1 Friction and Drag

1.1 Energy loss

Newton’s first law posits that a body in motion keeps moving. From changes in momentum we can infer the existence or nature of interaction forces. Alternatively interactions causes changes in momentum.

However, we live in a world were masses don’t usually keep coasting. Exceptions are pucks on an air hockey table, or on ice, or projectiles moving through vacuum.

Friction is a dissipative force between surfaces. Drag is due to hydrodynamic forces. Both of these depend upon velocity, unlike gravity or the electric force.
1.2 Friction

Consider a block on a horizontal surface. A force downward is exerted onto the block $F_N$. A horizontal force is also applied to the block $F_A$. If the horizontal component of the force is too low, the block will not move. If the force pushing it down is high, the block will not move. Because the block does not move, the friction force exactly opposes the applied horizontal force,

$$|F_{fr}| = |F_A|.$$ 

This is known as **static friction**. There is a maximum value for the static friction

$$F_{fr} \leq \mu_s F_N$$

defined with static friction coefficient $\mu_s$. The friction force adjusts itself to match $F_A$ up to a maximum value of $\mu_s F_N$ so as to keep the block from moving. Once the friction force reaches this maximum value, the block can start to slide.

We assume the block/surface contact is flat. We define a coordinate system with directions perpendicular and parallel to the surface. We construct a unit vector $\hat{n}$ that is perpendicular to the surface and a unit vector parallel to the surface $\hat{s}$. The force exerted by the surface on the block due is decomposed into normal and parallel components

$$\mathbf{F} = F_N \hat{n} + F_{fr} \hat{s}.$$ 

**Kinetic friction** or dynamic friction is the horizontal friction force when the block slides. It is a constant force

$$F_{fr} = \mu_k F_N$$

with dynamic friction coefficient $\mu_k$. The force is applied in the opposite direction of the direction of motion

$$\mathbf{F}_{fr} = -\mu_k F_N \text{sign}(v)\hat{s}.$$
with velocity \( v \).

This model for friction forces is known as the **Coulomb model**. The details of the surface are ignored in the Coulomb model for friction. The details of the number of contacts, the area of contacts and forces between contacts, surface deformation and how these depend on the normal force and speed are all ignored.

The coefficient of static friction is expected to be somewhat larger than the coefficient of kinetic friction.

The commonly assumed Coulomb friction description is not smooth! It is non trivial to numerically integrate Coulomb friction because the transitions are not smooth.

![Figure 2: The friction force as a function of applied horizontal force, \( F_A \).](image)

Figure 2: The friction force as a function of applied horizontal force, \( F_A \).

Friction is required for stopping a car, turning a car or motor cycle and many forms of locomotion.
Figure 3: The friction force is stronger if there a force pushing the sled downward (on left) rather than pulling it upward (on right).

1.3 Drag

Hydrodynamic or aerodynamic drag depend on the cross sectional area $A$, the velocity with respect to the fluid (or air) $v$, and the density $\rho$ of the fluid that is being displaced.

$$ F_D = \frac{1}{2} C_D \rho A v^2 $$

The rate that mass is swept up by the object is $\rho A v = dM/dt$. However momentum per unit volume is $\rho v$ so the rate that momentum is changed depends on $\rho A v^2$. This accounts for the $v^2$ dependence. The drag coefficient $C_D$ is sensitive to body shape and fluid viscosity (or Reynolds number). Drag is applied in the direction opposite to the motion (w.r.t to the fluid).

For a falling raindrop of mass $M$ we balance the drag force against the gravitational force

$$ \frac{1}{2} C_D \rho_{air} A v^2 = Mg $$

giving terminal velocity

$$ v_{\text{term}} = \sqrt{\frac{Mg}{\frac{1}{2} C_D \rho_{air} A}} $$

If the radius of the drop is $R$, then $M \propto R^3$ and $A \propto R^2$ and we expect a higher terminal velocity for larger raindrops.
Figure 4: A falling raindrop reaches terminal velocity when the aerodynamic drag force balances gravity.

Hydrodynamic drag limits the speed of boats and fish, and sets a terminal velocity for raindrops, where the drag force exactly balances the gravitational force.
1.4 Dashpots

Another common frictional type of force is with a dashpot. Here the force is proportional to the velocity and it is applied opposite to the velocity

\[ \mathbf{F} = -b \mathbf{v}. \]

Dashpots can be hydraulic or contain a viscous fluid or an air piston. Dashpots, drag and friction are velocity dependent (not position dependent) forces. Why does friction depend on velocity? It depends on velocity because it is always exerted in the direction opposite the motion.

2 Work, Energy and Power

The energy principle states that

\[ \Delta E_{\text{system}} = W_{\text{surr}} + Q \]

where \( \Delta E_{\text{system}} \) is the energy of the system, \( W_{\text{surr}} \) is the work done by the surroundings and \( Q \) is the energy flow (or heat flow) between system and surroundings due to a difference in temperature.

When a force \( \mathbf{F} \) is applied to an object and it produces a displacement \( \mathbf{d} \) the work done by the force is

\[ W = \mathbf{F} \cdot \mathbf{d} = Fd \cos \theta \]

where \( \theta \) is the angle between the displacement and force vectors. Work is a scalar, is in units of energy or Joules (J) and can be positive or negative or zero.

A Joule is N m is kg m\(^2\)/s\(^2\).

**Power** is the rate of work or work per unit time

\[ P = \frac{dW}{dt} \]

The units of power are J/s or W (Watts) or kg m\(^2\)/s\(^3\).

Consider an object in a circular orbit. The force is radial but the velocity is tangential. Because \( \mathbf{F} \cdot \mathbf{v} = 0 \), no work is done by the radial force.

2.1 Kinetic energy

We first discuss the non-relativistic setting and then generalize to the relativistic one.
2.1.1 Kinetic energy in the non relativistic limit

Consider a particle of mass $m$ initially at rest at $t = t_0$. We apply a constant force $F = ma$ for time $\Delta t = t_1 - t_0$. During the acceleration,

$$v = a(t - t_0).$$

The final velocity is

$$v_1 = a(t_1 - t_0).$$

We take $x_0$ and $x_1$ to be initial and final positions. To find the total work we integrate force times distance

$$W = \int_{x_0}^{x_1} F \cdot dx = \int_{t_0}^{t_1} F \cdot \frac{dx}{dt} \, dt = \int_{t_0}^{t_1} ma \cdot v \, dt = \int_{t_0}^{t_1} ma \cdot a(t - t_0) \, dt = ma^2 \left( \frac{t_1^2}{2} - t_0 t \right) \bigg|_{t_0}^{t_1} = ma \frac{(t_1 - t_0)^2}{2}.$$
If the initial velocity \( v_0 = 0 \) then
\[
W = \frac{mv_0^2}{2}
\]

We recognize this as the kinetic energy. When applying a constant force to an initially stationary object, the work done is equal to the kinetic energy.

What if the initial velocity is not zero? Under constant acceleration
\[
v_1 = a(t_1 - t_0) + v_0
\]
We insert this into our expression for the work and find that
\[
W = \int_{t_0}^{t_1} ma \cdot \mathbf{v} \, dt
\]
\[
= \int_{t_0}^{t_1} ma \cdot (a(t_1 - t_0) + v_0) \, dt
\]
\[
= \int_{t_0}^{t_1} m [a \cdot a(t - t_0) + a \cdot v_0] \, dt
\]
\[
= m \left[ a^2 \left( \frac{t^2}{2} - t_0 t \right) + ma \cdot v_0 t \right] \bigg|_{t_0}^{t_1}
\]
\[
= ma^2 \frac{(t_1 - t_0)^2}{2} + ma \cdot v_0 (t_1 - t_0)
\]
The difference or gain in kinetic energy is
\[
\Delta K = \frac{m}{2} \left[ v_1^2 - v_0^2 \right]
\]
\[
= \frac{m}{2} \left[ (a(t_1 - t_0) + v_0)^2 - v_0^2 \right]
\]
\[
= \frac{m}{2} \left[ a^2(t_1 - t_0)^2 + 2a \cdot v_0 (t_1 - t_0) \right]
\]
We recognize the work done as equivalent to the gain in kinetic energy, \( W = \Delta K \).

To summarize: If we put a constant force on a mass, the work after moving a distance \( d \) is equal to the gain in kinetic energy.

In summary, if a constant force is applied to a particle, the work done is equal to the change in kinetic energy.

2.1.2 Kinetic energy for relativistic particles

We now generalize for the relativistic setting. To make this calculation simpler we work in 1 dimension only. As before the work done across a distance \( dx \) is
\[
W = F \, dx.
\]
This means that the change in energy

\[ dE = F \, dx = \frac{dp}{dt} \, dx. \]

This means that

\[ \frac{dE}{dx} = \frac{dp}{dt} \tag{1} \]

Recall the relativistic relation for momentum

\[ p = \gamma m v \]

with

\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \]

Let us compute the time derivative

\[
\frac{dp}{dt} = \frac{d}{dt} \left( \frac{mv}{\sqrt{1 - v^2/c^2}} \right)
\]

\[
= \frac{m \frac{dv}{dt}}{\sqrt{1 - v^2/c^2}} - \frac{(mv)}{(1 - v^2/c^2)^{3/2}} (-\frac{v}{c^2} \, dv)
\]

\[
= \frac{m \frac{dv}{dt}}{(1 - v^2/c^2)^{3/2}} \left( 1 - \frac{v^2}{c^2} + \frac{v^2}{c^2} \right)
\]

\[
= \frac{m \frac{dv}{dt}}{(1 - v^2/c^2)^{3/2}} \, dt
\]

With

\[
\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v
\]

we can write

\[
\frac{dp}{dt} = \frac{mv}{(1 - v^2/c^2)^{3/2}} \frac{dv}{dx}
\]

We want a definition for \( E \) such that equation 1 is satisfied. It turns out that

\[
E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 \tag{2}
\]

satisfies this condition. Let’s compute \( dE/dx \) to check

\[
\frac{dE}{dx} = \frac{mc^2}{(1 - v^2/c^2)^{3/2}} (-1/2)(-2) \frac{v}{c^2} \frac{dv}{dx}
\]

\[
= \frac{mv}{(1 - v^2/c^2)^{3/2}} \frac{dv}{dx}
\]
and this is equivalent to our expression for $dp/dt$ above.

It is more work to show that

$$\nabla E = \frac{dp}{dt}$$

is also consistent with the definition for energy $E = \gamma mc^2$.

### 2.2 Rest mass, rest energy and kinetic energy

Consider our definition for energy $E = \gamma mc^2$ that is consistent with

$$\frac{dE}{dx} = \frac{dp}{dt}$$

with momentum $p = \gamma mv$. With $v = 0$, $\gamma = 1$ and

$$E = mc^2$$

This is known as the **rest energy**.

What is the kinetic energy?

We can update our definition of kinetic energy

$$K \equiv E - mc^2 = mc^2 \left( \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).$$

It is useful to know that $\sqrt{1 + x} \sim 1 + x/2$ for small $x$. In the limit of small $v$

$$K \approx mc^2 \left( 1 + \frac{v^2}{2c^2} - 1 \right) = \frac{mv^2}{2}$$

where I took the first term in a Taylor expansion.

Notice that our definitions for energy and momentum both depend on $v$. This means that both energy and momentum depend on the observer reference frame. The rest mass is a constant.

We can compute

$$E^2 - p^2c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2$$

$$= m^2 c^4 (1 - v^2/c^2)$$

$$= (mc^2)^2$$

Because $E^2 - p^2c^2$ depends only on the rest energy, it is frame independent. Also if you know the rest mass and $p$ you can compute $E$ and vice versa.

This relation follows from the Minkowski metric and considering $(E, p)$ as a four-vector. The length of the four vector is a **relativistic invariant**.

Because rest mass or rest energy is an invariant, in high energy astrophysics or particle physics, particle masses are often given in units of rest energy or $E_{\text{rest}} = mc^2$. 
3 Summary

• Work \( W = \int \mathbf{F} \cdot d\mathbf{x} \).

• Energy, momentum for relativistic particles.
  \[ E = \gamma mc^2, \quad p = \gamma m v, \quad E^2 - p^2 c^2 = (mc^2)^2. \]

• Kinetic energy \( K = E - mc^2 \) and \( K = mv^2/2 \) for \( v \ll c \).

• Energy principle.
  \[ \frac{dE}{dx} = \frac{dp}{dt} \]