1 Discrete Fourier Transforms

Consider a set of \( N \) data points \( y_j \) with \( j \in 0 \rightarrow N - 1 \). Suppose these are taken from measurements of some quantity \( y(t) \) and are taken so that the sampling spacing is even and each data point is separated in time by a time step \( dt \) corresponding to a sampling rate of \( 1/dt \). The interval over which we have data is \( Ndt = T \). A cosine that fits into the time interval \( T \) is

\[
\cos \left( \frac{2\pi t}{T} \right)
\]

and has a frequency

\[
f_1 = 1/T
\]

We can describe the data series as a sum of sines and cosines that fit into the interval \( T \)

\[
y(t) = \sum_{k=1}^{\infty} a_k \cos (2\pi f_k t) + \sum_{k=1}^{\infty} b_k \sin (2\pi f_k t) + a_0
\]

with

\[
f_k = k f_1 = k/T
\]

Assuming we have sines and cosines that fit into the interval makes the series discrete as only integer multiples of the base frequency \( f_1 \) are used. Implicitly we have assumed that our data series or function is periodic and repeats with a period of \( T \). Each consecutive
frequency differs by a frequency $\Delta f = f_1$ and this gives us the resolution of our Fourier transform and depends inversely on the length of the interval $T$. The longer the data stream the more precision we have with our ability to measure frequency $\Delta f = f_1 = 1/T$ giving us an uncertainty relation in our ability to measure anything in a time interval $\Delta t = T$ with precision $\Delta f$,

$$\Delta f \Delta t = 1$$

The same information in our transform is available in complex form

$$y(t) = \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi if_k t} \tag{1}$$

where $\gamma_k$ is complex.

$$\gamma_k = \begin{cases} \frac{1}{2}(a_{-k} + ib_{-k}) & \text{for } k < 0 \\ a_0 & k = 0 \\ \frac{1}{2}(a_k - ib_k) & k > 0 \end{cases} \tag{2}$$

and inverting this for $k \geq 0$

$$a_k = \gamma_k + \gamma_{-k}$$
$$a_0 = \gamma_0$$
$$b_k = i(\gamma_k - \gamma_{-k}) \tag{3}$$

Rewriting equation 1 at each time interval $t = jdt$

$$y_j = y(jdt) = \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi if_k jdt} = \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi i(k/T)(jdt)}$$

$$= \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi i k j/N}$$

1.1 Discrete Transform and Inverse Transform

We can’t actually get any information at very high frequencies or for $kf_1$ greater than our sampling rate $1/dt$. In this sense we consider a discrete transform that is truncated. We compute

$$c_k = \sum_{j=0}^{N-1} y_j e^{-2\pi i j k/N} = \gamma_k N \tag{4}$$
We can notice that this is a periodic function of $k$

$$c_{k+N} = \sum_{j=0}^{N-1} y_j e^{-2\pi ij (k+N)/N} = \sum_{j=0}^{N-1} y_j e^{-2\pi ijk/N} e^{-2\pi ij}$$

$$= \sum_{j=0}^{N-1} y_j e^{-2\pi ijk/N}$$

$$= c_k$$

Because $c_k$ is periodic we can chose to compute it for $k \in [-N/2, N/2]$ rather than $k \in [0, N - 1]$. This gives a maximum sensible frequency of $f_{k=N/2} = N/(2dt)$ also known as the Nyquist frequency.

Figure 1: By symmetry, the roots of unity on the complex plane sum to zero.

We can take an inverse transform showing that equation 4 gives

$$y_j = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{2\pi ijk/N}$$

(5)

The inverse transform works because

$$\sum_{k=0}^{N-1} e^{2\pi ijk/N} = e^{2\pi ijk/N}$$
is a sum of complex roots of unity for integers \( j \) (and via symmetry on the complex plane) we get either zero or \( N \). Another way to understand this is to compute using the geometric series

\[
\sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a}
\]
giving

\[
\sum_{k=0}^{N-1} e^{2\pi i jk/N} = \frac{1 - e^{2\pi i j/N}}{1 - e^{2\pi i j/N}}
\]

If the denominator is zero then we should not have used the geometric series and our sum is \( N \). Otherwise for integers, the nominator is zero and we get zero from the sum. So the sum is either zero or \( N \).

More specifically taking equation 4

\[
c_k = \sum_{j=0}^{N-1} y_j e^{-2\pi i jk/N} = \gamma_k N
\]

and taking the sum

\[
\sum_{k=0}^{N-1} e^{2\pi i j'k/N} c_k = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{2\pi i(j' - j)k/N} y_j
\]

\[
= N\delta(j - j')y_j
\]

\[
= Ny_{j'}
\]

and consistent with our stated inverse of equation 5.

### 1.2 Examples (sine and cosine)

Consider the function

\[
f(t) = \cos(2\pi f_1 t) = \frac{1}{2} \left[ e^{2\pi if_1 t} + e^{-2\pi if_1 t} \right]
\]

where \( f_1 = 1/T \) as before, giving a transform with

\[
\gamma_1 = \gamma_{-1} = \frac{1}{2}
\]

Now for a sine

\[
f(t) = \sin(2\pi f_1 t) = \frac{1}{2i} \left[ e^{2\pi if_1 t} - e^{-2\pi if_1 t} \right]
\]

giving

\[
\gamma_1 = -\gamma_{-1} = \frac{1}{2i}
\]
This is a nice setting to check what we think power is. If we integrate the square of the signal, average over the interval, and refer to that as power

\[ P = \frac{1}{T} \int dt \cos^2(2\pi f_1 t) = \frac{1}{2} \]

Adding squares of amplitudes of the Fourier coefficients

\[ \gamma_1^* \gamma_1 + \gamma_{-1}^* \gamma_{-1} = \frac{1}{2} \]

and this is consistent with the expectation that autocorrelation function gives the power as a function of frequency.

1.3 Summary: Nyquist frequency, precision of frequency measurement and an uncertainty relation

Consider a sampling spacing \( dt \), sampling frequency \( 1/dt \) and a window time interval \( T = N dt \) with \( N \) the number of data points.

1. The Nyquist frequency is \( f_{Ny} = 2/dt \) and this is the maximum frequency measured in the discrete Fourier transform. The output of the discrete Fourier transform gives values at frequencies \( f \in (-f_{Ny}, f_{Ny}) \).

2. The spacing between frequencies is \( \delta f = 1/T \) corresponding to the largest period that fits in the window and consistent with the uncertainty relation \( \delta f \delta t = 1 \) with \( \delta t = T \). The frequency \( \delta f \) gives the accuracy of frequency measurement from the transform and is dependent on the window size or number of data samples in the window.

3. What comes out of equation 4 for the transform is complex and includes a factor of \( N \). If you compute the power as function of frequency you need to divide this by \( N \).

4. To compute cosine and sine amplitudes from the complex frequency transform use relations in equation 3.

1.4 Convolution Theorem, Autocorrelation and Power

\[ \tilde{f}(\omega) = \int f(t)e^{-i\omega t}d\omega \]

A convolution of \( h \) with \( g \)

\[ f(t) = \int dt' h(t')g(t'+t) \]
\[ f(t) = \int dt' h(t') g(t' + t) \]
\[ \tilde{f}(\omega) = \int f(t) e^{-i\omega t} dt \]
\[ = \int dt'dt' h(t') g(t' + t) e^{-i\omega t} \]
\[ = \int dt'dt' d\omega' d\omega'' \tilde{h}(\omega') e^{i\omega' t'} \tilde{g}(\omega'') e^{i\omega'' (t' + t)} e^{-i\omega t} \]
\[ = \int dt'dt' d\omega' d\omega'' \tilde{h}(\omega') e^{i(\omega' + \omega'') t'} \tilde{g}(\omega'') e^{i(\omega'' - \omega) t} \]
\[ = \int d\omega' d\omega'' \tilde{h}(\omega') \delta(\omega' + \omega'' - \omega) \tilde{g}(\omega'') \delta(\omega'') \delta(\omega'' - \omega) \]
\[ = \tilde{h}(-\omega) \tilde{g}(\omega) \]

In frequency space we see a product.

Let us look at the autocorrelation

\[ a(t) = \int f(t') f(t' + t) dt \]
\[ a(\omega) = \tilde{f}(-\omega) \tilde{f}(\omega) \]

Here we associate \( f(\omega) \) with \( \gamma_k \) the Fourier component in equation 2. Notice from equation 2 that if \( a_k, b_k \) are real then \( \gamma_{-k} = \gamma_k^* \) the complex conjugate. Here that means if \( f \) is real that \( \tilde{f}(-\omega) = \tilde{f}^*(\omega) \). We can write

\[ p(\omega) = |\tilde{f}(\omega)|^2 + |\tilde{f}(-\omega)|^2 = 2a(\omega) \]

and interpret \( p(\omega) \) as the frequency power.

1.5 Some applications of the convolution theorem

Keep in mind that the Fourier transform of a narrow window is wide in Fourier space. And the transform of a wide window is narrow in Fourier space.

Window of an audio sample. If we have a narrow window then in Fourier space the signal is convolved with a wide function and we cannot make precise frequency measurements.

In contrast if the window is wide, the signal is convolved with a narrow function in Fourier space and we can make precise frequency measurements.

Aperture of a telescope giving the diffraction limit.