcation* using similar terminology to the one-dimensional dynamical system, while recognizing that the system is actually higher dimensional and the dynamics more complex than a one dimensional dynamical system.

Two dimensional non-area preserving dynamical systems can exhibit a wider range of phenomena than one-dimensional Hamiltonian systems. In the one-dimensional Hamiltonian system, fixed points are either stable (and like the stable fixed point for a harmonic oscillator) or unstable (and like the unstable fixed points in a pendulum model). However in more general two-dimensional dynamical systems, trajectories can spiral in or out of fixed points, fixed points can be attractors or repellers and orbits can decay to a periodic orbit.

2 Iteratively applied maps

Instead of considering a dynamical system $\dot{x} = f(x)$ we now consider a map $f(x)$ that we apply iteratively (over and over again). We could also write

$$x_{n+1} = f(x_n)$$

Given an initial condition x_0 we can look at orbits $f(x_0), f^2(x_0), f^3(x_0)...$ where $f^2(x) = f(f(x))$ and so forth.

A fixed point is a value x^* such that

$$x^* = f(x^*)$$

For the mapping if

$$x_{n+1} = f(x_n) > x_n$$

then the value of the iteratively applied map increases. This implies that the system is increasing when $f(x) > x$ or above a line on the plane $y = x$.

This setting is different than for the one-dimensional dynamical system ($\dot{x} = f(x)$). There fixed points satisfy $f(x) = 0$ and x increases when $f(x) > 0$.

A periodic point is one such that $f^n(x) = x$. We usually label the periodic point with the lowest natural n such that $f^n(x) = x$.

2.1 Cobweb plots

It is nice to make plots called cobweb plots where we connect points

$$(x_0, x_1), (x_1, x_1), (x_1, x_2), (x_2, x_2), (x_2, x_3), (x_3, x_3)....$$

and so on. The cobweb web can show attracting (stable) or repelling (unstable) fixed points, escape to infinity, attracting periodic orbits or chaotic behavior. See Figure 10 for an example of $x_{n+1} = c + x_n^3$ for different values of c showing attracting and repelling points. Figure 11 shows an orbit spiraling into a fixed point. Figure 12 shows an attracting period two orbit.

If $f(x)$ is near but above the line $y = x$ as shown in Figure 10 b) then it make take many iterations to get through the tight narrow region between the two curves. While the system is slowly going through the narrow neck it behaves as if it were at a fixed point. This will be important later when we discuss *intermittency* or *ghost orbits*.

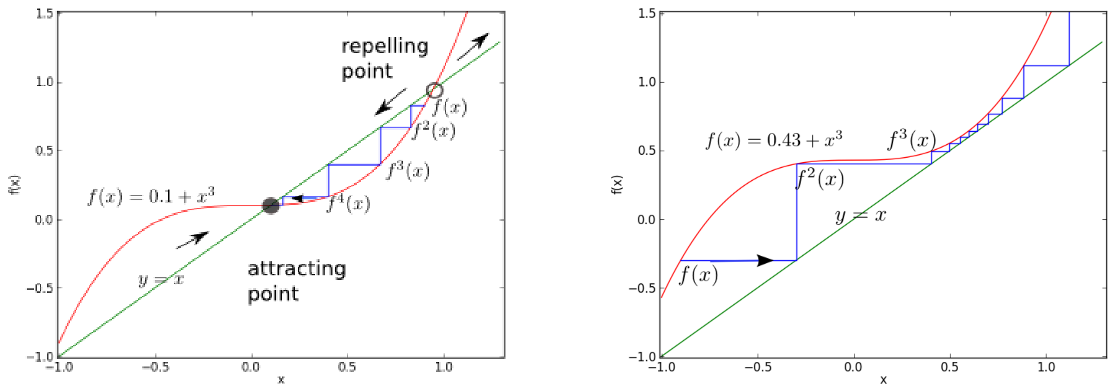

Figure 10: Graphical analysis or cobweb plots of $x_{n+1} = c + x_n^3$ a) with $c = 0.1$ and initial condition $x_0 = 0.9$. b) with $c = 0.43$ and initial condition $x_0 = -0.8$. There is a bifurcation in the behavior of the fixed points of the map that depends on the value of c. The plot on the left shows attracting and repelling fixed points. For the plot on the right, while the trajectory is passing through the narrow neck, it spends a long period of time behaving as if it were at a fixed point.

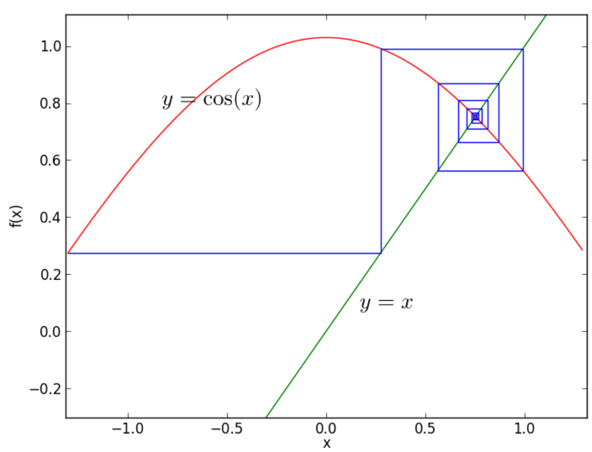

Figure 11: Cobweb plot for the mapping $x_{n+1} = \cos x_n$. Note the similarity with damped oscillations. The initial condition is $x_0 = -1.2$.

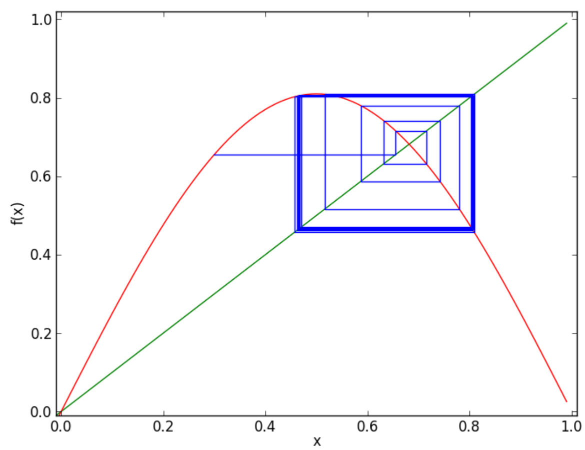

Figure 12: Cobweb plot for the mapping $x_{n+1} = \mu \sin(\pi x_n)$ and $\mu = 0.81$. We show an attracting period two orbit for initial condition $x_0 = 0.3$.

2.2 Stability of a fixed point

Suppose we have a fixed point x^*. What are the orbits like near it? We can consider $x_n = x^* + \eta$ and put this into our map

$$\begin{aligned} f(x^* + \eta) &= f(x^*) + f'(x^*)\eta + .. \\ &= x^* + f'(x^*)\eta + .. \end{aligned}$$

So $x_{n+1} - x^* = f'(x^*)\eta$. Which can be rewritten

$$\eta_{n+1} = f'(x^*)\eta_n$$

In this way $f'(x^*)$ is a multiplier. If $|f'(x^*)| < 1$ then the fixed point is stable and if $|f'(x^*) > 1$ then the fixed point is unstable. If $f'(x^*) = 1$ then higher order terms determine the stability.

2.3 The Logistic map

The famous logistic map is the innocent looking map

$$x_{n+1} = rx_n(1 - x_n)$$

typically defined over $0 \leq x_n \leq 1$ and $0 \leq r \leq 4$. The logistic map is motivated by a simple model for population growth that can be applied to simple systems such as colonies of bacteria in the lab. A population that is given infinite amount of food would grow as $\dot{N} = rN$ giving exponential growth. Exponential growth cannot continue, so when N reaches some limit (due to overcrowding), the population shrinks. This gives a model $\dot{N} = rN(1-N/K)$ where K is a parameter describing the death rate due to overpopulation. The logistic map is the discrete analog of the logistic dynamical system model.

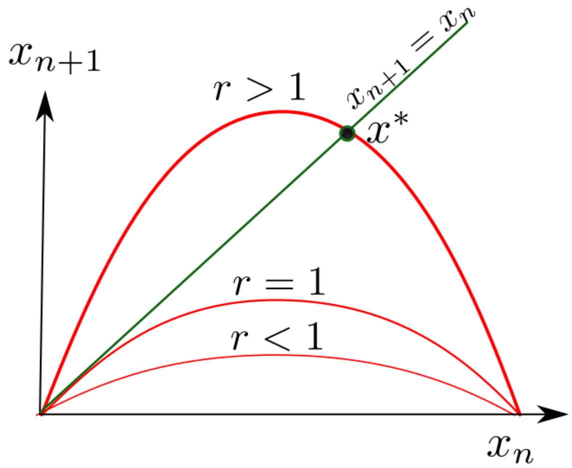

Figure 13: The logistic map has a single fixed point at the origin for $r < 1$. At $r = 1$ the fixed point at the origin bifurcates and there are two fixed points, one unstable (at the origin) and the other stable. The absolute value of slope at the stable fixed point increases as r increases. At large enough r (near $r = 3$) the slope at the fixed is steeper (lower) than -1 and the fixed point becomes unstable.

if we set $x_{n+1} = x_n$ and solve we find that $rx_n(1 - x_n) = x_n$ for fixed points. There is always a fixed point at the origin. There is an additional fixed point at $x^* = 1 - 1/r$. Constrained over $0 \leq x_n \leq 1$ and $0 \leq r \leq 4$, this additional fixed point only exists for $r \geq 1$. The derivative $f'(x_n) = r - 2rx_n$. For the fixed point at the origin $f'(x^* = 0) = r$ so it is stable for $r < 1$ and unstable at $r \geq 1$. At $x^* = 1 - 1/r$ the derivative $f'(x^*) = 2 - r$. For $1 < r < 3$ the $|f'(x^*)| < 1$ and the fixed point is stable. At $r = 1$ the fixed point at the origin bifurcates. At $r = 3$ the second fixed point becomes unstable!

2.4 Two-cycles and period doubling

The bifurcation at $r = 3$ is called a *flip* or *period-doubling* bifurcation. The fixed point spawns a period 2 cycle.

Let us look for solutions to $f^2(x) = x$ where $f^2(x) = f(f(x))$.

$$f^2(x) = r^2 x(1-x)(1 - rx + rx^2) = r^2(x - x^2)(1 - rx + rx^2)$$

This is a 4th order polynomial. The roots of $f^2(x) - x = 0$ include $x = 0, x = 1 - 1/r$, the previously known fixed points. There are two additional roots

$$x_2 = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

that are real for $r > 3$. So the logistic map has a two-cycle for all $r > 3$. It is convenient to label these roots as p, q. It's a two cycle so we know that $f(p) = q$ and $f(q) = p$.

Let us look at the stability of the two cycle where $f^2(x_2) = x_2$. Let us look at one of the points

$$f^2(p + \eta) = f(f(p + \eta) = f(f(p)) + f'(f(p))f'(p)\eta = p + f'(q)f'(p)\eta$$

Note the symmetry. We can evaluate $f^2(q+\eta) = q + f'(p)f'(q)\eta$. The p, q branches become unstable together.

As the original fixed point became unstable and spawned a 2-cycle, the 2-cycle becomes unstable and will span a 4-cycle.

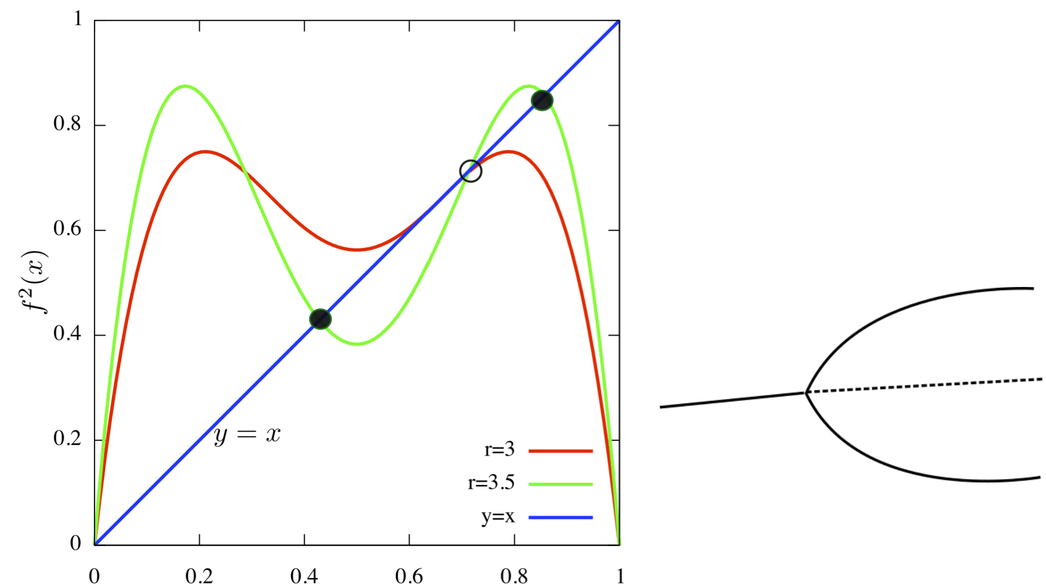

Figure 14: The map $f^2(x)$, the logistic map applied twice, for $r = 3$ and $r = 3.5$. At $r = 3.5$ there are three fixed points of the map, corresponding to two-cycles in the map $f(x) = rx(1 - x)$. Only the outer two of these are stable. This illustrates the period doubling bifurcation. Right hand side shows the bifurcation diagram with the bifurcation occurring at $r = 3$. Recall that the fixed point at origin became unstable at $r = 1$.

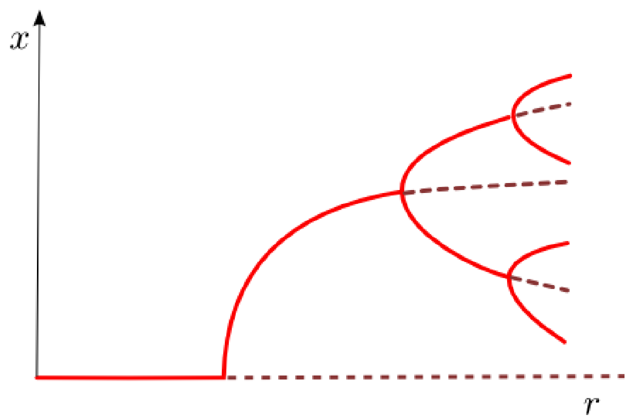

Figure 15: Bifurcation diagram for part of the logistic map ($f(x) = rx(1-x)$) showing both stable periodic and unstable periodic points. This region shows fixed points, period two and four orbits. Fixed point at origin becomes unstable at $r = 1$. The attracting fixed point for $r > 1$ becomes unstable to period doubling at $r = 3$. The period two attracting orbits become unstable at $r \sim 3.4$. Renormalization depends on two ratios: the ratio of the distance in r between period doublings (Feigenbaum's δ) and the ratio of the tine widths (Feigenbaum's α). Both ratios are computed in the limit of large number of doubling.

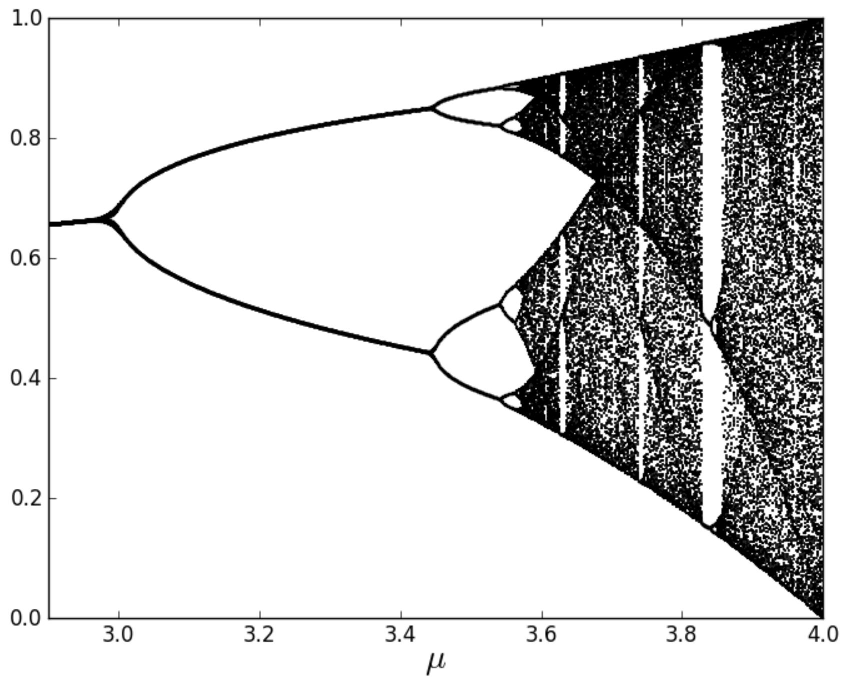

Figure 16: For different values of r, the 100-200th iterations of the logistic map, $f(x) = rx(1-x)$, are plotted starting with initial $x_0 = 0.5$. Note the period three window at about $r = 3.8$, and the period doubling at $r = 3, 3.4 - 3.6$.

2.5 Sarkovskii's Theorem and Ordering of periodic orbits

Theorem 2.1 *Sarkovskii's theorem: If a continuous function of the real numbers has a periodic point with prime period three then it has periodic points of all prime periods.*

'Prime period' three here means not prime as in prime number but prime period as in it is not a fixed point of the map $f(x)$ but is a fixed point of $f^3(x)$. An orbit of prime period n is a fixed point for $f^n(x)$ but not a fixed point for $f^m(x)$ for smaller m any natural number $1 \geq m > n$.

Definition Sarkosvskii's ordering for natural numbers:

$$3 \succ 5 \succ 7 \cdots\cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \cdots\cdots$$
$$\cdots 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \cdots\cdots 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \cdots$$
$$\cdots 2^3 \succ 2^2 \succ 2 \succ 1$$

Odd numbers come first, then two times every odd number, then 4 times every odd number, and so on. Then after all the 2^n times odds the sequence is completed by listing powers of two in inverse order.

A more general theorem by Sarkovskii: Suppose f has a periodic point with period prime period n. If $n \succ m$ in Sarkovskii's ordering, then there is also a periodic orbit with prime period m.

There is a nice proof of this theorem in Holmgren's book that is long enough that I am not trying to recount it here.

The ordering inspires one to look again at the periodic attractors of the logistic map. The ordering by Sarkovskii does not predict a progression of periodic orbits, just existence of periodic orbits. The order in which the periodic windows arise is explained by Metropolis, Stein, and Stein and is called an MSS or U-sequence and is valid for a wide class of *unimodal* maps.

Definition A *unimodal* map is a differentiable map from the unit interval onto the unit interval $f : [0, 1] \to [0, 1]$ with boundary conditions $f(0) = f(1) = 0$ that has a single maximum.

The logistic map is a *unimodal* map. The word unimodal is more commonly used for probability distributions. For example, a bimodal distribution has two peaks, rather than one.

The appearance of periodic orbits occurs in two ways. A fixed point can become unstable undergoing a period doubling bifurcation. The fixed point in f^2 can then undergo a period doubling, and so on giving a cascade of period doublings.

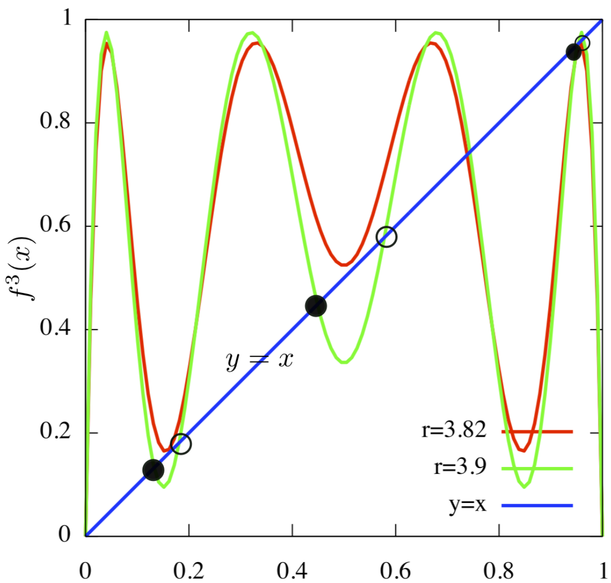

Figure 17: The map $f^3(x)$ for r below and above the location where the period three orbits appear in the logistic map. There are 6 period three points (where $f^3(x)$ crosses $y = x$), but only 3 of these are stable. The function $f^3(x)$ crosses $y = x$ at an additional point that is a fixed point for $f(x)$ and is not a prime period three point. This illustrates a *tangent* bifurcation giving birth to a period three cycles. The fixed point at $x \sim 0.75$ is a period 1 fixed point.

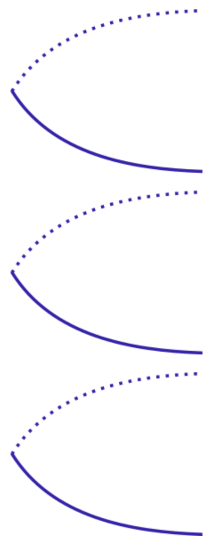

Figure 18: b) Bifurcation diagram for the appearance of period 3 orbits in the logistic map. This is known as a *tangent* bifurcation. The bifurcation happens at about $r = 3.82$.

The period three cycles do not appear this way. They appear with what is known as a *tangent* bifurcation. Each tangent bifurcation produces a stable cycle, that then also gives rise to a cascade of period-doubling bifurcations.

2.6 A Cantor set for $r > 4$

When $r > 4$ the logistic map sends some values of x to values greater than 1. Once an orbit reaches a value greater than 1, the next application of the map gives a negative number which is then sent to negative infinity as there are no fixed points for $x < 0$.

Definition A set $\Gamma \in R$ is called a *Cantor* set if

a) Γ is closed and bounded. (Sets of real numbers like this are called compact. Bounded means there is exists a positive number that is larger than all the elements in the set).

b) Γ contains no intervals. It is totally disconnected.

c) Every point in Γ is an accumulation or limit point of Γ.

Definition Let S be a subset of the real numbers. Then $x \in S$ is an *accumulation* or *limit* point in S if every neighborhood of x contains an element of S other than x.

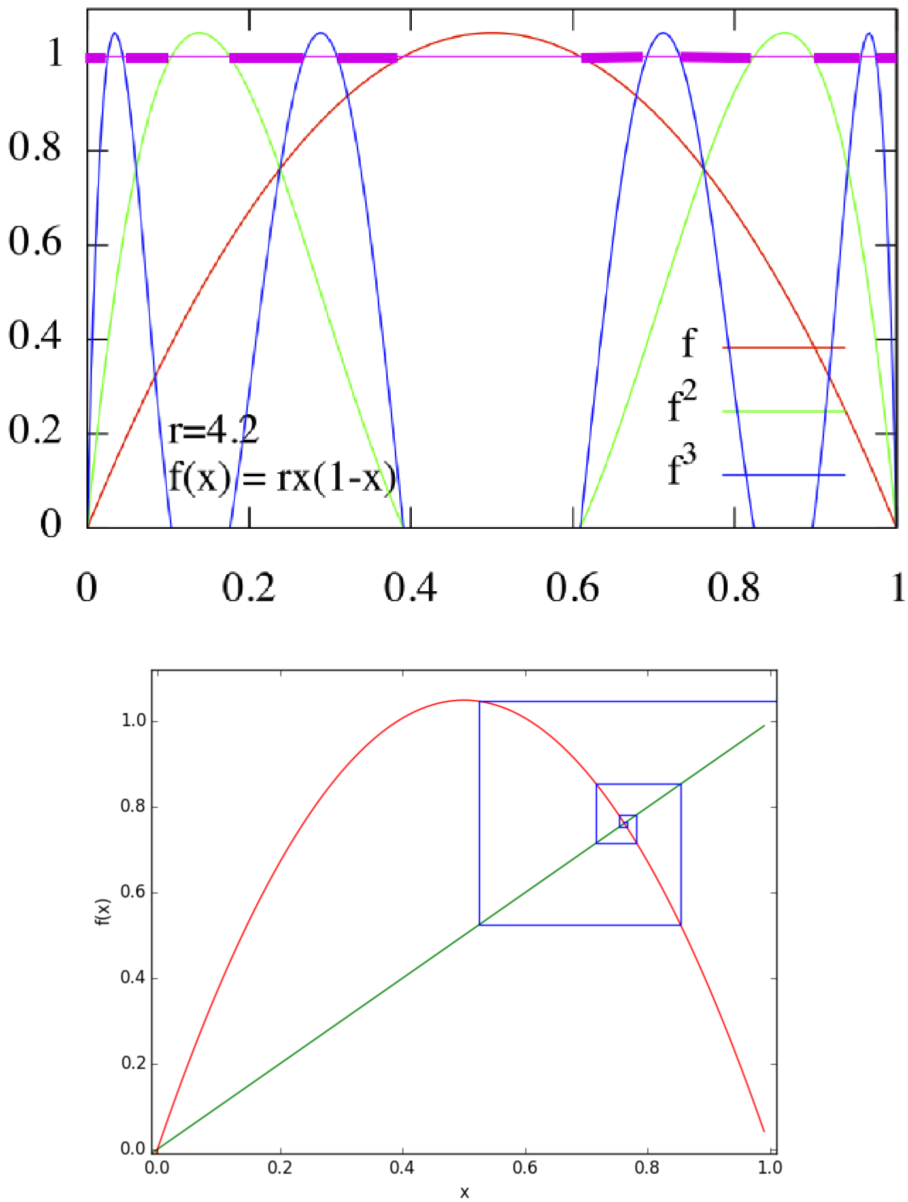

Figure 19: Top: The map $f(x)$, $f^2(x)$, $f^3(x)$ for the logistic map $f(x) = rx(1-x)$ and with $r = 4.2$. After a single application of the map a central region is removed. After a second application of the map, central regions from the two remaining regions are removed, leaving four regions. After a third application of the map, central regions from the four remaining regions are removed, leaving eight regions. The continued application of the map creates a fractal set similar to the Cantor middle thirds set. Points above $y = 1$ are sent to negative infinity and this is seen in the cobweb plot in the bottom figure that has initial condition $x_0 = 0.76$, and $r = 4.2$. This point is in the removed island of the f^3 map.

Consider the set of points in the unit interval that are left in the unit interval after repeated applications of the logistic map with $r > 4$. As is true for the Cantor set, we can construct the set iteratively. Denote Γ_1 as the set of points that are left in the unit interval after a single application of the logistic map. This set consists of two intervals and is missing the region near $x = 1/2$ (see Figure 19). Γ_2 is the set of points left after application of f^2, and so on, Γ_n is the set of points left in the unit interval after application of f^n. We can define the set $\Gamma = \cap_{i=1}^{n} \Gamma_n$.

To show that Γ is a Cantor set, it is useful to consider points on the end-points of intervals. The points $x = 0$ and 1 always remain in the unit interval. But so do the ends of any of the intervals of any of Γ_n. Because all the ends of intervals in each of Γ_n are in Γ, we can find a series of points that lead to (get closer to) any point. That gives part c of the definition for a Cantor set. For any separation defining an interval, we can always find an f^n that removes part of the interval. This gives part b of the definition. Part a) follows because the intersection of closed bounded sets is closed and bounded.

3 Lyapunov exponents

The logistic map exhibits aperiodic orbits that do not converge to a fixed point or to a periodic point. How is this chaotic? A chaotic orbit has extreme sensitivity to initial conditions. Orbits separate *exponentially* quickly. In other words, let us consider an initial condition x_0 and a nearby point $x_0 + \delta_0$. The separation between the two orbits at each iteration

$$\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$$

For the orbits to diverge exponentially

$$|\delta_n| \approx |\delta_0| e^{n\lambda}$$

where λ is called the Liapunov exponent. A positive Liapunov exponent is a signature of chaos. We can approximate

$$\begin{aligned}
\lambda &\approx \frac{1}{n} \ln \frac{|\delta_n|}{|\delta_0|} \\
&= \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \\
&= \frac{1}{n} \ln \left| \frac{d}{dx} f^n(x_0) \right|
\end{aligned}$$

We can expand the last expression. Consider

$$\frac{d}{dx} f^3(x_0) = \frac{d}{dx} f(f(f(x_0))) = f'(f(f(x_0))) f'(f(x_0)) f'(x_0) = f'(x_2) f'(x_1) f'(x_0)$$

so

$$\frac{d}{dx} f^n(x_0) = \prod_{i=0}^{n-1} f'(x_i) \tag{5}$$

and

$$\begin{aligned} \lambda &\approx \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \end{aligned}$$

We can take the limit as $n \to \infty$ and define the Lyapunov exponent for initial condition x_0

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right|$$

The exponent depends on x_0. The Lyapunov exponent should be the same for all points in a basin of a single attractor. For stable fixed points or cycles the exponent should be negative. In some cases λ can be calculated analytically.

Equation 5 can also be partly expanded, for example into multiples of three

$$\frac{d}{dx} f^n(x_0) = \prod_{i=0}^{n/3-1} \frac{d}{dx} f^3(x_{3i})$$

or into multiples of p

$$\frac{d}{dx} f^n(x_0) = \prod_{i=0}^{n/p-1} \frac{d}{dx} f^p(x_{pi}) \tag{6}$$

At a p cycle, since the sequence is periodic, all the terms in equation 6 are the same.

$$\frac{d}{dx} f^n(x_0) = \frac{n}{p} \frac{d}{dx} f^p(x_0)$$

Inserting this into the expression for the Lyapunov exponent The Lyapunov exponent for a p periodic point is

$$\lambda = \frac{1}{p} \ln \left| \frac{d}{dx} f^p(x_0) \right|$$

Recall that a stable p cycle is one that has $\left|\frac{d}{dx} f^p(x_0)\right| < 1$ and so has $\ln\left|\frac{d}{dx} f^p(x_0)\right| < 0$. Consequently a stable p cycle has negative Lyapunov exponent.

Remark Supercritical points that are periodic points $\frac{d}{dx} f^p(x) = 0$ have Lyapunov exponents going to negative infinity. These special points are used in renormalization because the function f^p locally looks parabolic.

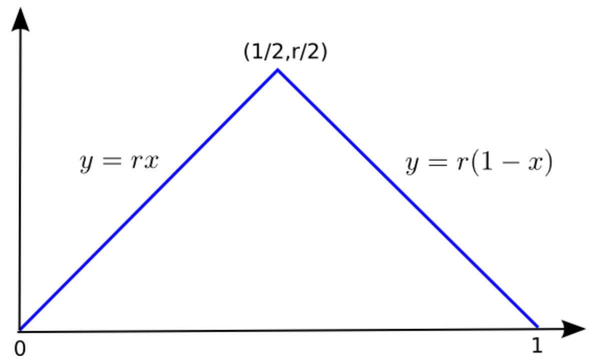

Figure 20: The tent map has a positive Lyapunov exponent.

3.1 The Tent Map

The tent map is

$$f(x) = \begin{cases} rx, & 0 \leq x \leq \frac{1}{2} \\ r(1-x), & \frac{1}{2} \leq x \leq 1 \end{cases}$$

for $0 \leq r \leq 2$ and $0 \leq x \leq 1$. This map has the advantage that it is very easy to compute the Lyapunov exponent. $|f'(x)| = r$ for all x so $\lambda = \ln r$.

The tent map can be said to be conjugate to the logistic map. In fact for $r = 2$ it can be transformed into the logistic map with parameter $r = 4$. There is a problem in the accompanying problem set exploiting this transformation to compute the Lyapunov exponent of the logistic map for $r = 4$.

3.2 Computing the Lyapunov exponent for a map

It is straight forward to compute the Lyapunov exponent numerically for a map such as the logistic map. Start with an initial condition and iterate for a few hundred iterates to allow transients to disappear. Next compute a large number (a few hundred more) additional iterates. From these iterates compute the average of the log of the slope or $\ln |f'(x_n)| = \ln |r - 2rx_n|$.

Instead of doing this for the logistic map we will do it for the map that has

$$x_{n+1} = r - x_n^2$$

and derivative $f'(x) = -2x$ and the result is shown in Figure 21. Not surprisingly there is a strong relation between the value of the computed Lyapunov exponent and the structure of the attracting orbits.

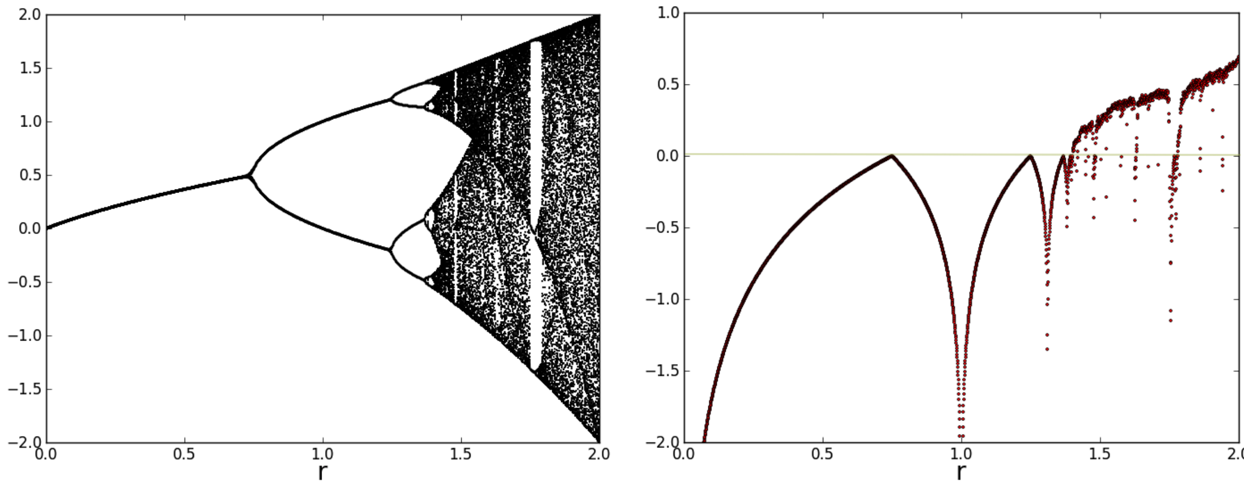

Figure 21: Attracting points (on left) and Lyapunov exponent (on right) computed numerically for the map $x_{n+1} = r - x_n^2$. System is chaotic where the Lyapunov exponent is greater than zero.

3.3 Entropy

The logarithmic form of the Lyapunov exponent is reminiscent of entropy! A positive Lyapunov exponent implies there is a loss of information about initial conditions. The loss of information means the system is becoming randomized, or entropy is increasing. Chaotic behavior is a source of the random-like behavior that is the heart of statistical physics.

3.4 The Maximum Lyapunov Exponent

In the previous example we discussed the Lyapunov exponent for a one-dimensional map. We can also consider similar exponential divergence but for multidimensional maps and for multidimensional dynamical systems.

We need to have a way of defining distance (a metric). Consider a trajectory $\mathbf{q}(t)$ with initial condition $\mathbf{q}_0$ and a nearby trajectory $\mathbf{q}^b(t)$ with initial condition $\mathbf{q}_0 + \delta\mathbf{q}_0$. We can look at the distance between the two trajectories

$$\|\delta\mathbf{q}(t)\| = \|\mathbf{q}(t) - \mathbf{q}^b(t)\|$$

Suppose that the distance between the two trajectories diverges so that

$$\|\delta\mathbf{q}(t)\| \sim \|\delta\mathbf{q}_0\| e^{\lambda t}$$

We can define a Maximum Lyapunov Exponent as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left[\frac{\|\delta \mathbf{q}(t)\|}{\|\delta \mathbf{q}_0\|} \right] \tag{7}$$

Note that the exponent is in units of inverse time.

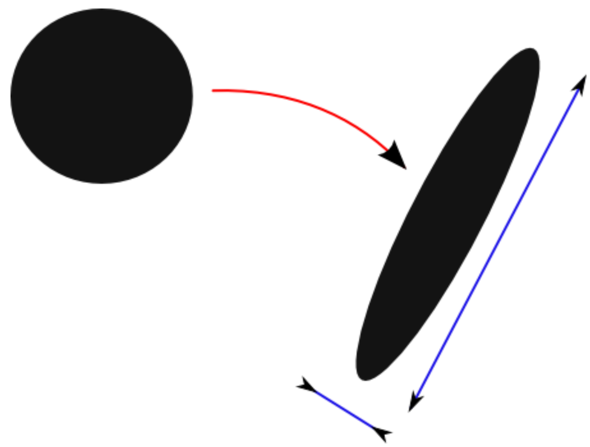

Figure 22: An initial phase space element is stretched in one direction but compressed in another. In a Hamiltonian system the area is conserved. In a dissipative system area decreases, but there may still be a direction where the element is stretched. A strange attractor is such a setting. The stretch rate gives the maximum Lyapunov exponent.

The exponent is denoted the *maximum* Lyapunov exponent. Why maximum? We could consider the divergence rate for each component $|\delta q_i(t)|$ separately and we could consider the divergence rate (for each of these components) for each direction for the initial separation $\delta q_{0,j}$. For each ij we can look at the function $\delta q_i(t, \delta q_{0,j})$ This gives us a matrix and the eigenvalues of this Jacobian matrix can be turned into Lyapunov exponents.

The divergence of the distance between the trajectories for a randomly chosen direction for $\delta \mathbf{q}_0$ is dominated by the maximum eigenvalue or exponent. The number of positive Lyapunov exponents indicates the number of directions in phase space along which the orbit exhibits chaotic or hyperbolic behavior.

Remark The Lyapunov exponent measures a timescale $T_L = \lambda^{-1}$ for memory of initial conditions to be lost.

Remark The Lyapunov exponent does not describe how far apart trajectories diverge, only how quickly they diverge.

3.5 Numerical Computation of the Maximal Lyapunov Exponent

What is meant by equation 7 is actually

$$\lambda = \lim_{\delta q_0 \to 0} \lim_{t \to \infty} \frac{1}{t} \ln \left[\frac{\|\delta \mathbf{q}(t)\|}{\|\delta \mathbf{q}_0\|} \right]$$

However it is impractical to set $\delta \mathbf{q}_0$ too small. After integrating for a while the two orbits may diverge sufficiently to nearly reach boundaries of the system (real systems don't tend to be infinitely big) or will grow to such a large value that computation is impractical (remember it's diverging exponentially).

We describe a numerical procedure for computing the maximal Lyapunov exponent. Start with an initial condition $\mathbf{q}$ and and a randomly chosen small offset $\delta \mathbf{q}_0$. Integrate both trajectories for a time T, long enough to measure some divergence but not long enough to reach a large size scale. Let

$$s_1 = \frac{\|\delta \mathbf{q}(T)\|}{\|\delta \mathbf{q}_0\|}$$

Now let us start with a new offset

$$\delta \mathbf{q}_1 = \frac{\delta \mathbf{q}(T)}{s_1}$$

Now we integrate again for time T and renormalize again with s_2. For each renormalization record the factors $s_1, s_2 ....$ Benettin et al.(1980) showed that the maximum Lyapunov exponent is equivalent to

$$\lambda = \frac{1}{nT} \lim_{n \to \infty} \sum_{j=1}^{n} \ln s_j$$

In practice one cannot integrate forever. How large an n is required before this gives a good approximation to the maximum Lyapunov exponent? Define an exponent that depends on how long on has been integrating

$$\lambda(nT) = \frac{1}{nT} \sum_{j=1}^{n} \ln s_j$$

and then plot it as a function of nT. As nT (the length of time integrated) increases, the estimate improves and converges to a constant value.

In practice the function $\lambda(nT)$ can be sensitive to initial conditions, intermittency, nearby resonances and can oscillate in time, making it difficult to get an accurate estimate of a long time scale limit. Also it is not obvious how to choose the timescale T. This has led to development of numerical indicators for chaotic behavior though these too involve adjustment of the scheme.

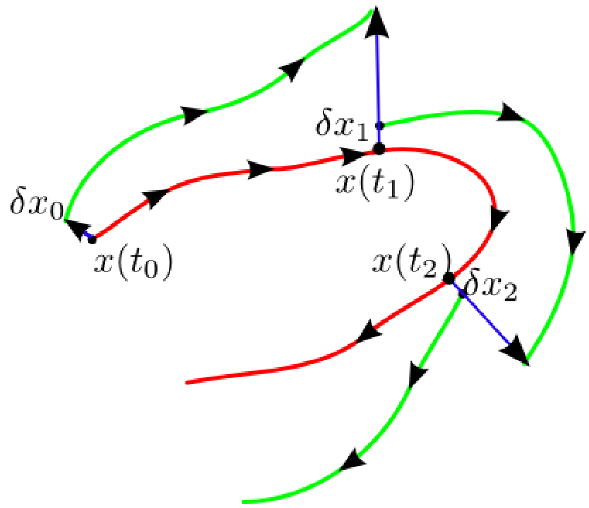

Figure 23: How to numerically calculate a maximum Lyapunov exponent. The separation is occasionally rescaled to keep the trajectory separation in a linearized flow regime.

3.6 Indicators related to the Lyapunov exponent

An indicator can be computed numerically that is different in chaotic systems than non-chaotic systems. The indicator may be loosely related to the maximum Lyapunov exponent. For example, the *Mean Exponential Growth factor of Nearby Orbits* (MEGNO) by Cincotta & Simó 2002, estimates the divergence rate of nearby orbits weighted by time rather than just computing the divergence rate.

The distance between two trajectories as a function of time

$$\delta(t) = \frac{d}{dt}\delta(t)dt$$

If we assume that $\delta(t) = e^{\lambda t}\delta_0$ then

$$\dot{\delta}(t) = \lambda\delta(t) \qquad \text{and} \qquad \lambda = \frac{\dot{\delta}(t)}{\delta(t)}$$

We can write the maximal Lyapunov exponent in integral form as

$$\lambda = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\dot{\delta}(t')}{\delta(t')}dt'$$

The quantity

$$Y(t) = \frac{2}{t}\int_0^t \frac{\dot{\delta}(t')}{\delta(t)}t'\ dt'$$

is known as the Mean Exponential Growth factor of Nearby Orbits or the MEGNO. While the Lyapunov exponent has units of inverse time Y is unit less. For a quasi periodic trajectory $\delta(t)$ grows linearly with time. Suppose $\delta(t) = at$, then $\dot{\delta}(t')t'/\delta(t') = 1$ and $Y(t)$ oscillates about a fixed value of 2. For a chaotic trajectory $\delta(t) = \delta_0 e^{\lambda t}$ we find $\dot{\delta}(t')t'/\delta(t') = \lambda t'$ The integral of this with respect to t' is $t^2/2$ so $Y(t)$ grows giving $Y(t)$ oscillating about λt. The time running average

$$\bar{Y}(t) = \frac{1}{t}\int_0^t Y(t')dt'$$

gives 2 for a quasi periodic trajectory and $\frac{\lambda t}{2}$ for a chaotic trajectory.

At first glance the time average of the Mean Exponential Growth factor of Nearby Orbits seems to be equivalent to the maximum Lyapunov exponent. However, the MEGNO could give a more accurate estimate for the Lyapunov time more quickly as it weights later times in the divergence of the trajectories more heavily.

Also popular is the *Fast Lyapunov Indicator* introduced by Froeschlé et al. 1997. The Fast Lyapunov Indicator is the length of time it takes two trajectories to diverge reaching a chosen value for separation R.