PHY256 Lecture notes: Introducing Quantum Systems, Measurement and the Qubit!

A. C. Quillen

April 19, 2021

Contents

1 Introduction

The goal of these notes is not to teach quantum mechanics but introduce the frame work for quantum mechanics well enough that we can explore concepts of quantum computing, quantum information and quantum simulation. Quantum computers are now a reality, and they continue to be improved. A significant fraction of research effort in physics is now devoted to various aspects of quantum computing. They have inspired new ideas in information theory and new types of algorithms. Quantum computers may turn out to be useful.

2 Basis vectors and quantum states

A Hilbert space is a complex vector space that has an inner product. We can describe a quantum state $\langle\psi|$ or $|\psi\rangle$ with a vector of unit length in a Hilbert space. With an orthonormal basis for the Hilbert space with basis vectors $\langle a_i|$, (satisfying $\langle a_i|a_j\rangle = \delta_{ij}$) we can write

$$
\begin{aligned}
|\psi\rangle &= \sum_i c_i \, |a_i\rangle \\
\langle\psi| &= \sum_i \langle a_i| \, c_i^*.
\end{aligned}
$$

Here the c_i are complex numbers. The wave vector or quantum state is normalized;

$$\langle\psi|\psi\rangle = \sum_i c_i c_i^* = 1.$$

The number $c_i c_i^*$ is associated with the probability to be in state $|a_i\rangle$. Quantum mechanics postulates that the world is described with probabilities of knowing something rather than absolutely knowing the position or state an object is in.

The ket or $|\psi\rangle = \sum_i c_i |a_i\rangle$ can also be written as a vector

$$|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ c_{N-1} \end{pmatrix}$$

where N is the dimension of the Hilbert space. The bra or $\langle\psi|$ can also be written as vector

$$\langle\psi| = \begin{pmatrix} c_0^* & c_1^* & c_2^* & ...c_{N-1}^* \end{pmatrix}$$

A linear operator on $|\psi\rangle$ can be written as a matrix. For example in a two state system a linear operator $\mathbf{A}$

$$\mathbf{A}|\psi\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} A_{00}c_0 + A_{01}c_1 \\ A_{10}c_0 + A_{11}c_1 \end{pmatrix}$$

with $|\psi\rangle = c_0|a_0\rangle + c_1|a_1\rangle = \sum_k c_k |a_k\rangle$. We can also write

$$\mathbf{A} = \sum_{ij} A_{ij} |a_i\rangle \langle a_j|$$

giving

$$\begin{aligned} \mathbf{A}|\psi\rangle &= \sum_{ij} A_{ij} |a_i\rangle \langle a_j| \sum_k c_k |a_k\rangle \\ &= \sum_{ijk} A_{ij} |a_i\rangle c_k \delta_{jk} = \sum_{ij} A_{ij} c_j |a_i\rangle . \end{aligned}$$

A Hamiltonian matrix is a **Hermitian** or **self-adjoint** matrix. A Hermitian matrix satisfies $\mathbf{A}^\dagger = \mathbf{A}$ where the dagger means taking both the transpose and complex conjugate of the matrix. A Hermitian matrix is equal to the complex conjugate of its transpose, it is diagonalizable and it has real eigenvalues.

Observables or measurements are represented by Hermitian matrices.

Hamiltonian evolution gives **unitary** evolution of the wave vector. However the Hamiltonian is not a unitary matrix, it is Hermitian.

A unitary matrix $\mathbf{U}$ satisfies $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$ with $\mathbf{I}$ the identity. A unitary transformation preserves the norm of the wave or state vector ψ. In other words, with $\langle\psi|\psi\rangle = 1$ and $|\psi'\rangle = \mathbf{U}|\psi\rangle$ and $\mathbf{U}$ unitary, then $\langle\psi'|\psi'\rangle = 1$.

Hamiltonian evolution is done using the exponential of the Hamiltonian via

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(t=0)\rangle$$

as this is consistent with Schroedinger's equation

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle$$

In a particular basis, the coefficients of $|\psi\rangle$, (c_i) are directly related to probabilities, with the probability of being in state $|a_i\rangle$ equal to $p_i = c_i c_i^*$. Unitary evolution is not random but deterministic. The coefficients are not randomly varied but updated using a Hamiltonian and evolved using unitary evolution.

The expectation value of an observable $\mathbf{A}$

$$\langle \mathbf{A}\rangle = \langle\psi|\mathbf{A}|\psi\rangle$$

In the orthonormal basis $|a_i\rangle$, the matrix $\mathbf{A}$ has coefficients

$$A_{ij} = \langle a_i|\,\mathbf{A}\,|a_j\rangle$$

and

$$\mathbf{A} = \sum A_{ij}\,|a_i\rangle\,\langle a_j|$$

The expectation value of an observable should be a real number and this implies observables should be Hermitian as Hermitian matrices have real eigenvalues.

For state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, aa_* can be interpreted as the probability that the state is in the $|0\rangle$ state and bb^* the probability that the state is in the $|1\rangle$ state. Expectation values are mean values of a system that is measured many times.

We can write a matrix in the form $\sum A_{ij}\,|a_i\rangle\,\langle a_j|$ as this gives a linear map taking a vector to another vector in our Hilbert space.

A projection operator $\mathbf{P}$ satisfies $\mathbf{P}^2 = \mathbf{P}$.

The probability p_i, that a wave vector ψ is in state $|a_i\rangle$ can be computed with the projection operator

$$\mathbf{P}_i = |a_i\rangle\,\langle a_i|$$

with $\psi = \sum_i c_i\,|a_i\rangle$,

$$\begin{aligned}
\langle\psi|\mathbf{P}_i|\psi\rangle &= \sum_{jk}\langle a_j|\,c_j^*\;\mathbf{P}_{i,jk}\;c_k\,|a_k\rangle \\
&= \sum_{ij}\langle a_j|\,c_j^*\;\;|a_i\rangle\,\langle a_i|\;\;c_k\,|a_k\rangle \\
&= \sum_{ij} c_j^*\delta_{ji}\delta_{ki}c_k \\
&= c_i^* c_i = p_i.
\end{aligned}$$

3 The qubit

A two state quantum system can be described with wave-vector

$$\begin{aligned} |\psi\rangle &= a\,|0\rangle + b\,|1\rangle \\ \langle\psi| &= \langle 0|\,a^* + \langle 1|\,b^* \end{aligned}$$

where $aa^* + bb^* = 1$ and $|0\rangle$ and $|1\rangle$ are the two energy states. You can also think of a particle with spin up and spin down as a two energy state object. Here we take $|0\rangle = |\uparrow\rangle$ and $|1\rangle = |\downarrow\rangle$. This is a building block for many quantum computer designs and is called a qubit or q-bit.

In vector notation the first state and second states are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1}$$

Coefficients a, b are both complex numbers, but the vector is normalized so that

$$aa^* + bb^* = 1.$$

A state ψ is described by two complex numbers a, b. Each one has a real and a complex part. This gives a four dimensional space. As $\langle\psi|\psi\rangle = 1$, the sum

$$\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2 + \mathrm{Re}(b)^2 + \mathrm{Im}(a)^2 = 1$$

restricting the space to a 3d spherical surface in this 4d space.

3.1 The Bloch Sphere

It is convient to define

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ |i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \\ |-i\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) \end{aligned}$$

These four and with $|0\rangle, |1\rangle$ are 6 equidistant poles on the Bloch sphere.

Global phase is unobservable whereas *relative phase* is observable. Global means we can multiply the entire state by $e^{i\alpha}$ and we not see any difference in our measurements.

The two states $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ have different *relative* phases. We could rotate the states and then measure their spin. We would measure different spin values in these rotated states. Relative phase is observable.

By adjusting the global phase we can take a wave vector

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$

and write it as

$$|\psi'\rangle = a'\,|0\rangle + b' e^{i\phi}\,|1\rangle$$

with a', b' both real and $a' \geq 0$. We do this by multiplying by $a^*/|a|$.

The map onto the Bloch sphere

$$|\psi\rangle \xrightarrow{\boldsymbol{\pi}} |\psi'\rangle$$

with map $\boldsymbol{\pi}$

$$a, b \xrightarrow{\boldsymbol{\pi}} a', b'$$

given by

$$\begin{aligned} a' &= |a| \\ b' &= \frac{|ba^*|}{|a|} \\ \phi &= \operatorname{arctan2}(\operatorname{Im}(ba^*), \operatorname{Re}(ba^*)) \end{aligned}$$

The map is a projection as $\boldsymbol{\pi}^2 = \boldsymbol{\pi}$. In other words $\boldsymbol{\pi}(\boldsymbol{\pi}(|\psi\rangle)) = \boldsymbol{\pi}(|\psi\rangle)$.

The wave vector has a norm of 1 so we we can find an angle θ with

$$|\psi'\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\phi}\,|1\rangle\,. \tag{2}$$

The factor of 2 within the cosine and sine lets us associate θ with a co-latitude on a sphere. The angles

$$\theta \in [0, \pi] \qquad \phi \in [0, 2\pi)$$

with ϕ acting like a longitude. With $\theta \in [0, \pi]$ the factor $\cos(\theta/2)$ ranges from 1 to 0. This means we have chosen a global phase that keeps a' positive!

We can describe the wave vector as a point on the sphere, or with unit vector

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{3}$$

This is known as the Bloch sphere. The projection $\boldsymbol{\pi}$ to the Bloch sphere can also be written as a map from complex numbers a, b to angles θ, ϕ,

$$a, b \xrightarrow{\boldsymbol{\pi}} \theta, \phi$$

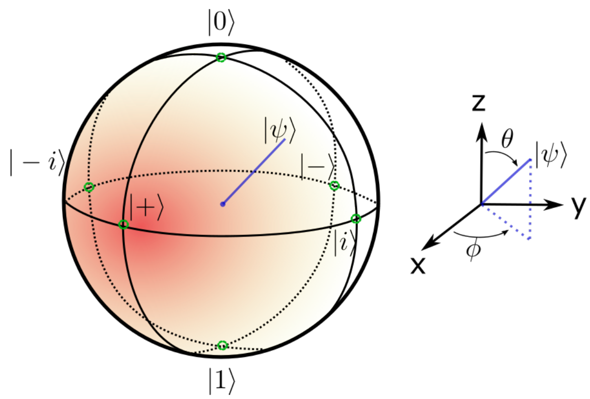

Figure 1: The Bloch sphere.

given by

$$\frac{\theta}{2} = \arctan2(b', a') = \arctan2(|b|, |a|)$$
$$\phi = \arctan2(\mathrm{Im}(ba^*), \mathrm{Re}(ba^*)).$$

A general qubit state $|\psi\rangle$ depends on two complex numbers, giving 4 degrees of freedom, when counting each real and complex part of the two complex numbers. However $\langle\psi|\psi\rangle = 1$ reduces the degrees of freedom by 1. There is a redundant phase. If we drop it by projecting onto the Bloch sphere, then this reduces the dimension again. This is why the Bloch sphere is a 2-d object, and is a sphere in 3d rather than a 3-sphere embedded in 4-dimensions, like the qubit prior to projection onto the Bloch sphere.

Points 180° apart on the Bloch sphere are orthogonal states. We have given three pairs of them, the spin up and down states, $|0\rangle, |1\rangle$, the states $|+\rangle, |-\rangle$, and the complex states $|i\rangle, |-i\rangle$. We could also rotate into another basis and find a different set of 6 extreme points. The spin up and down states are eigenvectors of σ_z.

Question: What are the θ, ϕ angles and x, y, z coordinates for the 6 states on the Bloch sphere?

Answer: Here is a table:

Points on the Bloch sphere				
State		θ	ϕ	(x, y, z)
$\|0\rangle$		0	-	(0,0,1)
$\|1\rangle$		π	0	(0,0,-1)
$\|+\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\pi/2$	0	(1,0,0)
$\|-\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\pi/2$	π	(-1,0,0)
$\|i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$	$\pi/2$	$\pi/2$	(0,1,0)
$\|-i\rangle$	$\frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$	$\pi/2$	$-\pi/2$	(0,-1,0)

Question: How do the Pauli spin matrices, $\sigma_x, \sigma_y, \sigma_z$ operate on the Bloch states?

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Answer: here is a table:

	Action of Pauli Matrices		
$\|\psi\rangle$	$\sigma_x\|\psi\rangle$	$\sigma_y\|\psi\rangle$	$\sigma_z\|\psi\rangle$
$\|0\rangle$	$\|1\rangle$	$i\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|0\rangle$	$-i\|0\rangle$	$-\|1\rangle$
$\|+\rangle$	$\|+\rangle$	$-i\|-\rangle$	$\|-\rangle$
$\|-\rangle$	$-\|-\rangle$	$i\|+\rangle$	$\|+\rangle$
$\|i\rangle$	$i\|-i\rangle$	$\|i\rangle$	$\|-i\rangle$
$\|-i\rangle$	$-i\|i\rangle$	$-\|-i\rangle$	$\|i\rangle$

Question: Find a matrix for which the complex states $|i\rangle$ and $|-i\rangle$ are eigenvectors. The answer is the Pauli-Y or σ_y.

Question: Find a matrix for which the states $|+\rangle$ and $|-\rangle$ are eigenvectors. Answer is the Pauli-X or σ_x.

Question: Is there a way to figure this out without guessing?

Answer: Yes. Make a unitary transformation U to transfer from $|0\rangle, |1\rangle$ basis to $|+\rangle, |-\rangle$. This can be done with the states themselves as these states are orthonormal and rows of unitary matrices are orthonormal. Then compute $U\sigma_z U^\dagger$.

3.2 Single qubit Gates

Quantum gates are unitary transformations. I give some examples of common gates that operate on a single qubit.

Hadamard	$-\boxed{\text{H}}-$	$\mathbf{H}$	$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
NOT or Pauli-X	$-\boxed{\text{X}}-$	σ_x	$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Pauli-Y	$-\boxed{\text{Y}}-$	σ_y	$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
Pauli-Z	$-\boxed{\text{Z}}-$	σ_z	$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Phase $= \sqrt{Z}$	$-\boxed{\text{S}}-$	S	$= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
$\pi/8 = \sqrt{S}$	$-\boxed{\frac{\pi}{8}}-$	$\frac{\pi}{8}$	$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

I have also include the quantum circuit symbols. The Hadamard gate takes a $|0\rangle$ and returns a $|+\rangle$ and takes a $|1\rangle$ and returns a $|-\rangle$. The Hadamard gate obeys $\mathbf{H}^2 = \mathbf{I}$. The NOT interchanges the $|0\rangle, |1\rangle$ states. The $\pi/8$ is the square root of the Phase Gate which itself is the square root of the Pauli-Z gate.

Unitary matrices satisfy $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$. The rows and columns of a unitary matrix are orthonormal. The above gates are unitary matrices.

The phase gate is sometimes called P or $P_{\frac{\pi}{2}}$. The $\pi/8$ is sometimes called $P_{\frac{\pi}{4}}$.

3.3 Exponentials of matrices

The exponential of a matrix $\mathbf{A}$ is

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 .... \frac{1}{i!}\mathbf{A}^i ... \tag{4}$$

We can generate a smooth trajectory in the space of matrices with

$$e^{t\mathbf{A}}$$

where t is like time. At $t = 0$ we recover the identity matrix $\mathbf{I}$ as

$$\lim_{t\to 0} e^{t\mathbf{A}} = \mathbf{I}.$$

When applied on a vector $|\psi(t)\rangle = e^{t\mathbf{A}}|\psi\rangle$, and starting at $t = 0$, the initial condition returns $|\psi\rangle$ itself. Then as t varies we have a trajectory in the Hilbert space in which $|\psi\rangle$ lives. Unitary evolution is more commonly written as e^{itH} where H is a Hermitian matrix.

Let us compute the exponentials of the Pauli matrices. Because $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}$ the exponentials of the Pauli matrices can be evaluated from the expansion in equation 4. The exponentials of the Pauli matrices are sometimes described like rotations $R_x(\alpha)$, $R_y(\alpha)$, $R_z(\alpha)$,

$$R_x(\alpha) = e^{i\alpha\sigma_x} = \cos\alpha\ \mathbf{I} + i\sin\alpha\ \sigma_x = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix} \tag{5}$$

$$R_y(\alpha) = e^{i\alpha\sigma_y} = \cos\alpha\ \mathbf{I} + i\sin\alpha\ \sigma_y = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{6}$$

$$R_z(\alpha) = e^{i\alpha\sigma_z} = \cos\alpha\ \mathbf{I} + i\sin\alpha\ \sigma_z = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \tag{7}$$

and they are functions of an angle α. They can be considered rotations in $U(2)$ that are generated exponentially from infinitesimal operators, the Pauli spin matrices.

The matrices $R_x(\alpha)$, $R_y(\alpha)$, $R_z(\alpha)$ are unitary transformations.

3.4 Some properties of unitary transformations

Some properties of *unitary transformations*:

- Reversible. They have an inverse. If U is a unitary transformation $U^{-1} = U^\dagger$.
- Deterministic. No random choices are required when a wave vector transforms via unitary transformation.
- They can be applied in a time continuous way with an exponential matrix.
- They preserve the wave vector norm, $\langle\psi|\psi\rangle = \langle U\psi|U\psi\rangle = \langle\psi| U^\dagger U |\psi\rangle = 1$.
- A unitary matrix has columns that are orthonormal and rows that are orthonormal.
- The determinant of a unitary matrix $|\det(U)| = 1$.
- A unitary matrix can be written as $U = e^{iH}$ where H is a Hermitian matrix.

4 Measurements

Measurements are

- Irreversible.
- Information is lost.
- Probabilistic.

- Involve collapse of wave-function. They can be discontinuous. (Though not necessarily in the setting of weak measurements).

Measurements can be described as a projection of the wave-function with a Hermitian projection operator that is chosen using a probability.

In contrast, unitary transformations always preserve the norm, are reversible, and involve no random choices.

Measurements involve a sum of probabilities that equal 1.

The Pauli spin matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The expectation

$$\langle \sigma_z \rangle = \langle \psi | \sigma_z | \psi \rangle$$

gives us the expectation value of the spin value with a measured 1 being spin up and a measured -1 being spin down. (For a spin 1/2 particle the spin operator is $\mathbf{J} = \hbar \boldsymbol{\sigma}/2$ so a measurement of spin in the z direction actually gives $\pm\hbar/2$). With $|\psi\rangle = |0\rangle$ the spin is up and with $|\psi\rangle = |1\rangle$ the spin is down. With $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, the probability of measuring spin up is aa^* and the probability of measure spin down is $bb^* = 1 - aa^*$. After measuring a spin up, the wave vector collapses and becomes a complex number of magnitude 1 times $|0\rangle$. After measuring a spin down, the wave vector collapses and becomes $|1\rangle$ times a complex number of magnitude 1. Is the phase important? If you are later on carrying out an interference experiment, the phase could be important.

We can collapse the wave function with two projection operators

$$\mathbf{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The projection operators can also be written

$$\mathbf{P}_0 = |0\rangle \langle 0| \qquad \mathbf{P}_1 = |1\rangle \langle 1|$$

The projection operators satisfy $\mathbf{P}_0^2 = \mathbf{P}_0$, $\mathbf{P}_1^2 = \mathbf{P}_1$. After a measurement giving spin up, the wave-vector must be normalized

$$|\psi'\rangle = \frac{\mathbf{P}_0\,|\psi\rangle}{\langle\psi|\,\mathbf{P}_0\,|\psi\rangle}$$

The probability that a wave vector $|\psi\rangle$ is in the spin up state is $\langle\psi|\mathbf{P}_0|\psi\rangle$ and that it is in the spin down state is $\langle\psi|\mathbf{P}_1|\psi\rangle$.

The measurement operator σ_z can be written as

$$\sigma_z = m_0\mathbf{P}_0 + m_1\mathbf{P}_1$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

where $m_0 = 1$ is the measurement associated with the $\mathbf{P}_0$ projection operator and $m_1 = -1$ is the measurement associated with the $\mathbf{P}_1$ projection operator.

To simulate measurement of a single state, you need a set of projection operators, $\mathbf{P}_i$ each associated with each possible measurement value m_i. To simulate a measurement, project the state with each projection operator. For example if $|\psi\rangle = a|0\rangle + b|1\rangle$ then

$$\mathbf{P}_0 |\psi\rangle = a|0\rangle$$
$$\mathbf{P}_1 |\psi\rangle = b|1\rangle$$

The probability of getting spin up is $aa*$ and the probability of getting spin down is $bb*$. Now you would choose a possible measurement value randomly, using these two probabilities. After you know what state resulted from the measurement you then would normalize the state vector. If spin up was measured then the resulting statevector after measurement is $\frac{a}{|a|}|0\rangle$, otherwise it is $\frac{b}{|b|}|1\rangle$. Here I am keeping track of the phase. If you don't want to keep track of the phase then after measurement you get $|0\rangle$ or $|1\rangle$.

Simulation of measurement on a single state involves making a random choice. A series of measurements would then be a series of random choices each giving you a new statevector. Mimicking a series of measurements is a Markov chain Monte Carlo (MCMC) model.

If you have an experiment that you run many times, then the expectation value of the spin would be

$$\langle\psi| \sigma_z |\psi\rangle$$

with 1 corresponding to spin up and -1 corresponding measuring spin down. This expectation value would be the average over many possible measurements of the state vector $|\psi\rangle$.

4.1 Measurement Postulates of Quantum Mechanics

- A measurement can be specified via a Hermitian operator $\mathbf{A}$ which can also be called an *observable*.
- The eigenvalues m_i of the operator are the possible measured values. Because $\mathbf{A}$ is Hermitian, the measurement values m_i are real numbers.
- The eigenvectors v_i of the operator can be used to construct a set of orthogonal projection operators. The measurement operator can be written as

  $$\mathbf{A} = \sum_i m_i \mathbf{P}_i$$

  where m_i are the possible measurement values and $\mathbf{P}_i = |v_i\rangle \langle v_i|$ are the projection operators.

- The dot product of a normalized eigenvector v_i with the wave function $|\psi\rangle$ gives the probability of a particular measurement value.

$$p_i = |\langle v_i|\psi\rangle|^2 = \langle\psi|\mathbf{P}_i|\psi\rangle.$$

- The expectation value of the measurement $\langle\mathbf{A}\rangle = \langle\psi|\mathbf{A}|\psi\rangle$. If you redid the measurement many times (on the same wave vector) the expectation value is the average of all the measurements.

- After a single measurement the wave function is collapsed using one of the projection operators. If the measured value is m_i then the wave function becomes

$$\psi \to \frac{\mathbf{P}_i\psi}{\langle\psi|\mathbf{P}_i|\psi\rangle}.$$

  After measurement, the wavefunction is a normalized eigenvector of the measurement operator.

4.2 The Quantum Zeno effect

The **Quantum Zeno** effect describes what happens if you measure a state over and over again. It is nearly frozen into a single measured state. With a slowly drifting system, the process of repeated measurement and wave vector collapse keeps the probability low that the system can evolve into a different state and that a subsequent measurement will give a different value.

Consider a single qubit that is initially in the spin up state $|\psi\rangle = |0\rangle$. We operate on it with a smoothly varying unitary operation $U(t) = R_x(\beta t) = e^{i\beta\boldsymbol{\sigma}_x t}$ where t is time and β is small and specifies how fast U varies. We consider the state at time intervals of

$$\delta t = \frac{2\pi}{N\beta}.$$

Here N is the number of time intervals required for a complete revolution of 2π. At time interval $j\delta t$, the unitary transformation $U_j = e^{\frac{i2\pi j\boldsymbol{\sigma}_x}{N}}$ rotates the state by angle $\alpha = \frac{2\pi j}{N}$. We evaluate this matrix using equation 5 to find

$$U_j = \begin{pmatrix} \cos\frac{2\pi j}{N} & i\sin\frac{2\pi j}{N} \\ i\sin\frac{2\pi j}{N} & \cos\frac{2\pi j}{N} \end{pmatrix} \tag{8}$$

With a single time interval dt the unitary transformation is

$$U_1 = \begin{pmatrix} \cos\frac{2\pi}{N} & i\sin\frac{2\pi}{N} \\ i\sin\frac{2\pi}{N} & \cos\frac{2\pi}{N} \end{pmatrix} \tag{9}$$

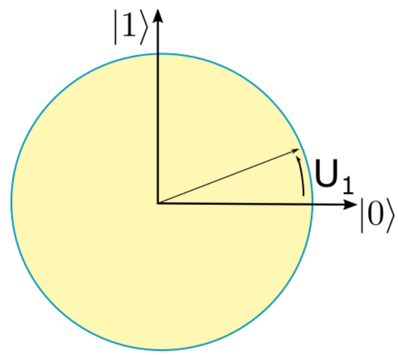

Figure 2: A qubit drifts in the counter clockwise due to continuous unitary evolution. Measurement by σ_z projects the state either into $|0\rangle$ or $|1\rangle$. If the state originates at $|0\rangle$ repeated measurements tend to keep it near $|0\rangle$. This is called the Quantum Zeno effect.

We evolve with U_1, then measure the state, then evolve it again, and so on, alternating between continuous unitary evolution and measurement.

After evolving via U_1 the wave vector becomes

$$|\psi\rangle = \cos\frac{2\pi}{N}\,|0\rangle + i\sin\frac{2\pi}{N}\,|1\rangle$$

We measure it with $\boldsymbol{\sigma}_z$. The probability that the state has a spin up is $\cos^2\frac{2\pi}{N} = 1-\sin^2\frac{2\pi}{N}$ and that the state has spin down is $\sin^2\frac{2\pi}{N}$. Let $\epsilon = \sin\frac{2\pi}{N}$ and we assume that ϵ is small, which is equivalent to assuming that N large. The probability that a spin up is measured is $1-\epsilon^2$.

Suppose we alternate between unitary evolution by dt and measurement by σ_z. We do this M times. We estimate theprobability that a spin up is measured after M repeats is

$$P_{UM} \sim (1-\epsilon^2)^M \sim 1 - M\epsilon^2.$$

The probability that a spin up is measured after M unitary evolutions by U_1 and measurements by σ_z is actually higher than this since one of the intermediate measurements could have been spin down.

Suppose instead the system is allowed to evolve without measurement during the M time intervals. The wave-vector becomes

$$|\psi\rangle = \cos\frac{2\pi M}{N}\,|0\rangle + i\sin\frac{2\pi M}{N}\,|1\rangle$$

The probability that spin up is measured after time $T = M\delta t$ is

$$P_U \sim \cos^2\frac{2\pi M}{N}.$$

Suppose we chose number of evolution steps to be $M = N/4$. This gives probably of spin up $P_U = 0$ (without measurement) because the state evolves to $|1\rangle$. In contrast the probability of spin up when measurements are taken

$$P_{UM} \sim 1 - M\frac{4\pi^2}{N^2} \sim 1 - \frac{\pi^2}{N}.$$

This is much higher than 0 if N is large. The probability can be high that the spin up is measured after the evenly spaced M measurements. The frequent measurements keep the system near the spin up position! The **Qauntum Zeno effect** is when rapid measurements keep a system from evolving.

5 Product spaces and 2 qubits

With two qubits we have a Hilbert space H_{AB} that is a product of two Hilbert spaces H_A and H_B.

A state in the Hilbert space $H_{AB} = H_A \otimes H_B$ can be written in terms of basis vectors for H_A and H_B,

$$|\psi\rangle = \sum_{ij} a_{ij} \left|i\right\rangle_A \otimes \left|j\right\rangle_B .$$

where $|i\rangle_A$ is in H_A and $|j\rangle_B$ is in H_B. Shorthand includes

$$|i\rangle_A \otimes |j\rangle_B = |i\rangle\,|j\rangle = |ij\rangle .$$

With two qubits, a state looks like this

$$|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|01\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle$$

with four complex coefficients. Our Hilbert space has 4 elements in its basis and must be normalized so that $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$.

We could also write $|00\rangle$ as $|0\rangle \otimes |0\rangle$ making it clearer that our Hilbert space is a product of two complex vector spaces. The product space contains elements like $a_{ij}\,|i\rangle \otimes |j\rangle$ or $a_{ij}\,|ij\rangle$. The product is known as a *tensor* product.

A single qubit has wavevector in the form $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$. We can make a tensor product with two single qubits (each in their own 2d Hilbert space) with

$$\begin{aligned}(a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle) &= ac\,|0\rangle\,|0\rangle + bd\,|1\rangle\,|1\rangle + ad\,|0\rangle\,|1\rangle + bc\,|1\rangle\,|0\rangle \\ &= ac\,|00\rangle + bd\,|11\rangle + ad\,|01\rangle + bc\,|10\rangle . \end{aligned} \tag{10}$$

However not every state in the product space can be written in the form on the left hand side. In others words, there are wave vectors $|\psi\rangle_{AB} \in H_A \otimes H_B$ where there does not exist $|\phi\rangle_A \in H_A$ and $|\phi\rangle_B \in H_B$ such that $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\phi_B\rangle$.

5.1 Entanglement

We consider the product space of 2 qubits. Not all states in the full 2-qubit Hilbert space can be written as a tensor product. For example consider the state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Can we find a, b, c, d

$$(a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$$

that would allow us to write the wavevector as a tensor product? We need to find a, b, c, d that would satisfy

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle)$$

Using equation 10 we find that gives $ac = 1/\sqrt{2} = bd$ and $ad = bc = 0$. The second condition implies that one of a, d must be zero but neither can be zero according to the first condition. There is no solution.

States that cannot be written as a tensor product are called **entangled**.

We consider a state-vector $|\psi\rangle = \sum_{ij} a_{ij}\,|i\rangle\,|j\rangle$. There might be a basis $|\tilde{i}\rangle \otimes |\tilde{j}\rangle$ in which we can write the state-vector as

$$|\psi\rangle = \sum_{ij} a_i a_j\,|i\rangle\,|j\rangle = \left(\sum_i b_i\,|\tilde{i}\rangle\right) \otimes \left(\sum_j c_j\,|\tilde{j}\rangle\right).$$

If there is no-such basis, then the state is described as **entangled**.

5.2 The Bell pair state

The state

$$|\psi\rangle_{\text{Bell}} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is called a **Bell pair** or EPR pair state. Consider making a Bell pair state and then sending the first qubit to Alice and the second qubit to Bob. Alice and Bob perform measurements on their qubits. We now consider creating a sequence of Bell pair states and sending the first qubit in each pair to Alice and the second qubit in each pair to Bob. Alice makes a series of measurements and so does Bob. Looking at her results, Alice sees a sequence that appears to be randomly distributed with 0 and 1 states given with equal probability. Similarly Bob measures a sequence that appears to be random. However, when Alice and Bob compare their sequences they notice that they are highly *correlated*. Alice and Bob measure the same value at each iteration in the sequence.

5.3 Operations on two qubits

For the product of two qubits, we can order matrices and vectors using the order above so

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{11}$$

The order of these states is consecutive in base two: 00 is 0, 01 is 1, 10 is 2 and 11 is 3.

The identity for the 2 qubit system can be written as $\mathbf{I} \otimes \mathbf{I}$,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Many operators can be written as direct products, like $\mathbf{A} \otimes \mathbf{I}$ where $\mathbf{I}$ is the identity for a single qubit and $\mathbf{A}$ is an operator for a single qubit. We can construct operators (or gates) in the full space that are products of gates in the subspaces. For example $\mathbf{H} \times \mathbf{H}$ where $\mathbf{H}$ is the Hadamard gate or $\mathbf{H} \otimes \mathbf{I}$.

$$\mathbf{I} \otimes \mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathbf{H} \otimes \mathbf{I} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

For example

$$\begin{aligned} \mathbf{H} \otimes \mathbf{I}\,|00\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\ \mathbf{H} \otimes \mathbf{I}\,|01\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \\ \mathbf{H} \otimes \mathbf{I}\,|10\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\ \mathbf{H} \otimes \mathbf{I}\,|11\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle). \end{aligned}$$

The product of $\mathbf{I} \otimes \mathbf{H}$ and $\mathbf{H} \otimes \mathbf{I}$ is

$$\mathbf{H} \otimes \mathbf{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It may help to know that unitary matrices have rows and columns that have norm 1 and are orthogonal. It is also useful to check that the conjugate transpose of the matrix times itself gives the identity.

5.4 Controlled NOT gate

An interesting new gate that operates on 2 qubits is the **Controlled NOT** or **CNOT** where the second bit is flipped only if the first bit is 1.

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

With basis as defined in equation 11.

$$\begin{aligned} \mathrm{CNOT}\,|00\rangle &= |00\rangle \\ \mathrm{CNOT}\,|01\rangle &= |01\rangle \\ \mathrm{CNOT}\,|10\rangle &= |11\rangle \\ \mathrm{CNOT}\,|11\rangle &= |10\rangle\,. \end{aligned}$$

The CNOT *cannot* be written as a tensor product.

The CNOT can also be written as

$$\mathrm{CNOT}\,|xy\rangle = |x, x+y\rangle$$

where x, y are either 0 or 1 and the sum $x+y$ is mod 2. Treating the states as classical bits, $x+y$ mod 2 is also **XOR** applied to bit x, y, which gives 0 if both bits are the same but 1 otherwise. The XOR is sometimes written as $x \otimes y$.

I can start with a state that is a tensor product and apply the CNOT

$$\begin{aligned} \mathrm{CNOT}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle &= \mathrm{CNOT}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned}$$

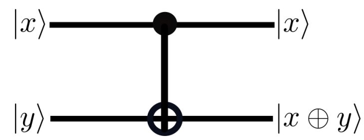

Figure 3: Common convention (though there are exceptions to this rule) with quantum circuit drawings is that the first qubit is on the top. The operation is the CNOT but with first (top) bit as control and second (bottoom) bit as target. The CNOT looks like a plus on the second qubit because the CNOT can be written as $|x, x+y\rangle$ with $x, y \in \{0, 1\}$.

The result is an entangled state (as we showed above that this state could not be written as a tensor product). So the CNOT takes a tensor product state that is not entangled, and turns it into an entangled state.

The CNOT is said to induce **correlations**. With $|{+0}\rangle$ the original state, there is 50% chance for the first qubit to be measured in the 0 state, 50% chance of it being measured in the 1 state and 100% chance that the second qubit is measured in the 0 state. The final entangled state when measured has a 50% chance of both qubits being zero and 50% chance of both qubits being 1.

Example: Starting from $|00\rangle$ construct a Bell pair state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ using simple gates. See Figure 4 for an illustration.

Here's how we do it.

- Apply $\mathbf{H} \otimes \mathbf{I}$, the Hadamard gate to the first qubit.

$$\mathbf{H} \otimes \mathbf{I}\,|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle).$$

- Then apply the CNOT where the control bit is the first qubit and the 2nd qubit is the target

$$\text{CNOT}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

A diagram of the quantum circuit for the creating the Bell pair state is shown in Figure 4. Convention seems to be putting the first qubit on the top of the diagram (following QuTip and Presskill's book but not Reifel & Polack's books). Inputs are on the left and outputs are on the right.

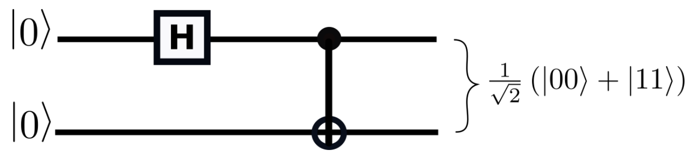

Figure 4: The first quantum operation is a Hadamard on the bottom qubit. This corresponds to the unitary transformation $\mathbf{H} \otimes \mathbf{I}$. The second operation (on the right) is the CNOT but with first (bottom) bit as control and second (top) bit as target. The CNOT looks like a plus on the second qubit because the CNOT can be written as $|x, x+y\rangle$. Starting with state $|00\rangle$ on the left, the result, on the right is the Bell pair state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

5.5 Other controlled 2 qubit operators

We can consider a 2 qubit gate with the first bit a control bit and the second bit a target bit. Instead of flipping the target bit if the first bit is 1, we can execute a gate on the target bit if the first bit is 1 and not change it if the first bit is 0. Any one bit gate can be controlled. We write the CNOT gate as $\Lambda(\mathbf{X})$, where $\mathbf{X}$ is the NOT gate and equivalent to the Pauli-X gate. We can use the Λ symbol to denote other types of controlled gates.

For the first qubit we use projection operators

$$\mathbf{P}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{P}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the tensor product space

$$\mathbf{P}_0 \times \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{P}_1 \times \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

We can write CNOT gate as

$$\text{CNOT} = \Lambda(\mathbf{X}) = \mathbf{P}_0 \otimes \mathbf{I} + \mathbf{P}_1 \otimes \mathbf{X}.$$

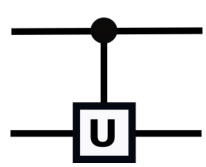

Figure 5: A controlled gate. The control bit is the top one and if it is 1 then the gate U operates on the bottom bit.

For example a controlled phase gate $\Lambda(\mathbf{S})$ gate operates on the second bit with the phase gate if the first bit is 1. Recall that the phase gate looks like

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The controlled phase gate

$$\begin{aligned}\Lambda(\mathbf{S}) &= \mathbf{P}_0 \otimes \mathbf{I} + \mathbf{P}_1 \otimes \mathbf{S} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.\end{aligned}$$

5.6 Partial measurement on two qubits

Supose we have

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

Question: We measure the first qubit. What is the probability to get a 0 (or spin up)?
Answer: The probability to get 0 in a measurement of the first qubit is $|\alpha|^2 + |\beta|^2$.
Question: We measure a 0. What does the wave vector look like now?
Answer:

$$|\psi'\rangle = \alpha'\,|00\rangle + \beta'\,|01\rangle$$

with coeffcients

$$\alpha' = \frac{\alpha}{|\alpha|^2 + |\beta|^2} \qquad \beta' = \frac{\beta}{|\alpha|^2 + |\beta|^2}.$$

What are the projection operators associated with measuring the first bit? They look like $\mathbf{P}_0 \otimes \mathbf{I}$ and $\mathbf{P}_1 \otimes \mathbf{I}$ as shown in equation 12.

5.7 The no-cloning theorem

Cloning means taking taking an arbitrary state in the first qubit, a specific state in the second one and making the second qubit the same as the first qubit and keeping the state unentangled. In other words finding a transformation

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |0\rangle \to (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (\alpha\,|0\rangle + \beta\,|1\rangle)$$

with a unitary transformation and for **any** α, β.

In matrix form, the transformation would satisfy

$$U \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^2 \\ \alpha\beta \\ \alpha\beta \\ \beta^2 \end{pmatrix}$$

The transformation is non-linear. There is no way to solve for a matrix U that does not depend on α, β. So, it is not possible to do this with a unitary transformation.

Can you make a copy of a state with a CNOT? Let's recall Figure 3 showing the CNOT operation $|x, y\rangle \to |x, x + y\rangle$ with $x, y \in \{0, 1\}$. Let's apply the CNOT operation to $(\alpha |0\rangle + \beta |1\rangle) \otimes |0\rangle$

$$CNOT(\alpha |0\rangle + \beta |1\rangle) \otimes |0\rangle = \alpha |00\rangle + \beta |11\rangle .$$

This result is not the same as $(\alpha |0\rangle + \beta |1\rangle) \otimes (\alpha |0\rangle + \beta |1\rangle)$.

In general a cloning device can only simultaneously clone a set of states which are orthogonal to one another and a general quantum cloning device is impossible. In other words suppose we have two states $|\psi\rangle$ and $|\phi\rangle$ in H_A and a state $|s\rangle \in H_B$ giving states in $H_A \times H_B$

$$|\psi\rangle \otimes |s\rangle \qquad \text{and} \qquad |\phi\rangle \otimes |s\rangle .$$

Now suppose we have a unitary transformation U

$$\begin{aligned} U |\psi\rangle \otimes |s\rangle &= |\psi\rangle \otimes |\psi\rangle \\ U |\phi\rangle \otimes |s\rangle &= |\phi\rangle \otimes |\phi\rangle . \end{aligned}$$

We take the inner product of these two equations

$$\begin{aligned} \langle\psi| \otimes \langle s| U^\dagger U |\phi\rangle \otimes |s\rangle &= (\langle\psi| \otimes \langle\psi|)(|\phi\rangle \otimes |\phi\rangle) \\ (\langle\psi| \otimes \langle s|)(|\phi\rangle \otimes |s\rangle) &= \langle\psi|\phi\rangle \langle\phi|\psi\rangle \\ \langle\psi|\phi\rangle &= | \langle\psi|\phi\rangle |^2. \end{aligned}$$

The only solutions of $x = x^2$ are 0, 1. So if $|\psi\rangle \neq |\phi\rangle$ then $\langle\psi|\phi\rangle = 0$ and they are orthogonal. The states can only be simultaneously cloned if they are orthogonal.

5.8 Qutrits instead of Qubits

So far we have discussed bipartite systems of two qubits. A **qutrit** is a three state system with $|0\rangle , |1\rangle , |2\rangle$. We could make a Hilbert space that is a product of a qubit and a qutrit or a product of a qutrit and a quitrit. And so on!

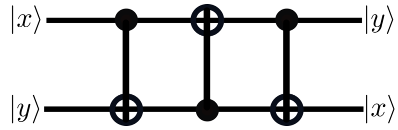

Figure 6: A Quantum circuit that swaps two bits. Here $x, y \in \{0, 1\}$.

5.9 Some quantum circuits

5.9.1 A swap circuit

Let's construct a quantum circuit that flips the states of 2 qubits. The first qubit becomes whatever the other one was and vice versa. In other words we want a circuit that does this transformation

$$\begin{aligned} \text{SWAP}: \qquad & |00\rangle \to |00\rangle \\ & |11\rangle \to |11\rangle \\ & |10\rangle \to |01\rangle \\ & |01\rangle \to |10\rangle\,. \end{aligned} \tag{13}$$

A circuit that does this is shown in Figure 6 and involves 3 CNOTs applied consecutively. Let's show that it works. We start with $|x, y\rangle$ where x can be 0 or 1 and y can be 0 or 1. The first CNOT changes the second bit if the first one is 1. This can be written as

$$\text{First CNOT} \qquad |x, y\rangle \to |x, x + y\rangle$$

where $x + y$ is 1 if one of them is 1 and is 0 if both are 1 or both are 0. The second CNOT flips the first bit if the second one is 1.

$$\text{Second CNOT} \qquad |x, x + y\rangle \to |x + x + y, x + y\rangle\,.$$

Let's look at $x + x$. If $x = 0$ then $x + x$ is 0. If $x = 1$ then $x + x$ is 0 mod 2. So the second CNOT does this

$$\text{Second CNOT} \qquad |x, x + y\rangle \to |y, x + y\rangle\,.$$

Now we apply the third CNOT which flips the second bit if the first one is 1.

$$\text{Third CNOT} \qquad |x, x + y\rangle \to |y, y + x + y\rangle = |y, x\rangle\,.$$

Altogeter our circuit does this

$$\text{SWAP}: \qquad |x, y\rangle \to |y, x\rangle$$

which is consistent with our desired transformation in equation 13.

5.10 Dense Coding. Sending two classical bits via sending one qubit and sharing a Bell pair

Alice wishes to send two classical bits of information to Bob. She and Bob start with a Bell pair state that is shared between them (Alice has the first qubit and Bob has the second)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

The datum Alice wants to send to Bob has one of 4 values: 00, 01, 10, or 11 in base 2 — or 0, 1, 2, or 3 in base 10. The operation she applies to the Bell pair depend upon which datum number she wants to send. She operates on her half of the Bell pair using the following recipe:

Al's Data and Operations		
Value	Operation	Resulting state
00	$\mathbf{I}\otimes\mathbf{I}$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
01	$\boldsymbol{\sigma}_x\otimes\mathbf{I}$	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$
10	$\boldsymbol{\sigma}_z\otimes\mathbf{I}$	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
11	$\boldsymbol{\sigma}_y\otimes\mathbf{I}$	$\frac{i}{\sqrt{2}}(-\|10\rangle + \|01\rangle)$

Alice then sends her qubit to Bob. Bob applies the following operations to the 2 qubits, $(H\otimes I)\text{CNOT}\,|\psi\rangle$. In other words, he first applies a CNOT with control bit the first one and target the second qubit, and then he performs a Hadamard op on the first qubit. Bob's operations are shown in quantum circuit form in Figure 7, Afterwards he measures both qubits.

Bob's Operations and Measurements			
$\|\psi\rangle_{\text{received}}$	CNOT(0,1) $\|\psi\rangle$	$\mathbf{H}\otimes\mathbf{I}$ CNOT(0,1)$\|\psi\rangle$	Measurement
$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle+\|10\rangle)=\frac{1}{\sqrt{2}}(\|0\rangle+\|1\rangle)\otimes\|0\rangle$	$\|0\rangle\otimes\|0\rangle$	00
$\frac{1}{\sqrt{2}}(\|10\rangle+\|01\rangle)$	$\frac{1}{\sqrt{2}}(\|11\rangle+\|01\rangle)=\frac{1}{\sqrt{2}}(\|1\rangle+\|0\rangle)\otimes\|1\rangle$	$\|0\rangle\otimes\|1\rangle$	01
$\frac{1}{\sqrt{2}}(\|00\rangle-\|11\rangle)$	$\frac{1}{\sqrt{2}}(\|00\rangle-\|10\rangle)=\frac{1}{\sqrt{2}}(\|0\rangle-\|1\rangle)\otimes\|0\rangle$	$\|1\rangle\otimes\|0\rangle$	10
$\frac{i}{\sqrt{2}}(-\|10\rangle+\|01\rangle)$	$\frac{i}{\sqrt{2}}(-\|11\rangle+\|01\rangle)=\frac{i}{\sqrt{2}}(-\|1\rangle+\|0\rangle)\otimes\|1\rangle$	$i\,\|1\rangle\otimes\|1\rangle$	11

Here CNOT(0,1) is the CNOT with control bit 0 (the first qubit) that operates on bit 1(the second qubit). I am using the order and notation for the control and operation bits that is used in the `addgate` routine in the python package `qutip`.

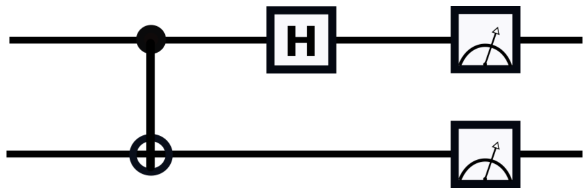

Figure 7: Bob's operations after receiving the 2 qubit state from Alice.

Why is this called *dense coding*? We used 2 qubits to send 2 classical bits of information. Notice that Alice only operated on one of the qubits. The original Bell pair could have been created and then a single qubit sent to Alice and the other sent to Bob. Then Alice operates on her qubit. She does not perform any measurements on it so it remains entangled. She then sends her qubit to Bob who decodes the information she wanted to send by measuring both qubits. Technically Alice only sent 1 qubit to Bob, though a Bell pair was shared prior to the information transfer.

6 Interpretations of Quantum Mechanics

6.1 Thoughts on the EPR paradox

The **EPR paradox** refers to a paper by Albert Einstein, Boris Podolsky, and Nathan Rosen entitled "Can quantum-mechanical description of physical reality be considered complete?"

We start with two particles in a locally generated Bell pair state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Then the two qubits are spit up and Alice is given one of them and Bob is given the other. If Alicee measures 0 then Bob's particle must instantaneously be put in the state $|0\rangle$ and she would then also measure 0. This could be interpreted as transfer of information over large distances in an infinitely small period of time. It may seem like the particles are communicating faster than the speed of light.

What if both Alice and Bob are given a newspaper? They both can simultaneously know the same information. In both cases the information is not actually transferred instantaneously. Hence information is not necessarily transmitted faster than the speed of light. The experimental results can be explained equally well by Bob measuring first and then Alice as in the opposite order. This symmetry shows while there is a correlation between the two measurements, it is not *causal* and Alice and Bob are not communicating faster than the speed of light. This conundrum is known as the **EPR paradox**.

What if particles are not really described by probabilities but rather the uncertainty arises due to local **hidden variables**. In this case particles have an internal hidden state that determines the result of measurements. The hidden state is identical in two particles when the Bell pair is generated. However, the hidden variable is not the same for each generated Bell pair.

This local hidden variables interpretation can be ruled out via Bell's inequalities which we discuss below.

So far we have divided up our operations on quantum states into two categories

1. Unitary evolution. Nobody is watching. No information loss.

2. Measurement. The wave-function is collapsed to a single state which is chosen based on a probability described by the wave-function.

This division also presents some paradoxes. If physical laws are based on quantum mechanics why can't everything be described via unitary evolution only? When can we approximate coupling between systems as a measurement?

6.2 Copenhagen interpretation

It's not all that easy to pin down the Copenhagen interpretation. It is more like a set of guiding principles.

In the appropriate limit, quantum theory should resemble classical physics and reproduces the classical predictions. Quantum mechanics obeys different rules than classical physics. The results provided by measuring devices are essentially classical. Measurement involves an interaction between the system and a laboratory device and this interaction 'collapses' the wave function. A wave function is a mathematical entity that provides a probability distribution for the outcomes of each possible measurement on a system.

The **Born rule**: The wave function gives probabilities for the outcomes of measurements.

The **correspondence principle**: In the appropriate limit, quantum theory should give predictions consistent with classical mechanics.

Complementarity: Certain properties cannot be simultaneously measured on a particular system (this is related to the Heisenberg uncertainty principle).

There are now numerous other *interpretations*.

6.3 The many worlds interpretation

For example the *many worlds interpretation* describes the universe with a single wave function that evolves deterministically via unitary evolution. Interactions of objects within the universe can behave like measurements. The subjective appearance of wavefunction collapse is explained by the mechanism of *quantum decoherence*.

The universe's wave function then describes probabilities for ensembles of many universes. As this idea is not testable, some people think that it is not really a theory. In contrast, quantum mechanics is accurately predictive and has been overwhelmingly verified experimentally. In this sense quantum mechanics is a very good theory.

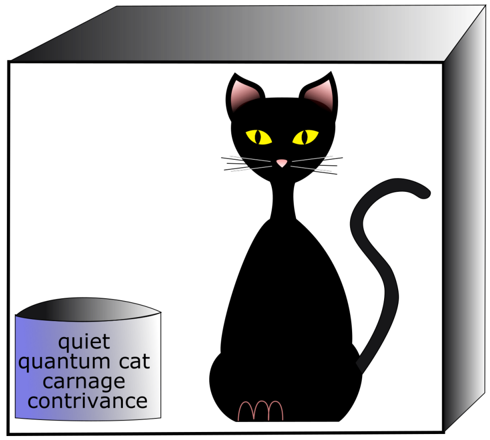

Figure 8: Shrödinger's cat is in a box. A mechanism, here the 'quiet quantum cat carnage contrivance', has probability 1/2 that it will kill the cat. The state of the cat is $|\psi\rangle_{\text{cat}} = \frac{1}{\sqrt{2}}(|\text{dead}\rangle + |\text{alive}\rangle)$. The observer cannot see into the box until after the experiment is done. The observer cannot tell from listening (or any other probe) what is going on inside the box until he or she opens it.

6.4 Shrödinger's cat

Schrödinger's cat is a thought experiment that illustrates an apparent paradox caused by quantum superposition. A cat is in an opaque box. There is a Geiger counter next to some radioactive material. In a single hour, the probability that the Geiger counter counts 1 radioactive decay is 1/2 and counts 0 decays is 1/2. If the Geiger counter counts a decay, then the cat is automatically killed.[1] After the hour is up, the box is opened to see if the cat is alive or dead. During this hour, the cat can be considered to be in a superposition of dead and alive states;

$$|\psi\rangle_{\text{cat}} = \frac{1}{\sqrt{2}}\left(|\text{dead}\rangle + |\text{alive}\rangle\right).$$

In the Copenhagen interpretation, the superposition of states exists only while the box is closed. Only when the box is opened and the cat inside is observed is the cat's wave function collapsed.

However Niels Bohr suggested instead that effectively irreversible processes causes the decay of quantum coherence which imparts the classical behavior of *observation* to the cat. The cat is either alive or dead before the box is opened.

In the many worlds interpretation the universe with the observer and the possibly-dead cat split into a universe with an observer looking at a box with a dead cat, and a universe

[1] Apparently Shrödinger's version involved hydrocyanic acid whereas Einstein's version involved gunpowder.

with an observer looking at a box with a live cat. Since the dead and alive states are decoherent, there is no effective communication or interaction between the two universes.

6.5 EPR polarization measurements

Bell's inequalities which are used to rule out hidden variable interpretations are often discussed in terms of photon polarization. The polarization of a single photon can be described in terms of basis states $|\uparrow\rangle$, corresponding to vertical polarization and $|\rightarrow\rangle$ corresponding to horizontal polarization. A photon can be in a superposition $|\psi\rangle = a\,|\uparrow\rangle + b\,|\rightarrow\rangle$ where aa^* is the probability that a vertical polarizer allow the photon to pass through it. A photon that passes through a vertical polarizer becomes vertically polarized, $|\psi\rangle \to \frac{a}{aa^*}\,|\uparrow\rangle$.

Consider a photon source, called an EPR source (for the Einstein-Podolsky-Rosen paradox) that generates two entangled photos,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\rightarrow\rightarrow\rangle\right).$$

One photon travels to Alice and the other travels to Bob. Both Alice and Bob can measure the photon polarization, but each of them can measure the polarization in one of three orientations. They can measure the polarization in the vertical direction, at 60° from vertical or at -60° from vertical. The different directions are chosen by changing the orientation of the polarizer.

If they both measure polarization in the same orientation, they will 100% of the time measure the same polarization (whether the photon passes through or is absorbed by the polarizer). This follows if we consider rotations of the Bell state, $R(\theta) \otimes R(\theta)$, for both qubits by the same angle θ. This rotation transfers both qubits to the same new basis. To make a polarization measurement that is an angle, we can rotate the measurement and projection operators and keep the wave vector in the same basis or equivalently we can rotate the basis of the wave vector and use diagonal matrices for measuring the polarization.

What happens if Alice measures the polarization of the first qubit in the vertical direction and Bob measures it at angle θ? In the $|\uparrow\rangle, |\rightarrow\rangle$ basis,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\rightarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We rotate the basis of the second qubit

$$\begin{aligned}
|\psi\rangle' &= \mathbf{I} \otimes R(\theta) \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\rightarrow\rightarrow\rangle\right) \\
&= \mathbf{I} \otimes \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\rightarrow\rightarrow\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\cos\theta\,|\uparrow\uparrow\rangle + \sin\theta\,|\rightarrow\uparrow\rangle - \sin\theta\,|\uparrow\rightarrow\rangle + \cos\theta\,|\rightarrow\rightarrow\rangle\right).
\end{aligned}$$

This rotation let's us mimic measurement of the second photon with a polarizer rotated by angle θ.

We measure the first photon with a vertical polarizer. The probability is 1/2 that the polarization of the first photon is vertical and after measurement the wavevector is is $|\psi\rangle = \cos\theta\,|\uparrow\uparrow\rangle - \sin\theta\,|\uparrow\rightarrow\rangle$. The probability is $\cos^2\theta$ that the second qubit passes (P) through the polarizer aligned with θ and $\sin^2\theta$ that it is absorbed (A).

Polarization Measurements of the EPR source					
State	$\frac{1}{\sqrt{2}}\left(\cos\theta\,|\uparrow\uparrow\rangle + \sin\theta\,|\rightarrow\uparrow\rangle - \sin\theta\,|\uparrow\rightarrow\rangle + \cos\theta\,|\rightarrow\rightarrow\rangle\right)$				
Measurement First photon	New Wavevector	Probability	Measurement Second photon	New Wavevector	Probability
P	$\cos\theta\,|\uparrow\uparrow\rangle - \sin\theta\,|\uparrow\rightarrow\rangle$	$\frac{1}{2}$	P	$\cos\theta\,|\uparrow\uparrow\rangle$	$\cos^2\theta$
P	$\cos\theta\,|\uparrow\uparrow\rangle - \sin\theta\,|\uparrow\rightarrow\rangle$	$\frac{1}{2}$	A	$-\sin\theta\,|\uparrow\rightarrow\rangle$	$\sin^2\theta$
A	$\sin\theta\,|\rightarrow\uparrow\rangle + \cos\theta\,|\rightarrow\rightarrow\rangle$	$\frac{1}{2}$	P	$\sin\theta\,|\rightarrow\uparrow\rangle$	$\sin^2\theta$
A	$\sin\theta\,|\rightarrow\uparrow\rangle + \cos\theta\,|\rightarrow\rightarrow\rangle$	$\frac{1}{2}$	A	$\cos\theta\,|\rightarrow\rightarrow\rangle$	$\cos^2\theta$

Similarly the probability is 1/2 that the polarization of the first photon is horizontal and is absorbed (A) by the polarizer. After measurement $|\psi\rangle = \sin\theta\,|\rightarrow\uparrow\rangle + \cos\theta\,|\rightarrow\rightarrow\rangle$ and the probability is $\sin^2\theta$ that the second photon has polarization aligned with θ.

The probability that Alice and Bob's measurements agree (either both passing through PP or both absorbed by the polarizers AA) is

$$P_{\text{agree}} = \frac{1}{2}\cos^2\theta + \frac{1}{2}\cos^2\theta = \cos^2\theta. \tag{14}$$

If $\theta = 60^\circ$, the probability that their measurements agree is 1/4. Here we asserted that the polarizers were like this $\uparrow\nearrow$ as Al's was up and Bob's was at 60°. We summarize:

$$\text{QM : for Polarizers } \uparrow\nearrow,\ p_{\text{agree}} = \frac{1}{4}. \tag{15}$$

Exploiting the symmetry of the problem, we make a table of the possible orientations for Alice and Bob's polarizers and the probability that their measurements agree. Here $\nearrow, \nwarrow$ are the $\pm 60^\circ$ orientations.

Quantum EPR Measurements		
Alice's polarizer	Bob's polarizer	probability meas. agree
$\uparrow$	$\uparrow$	1
$\uparrow$	$\nearrow$	1/4
$\uparrow$	$\nwarrow$	1/4
$\nearrow$	$\uparrow$	1/4
$\nearrow$	$\nearrow$	1
$\nearrow$	$\nwarrow$	1/4
$\nwarrow$	$\uparrow$	1/4
$\nwarrow$	$\nearrow$	1/4
$\nwarrow$	$\nwarrow$	1

One third of the possible rows in this table always give agreement. The probability is 1/3 that they will measure have the same polarizer orientation and it is 2/3 that they will not. Altogether the probability that their measurements agree is

$$p_{agree} = \frac{1}{3} + \frac{2}{3} \times \frac{1}{4} = \frac{4}{12} + \frac{2}{12} = \frac{1}{2}.$$

The measurements should agree half of the time!

6.6 EPR polarization measurements with hidden variables

Suppose there is some hidden state associated with each photon that determines the result of measuring the photon with a polaroid in each of the three possible settings. We list the possible polarization measurements with P for pass and A for absorb. We can refer to these possibilities as covering the possible range of hidden states.

The idea is that the choice of the hidden variables determines the polarization measurement results ahead of time. Later on Alice and Bob chose an orientation and measure the polarization of their photons but the outcome of these measurements would have been determined ahead of time from the hidden variables. The hidden variables are created or set when the two photons are created.

Hidden variables states			
$\nearrow$	$\uparrow$	$\nwarrow$	p_{agree}
P	P	P	1
P	P	A	
P	A	P	
P	A	A	
A	P	P	
A	P	A	
A	A	P	
A	A	A	1

We expect that two polarization measurements will agree for the two photon Bell state if the measurements of individual photons are the same or if the hidden states are PPP or AAA. I added a third column in the above table giving $p_{agree} = 1$ for these two possible hidden variable states.

Let us consider the second line with PPA and list the possibilities for each possible measurement.

Hidden variables are PPA			
Alice's polarizer	Bob's polarizer	PPA meas. Al/Bob	meas. agree
$\uparrow$	$\uparrow$	PP	yes
$\uparrow$	$\nearrow$	PP	yes
$\uparrow$	$\nwarrow$	PA	no
$\nearrow$	$\uparrow$	PP	yes
$\nearrow$	$\nearrow$	PP	yes
$\nearrow$	$\nwarrow$	PA	no
$\nwarrow$	$\uparrow$	AP	no
$\nwarrow$	$\nearrow$	AP	no
$\nwarrow$	$\nwarrow$	AA	yes

We see that there are 5 cases where the measurements agree so the probability is 5/9 of measurements agreeing if the photon variables were in the PPA state. Using the symmetry of the problem, we can fill in the table of hidden states.

Hidden variable states			
$\nearrow$	$\uparrow$	$\nwarrow$	p_{agree}
P	P	P	1
P	P	A	5/9
P	A	P	5/9
P	A	A	5/9
A	P	P	5/9
A	P	A	5/9
A	A	P	5/9
A	A	A	1

What is the probability that the measurements agree?

$$p_{agree} = \frac{1}{8}\left(2 \times 1 + 6 \times \frac{5}{9}\right) = \frac{1}{8}\left(\frac{6}{3} + \frac{10}{3}\right) = \frac{16}{3 \times 8} = \frac{2}{3}$$

This exceeds the 1/2 expected (and verified experimentally) from the Quantum measurement in section 6.5.

6.7 Bell's inequality

Bell's inequality is a generalization of the preceding two sections. Polarizers can be set at any triple of three distinct angles a, b, and c. P_{ab} is the sum of probability that both photons both pass through or both are absorbed by the polarizers, if the first polarizer is at angle a and the second is at angle b and the probability that the first polarizer is at angle b and the second is at angle a.

For any local hidden variable theory,

$$P_{ab} + P_{ac} + P_{bc} \geq 1. \tag{16}$$

This is **Bell's inequality**.

Quantum mechanics allows Bell's inequality to be violated. For example, in section 6.5 we determined that the probability that Alice's and Bob's polarization measurements agree if Alice's polarizer is vertical and Bob's polarizer is at 60° is 1/4 (see equation 15). This means that $P_{\uparrow\nearrow} = \frac{1}{4}$. Because of symmetry, $P_{\uparrow\nwarrow} = \frac{1}{4}$. With the two polarizers separated by angle θ, the probability that the two measurements agree is equal to $\cos^2\theta$ (equation 14). For $P_{\nearrow\nwarrow}$ the two polarizers differ by 120° and the probability is also 1/4 as $\cos 120° = -1/2$.

The sum of

$$QM: \qquad P_{\uparrow\nearrow} + P_{\uparrow\nwarrow} + P_{\nearrow\nwarrow} = 3/4,$$

and as this is less than 1, the inequality is violated.

We now show how equation 16 is derived. According to a local hidden-variable theory, the result of measuring a photon by a polarizer in each of the three possible settings is determined by a local hidden state h of the photon. Any measurement has only two possible outcomes P (for pass) and A (for absorb). We assume if Alice and Bob's polarizers are oriented the same, their measurements or outcomes will agree on an EPR pair. This implies that for an EPR pair, the hidden state is the same for both photons.

Let P^h_{ab} be 1 if the measurements agree for hidden state h and be zero otherwise. The two photons in the EPR pair are assumed to be in the same hidden state h. Because P^h_{ah} can only be 1 or zero and outcomes can only be P or A one of P_{ab}, P_{bc}, P_{ac} must be 1 (with agreement for the outcomes). This implies that for state h

$$P^h_{ab} + P^h_{bc} + P^h_{ac} > 1.$$

Consider a probability distribution for the hidden variable states h, where w_h is the probability that the EPR source emits photons of kind h. As w_h gives the probability of each kind, $\sum_h w_h = 1$. The sum

$$P_{ab} + P_{bc} + P_{ac} = \sum_h w_h(P^h_{ab} + P^h_{bc} + P^h_{ac}).$$

The weighted average of a sum of terms that is greater than 1 must also be greater than 1. Hence

$$P_{ab} + P_{bc} + P_{ac} > 1.$$

We have shown that a local hidden variable theory satisfies Bell's inequality.

Let's check how this relates to the hidden variable assumptions we made in section 6.6.

Hidden variable states - Probabilities for pairs						
Hidden vars $\nwarrow\uparrow\nearrow$	a b $\uparrow\nearrow$	agree	b c $\nwarrow\nearrow$	agree	c a $\nwarrow\uparrow$	agree
PPP	PP	y	PP	y	PP	y
PPA	PP	y	PA	n	AP	n
PAP	AP	n	PP	y	PA	n
PAA	AP	n	PA	n	AA	y
APP	PA	n	AP	n	PP	y
APA	PA	n	AA	y	AP	n
AAP	AA	y	AP	n	PA	n
AAA	AA	y	AA	y	AA	y
	P_{ab}		P_{bc}		P_{ca}	
	$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$	

For the hidden variable states we assumed in section 6.6

$$P_{ab} + P_{bc} + P_{ac} = \frac{3}{2}$$

which exceeds 1, as expected from Bell's inequality.

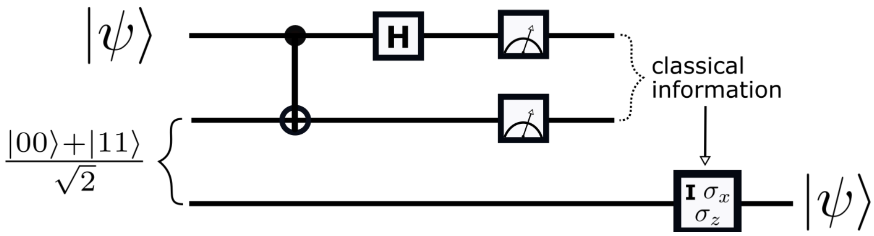

Figure 9: A recipe for teleporting a qubit $|\psi\rangle$ using two additional entangled qubits, a CNOT, a Hadamard operation and two measurements. Alice and Bob share an entangled state. The transmitter (Alice) applies the CNOT and the Hadamard and makes the two measurements. The transmitter then tells the receiver the results of the two measurements. The receiver (Bob) applies a transformation on the third qubit that is based on the measurements of the first two qubits. The receiver then holds the third qubit which has become identical to $|\psi\rangle$, the original state of the first qubit.

7 Teleportation with a Bell pair

We describe how an unknown state can be transferred from one qubit to another with a series of measurements and two qubits in an entangled state.

We have three qubits. Two of them are in an entangled Bell pair state. Alice has a single qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ where a, b are not known. She wants to teleport its state to Bob. Alice can measure 1 qubit of the Bell pair and Bob can measure the other qubit in the Bell pair.

- Alice performs a transformation on a qubit $|\psi\rangle$ that entangles it with the Bell pair.
- Alice measures the transformed qubit **and** her qubit that is part of the entangled pair.
- Alice tells Bob (via classical communication) what she has measured.
- Bob performs a transformation on his half of the entangled qubit pair that depends on these measurements.
- Bob now holds the information that was in $|\psi\rangle$.
- Alice no longer holds the information that was in $|\psi\rangle$.
- The information that was in $|\psi\rangle$ (the values of a, b) has teleported to Bob.

We start with total state

$$|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|011\rangle + b\,|100\rangle + b\,|111\rangle\right)$$

Alice performs the CNOT gate on the total state with target the second qubit and control the first qubit. The CNOT gate flips the second qubit if the first one is 1. The total state becomes

$$\frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|011\rangle + b\,|110\rangle + b\,|101\rangle\right).$$

Alice performs a Hadamard operation on the first qubit. The total state becomes

$$\frac{1}{2}\left(a\,|000\rangle + a\,|100\rangle + a\,|011\rangle + a\,|111\rangle - b\,|110\rangle + b\,|010\rangle - b\,|101\rangle + b\,|001\rangle\right).$$

Alice now measures the first two bits. She can measure 0,0 with a probability of $1/4(aa^* + bb^*) = 1/4$. If this happens the state becomes

$$a\,|000\rangle + b\,|001\rangle\,.$$

All her possible measurements are equally likely.

Let's make a table summarizing all possible measurements.

Alice measures the first two qubits				
Alice measures	00	01	10	11
The state becomes	$a\,|000\rangle + b\,|001\rangle$	$a\,|011\rangle + b\,|010\rangle$	$a\,|100\rangle - b\,|101\rangle$	$a\,|111\rangle - b|110\rangle$

Alice sends her measurements to Bob who then performs the following transformations on the last qubit.

If the first bit that Alice measures is 1, Bob applies σ_z

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to the last qubit. This flips the sign of $|1\rangle$.

If the second bit that Alice measures is 1, Bob applies σ_x to the last qubit,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This sends $|1\rangle \to |0\rangle$ and vice versa.

Let's apply these transformations to the above table.

Alice measures	00	01	10	11
The state is now	$a\,\lvert 000\rangle + b\,\lvert 001\rangle$	$a\,\lvert 011\rangle + b\,\lvert 010\rangle$	$a\,\lvert 100\rangle - b\,\lvert 101\rangle$	$a\,\lvert 111\rangle - b\,\lvert 110\rangle$
Bob applies	$\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}$	$\mathbf{I}\otimes\mathbf{I}\otimes\sigma_x$	$\mathbf{I}\otimes\mathbf{I}\otimes\sigma_z$	$\mathbf{I}\otimes\mathbf{I}\otimes(\sigma_x\sigma_z)$
The state becomes	$a\,\lvert 000\rangle + b\,\lvert 001\rangle$	$a\,\lvert 010\rangle + b\,\lvert 011\rangle$	$a\,\lvert 100\rangle + b\,\lvert 101\rangle$	$a\,\lvert 110\rangle + b\,\lvert 111\rangle$
Which is equal to	$\lvert 00\rangle\otimes(a\lvert 0\rangle+b\lvert 1\rangle)$	$\lvert 01\rangle\otimes(a\lvert 0\rangle+b\lvert 1\rangle)$	$\lvert 10\rangle\otimes(a\lvert 0\rangle+b\lvert 1\rangle)$	$\lvert 11\rangle\otimes(a\lvert 0\rangle+b\lvert 1\rangle)$

Examining this table we see that the wave vector that was initially in the first bit has been transported (teleported) to the third bit.

Note: Alice never knew what a, b were. At the end, there is nothing left of interest in the first two bits as both Alice and Bob know what they are. The Bell pair is 'used up'.

With more Bell pairs, more qubit states can be teleported. If two qubits are teleported and they are initially entangled, the resulting teleported state will also be entangled.