Resonance Capture

Alice Quillen

University of Rochester

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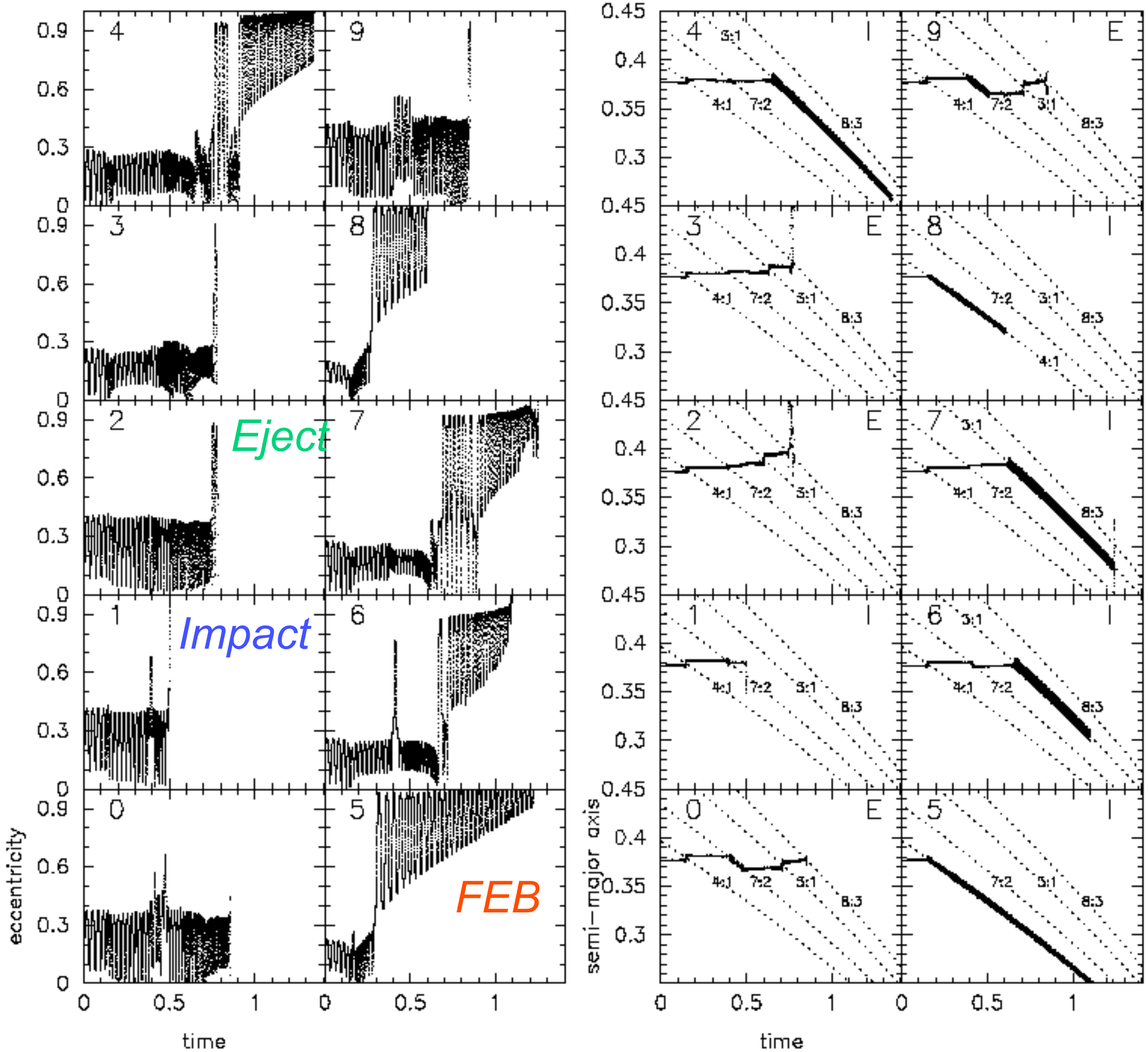
This Talk

- Resonance Capture – extension to non-adiabatic regime
- Astrophysical settings:
  - Dust spiraling inward via radiation forces (PR drag) when collisions are not important
  - Neptune or exoplanet or satellite migrating outward
  - Jupiter or exoplanet or satellite migrating inward
  - Multiple exoplanets drifting into resonance

  Resonances are thin, but sticky (random drift leads to capture, in some cases increased stability)

Example of Resonance Capture



Production of Star Grazing and Star-Impacting Planetesimals via Planetary Orbital Migration

Quillen & Holman 2000, but also see Quillen 2002 on the Hyades metallicity scatter

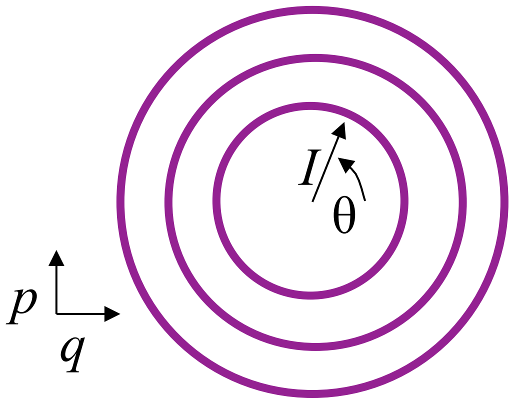
Simple Hamiltonian systems

Harmonic oscillator

$$H(p,q) = \frac{p^2}{2} + \frac{q^2}{2} \rightarrow H(I,\theta) = I\omega$$

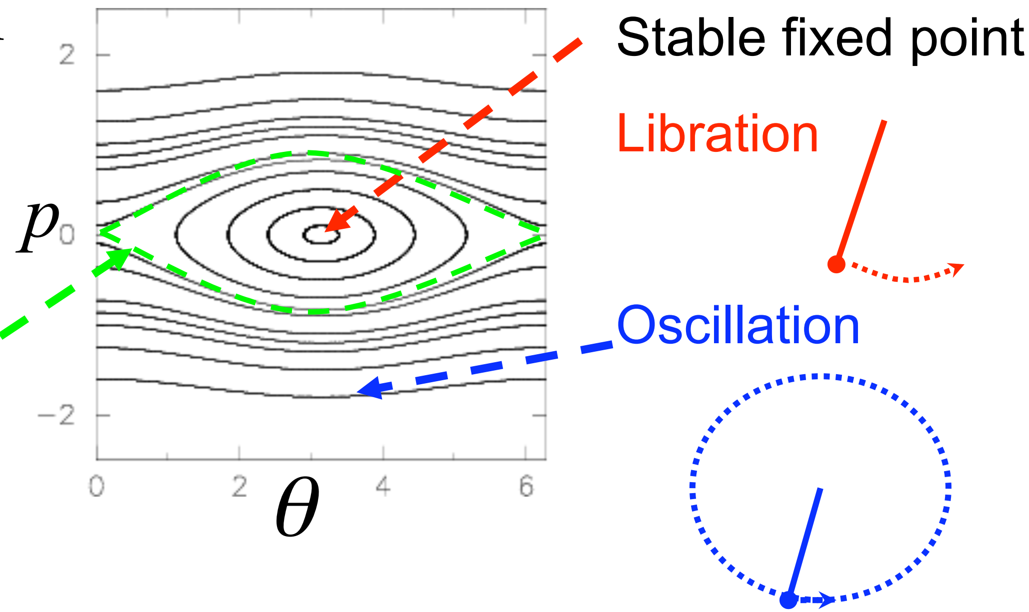
$$\frac{\partial H}{\partial I} = \frac{d\theta}{dt} = \omega \quad \text{is constant}$$

$$\frac{\partial H}{\partial \theta} = -\frac{dI}{dt} = 0 \quad I \text{ is conserved}$$

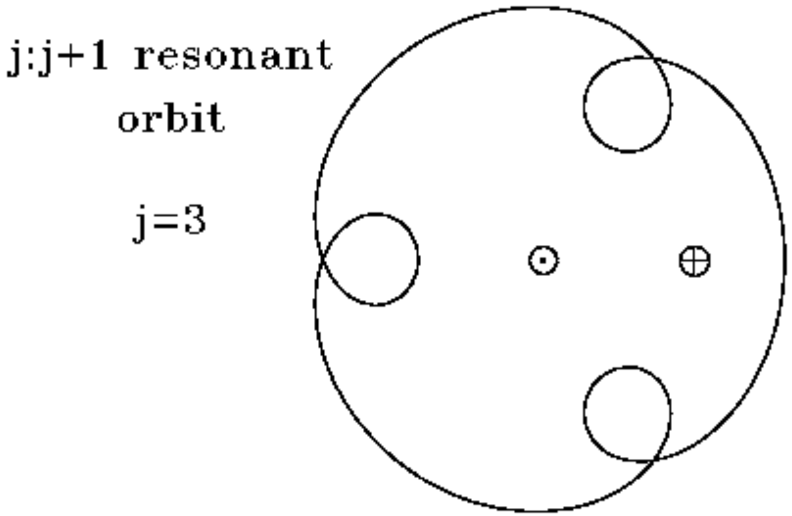


Pendulum

$$H(p,\theta) = \frac{p^2}{2} + \epsilon \cos\theta$$



Resonant angle



In the frame rotating with the planet

$$jn_p - (j+1)n = 0$$

mean motions are integer multiples

$$\phi = j\lambda_p - (j+1)\lambda = \text{constant like } 0 \text{ or } \pi$$

Resonant angle remains fixed

Librating resonant angle ←→in resonance

Oscillating resonant angle ←→outside resonance

Image by Marc Kuchner

Drifting Pendulum and distance to resonance

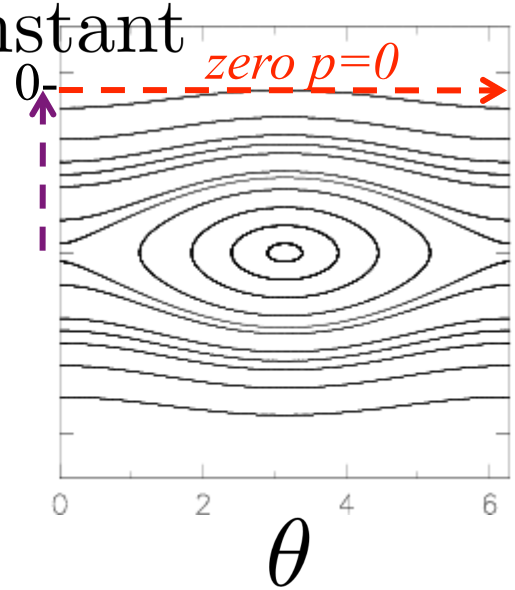
$$H(p,\theta) = a\frac{p^2}{2} + bp + \epsilon\cos\theta$$

Fixed points at $\frac{\partial H}{\partial p} = ap + b = 0$

Canonical transformation with $p' = p + b/a$

$$H(p',\theta) = a\frac{p'^2}{2} + \epsilon\cos\theta + \text{constant}$$

$$\boxed{\Delta p = b/a}$$



b sets distance to resonance

A drifting system can be modeled with time varying *b* or *db/dt* setting drift rate

Dimensional Analysis on the Pendulum

$$H(p,\theta) = a\frac{p^2}{2} + bp + \epsilon\cos\theta$$

• *H* units	$cm^2\ s^{-2}$
• Action variable *p*	$cm^2\ s^{-1}$
	(H=Iω) and ω with 1/s
• *a*	cm^{-2}
• *b*	s^{-1}
• Drift rate *db/dt*	s^{-2}
• ε	$cm^2\ s^{-2}$

Ignoring the distance from resonance we only have two parameters, *a*,ε

- Only one way to combine to get momentum $\boxed{\sqrt{\epsilon/a}}$
- Only one way to combine to get time $\boxed{1/\sqrt{a\epsilon}}$

Resonant width and Libration period

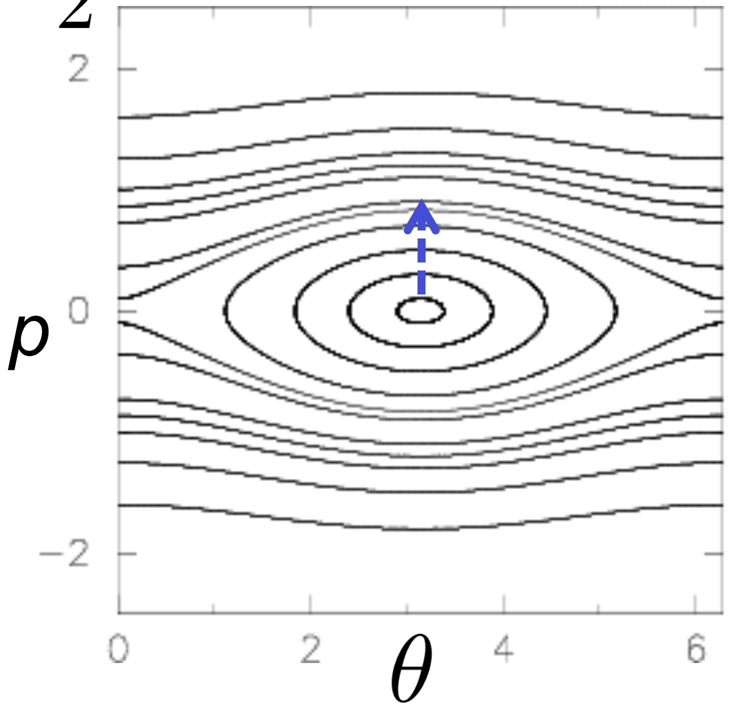
$$H(p,\theta) = a\frac{p^2}{2} + \epsilon\cos\theta$$

- Resonant width solve H(p,θ) =0 for maximum *p.* Distance to separatrix

$$\Delta p = \sqrt{\epsilon/a}$$

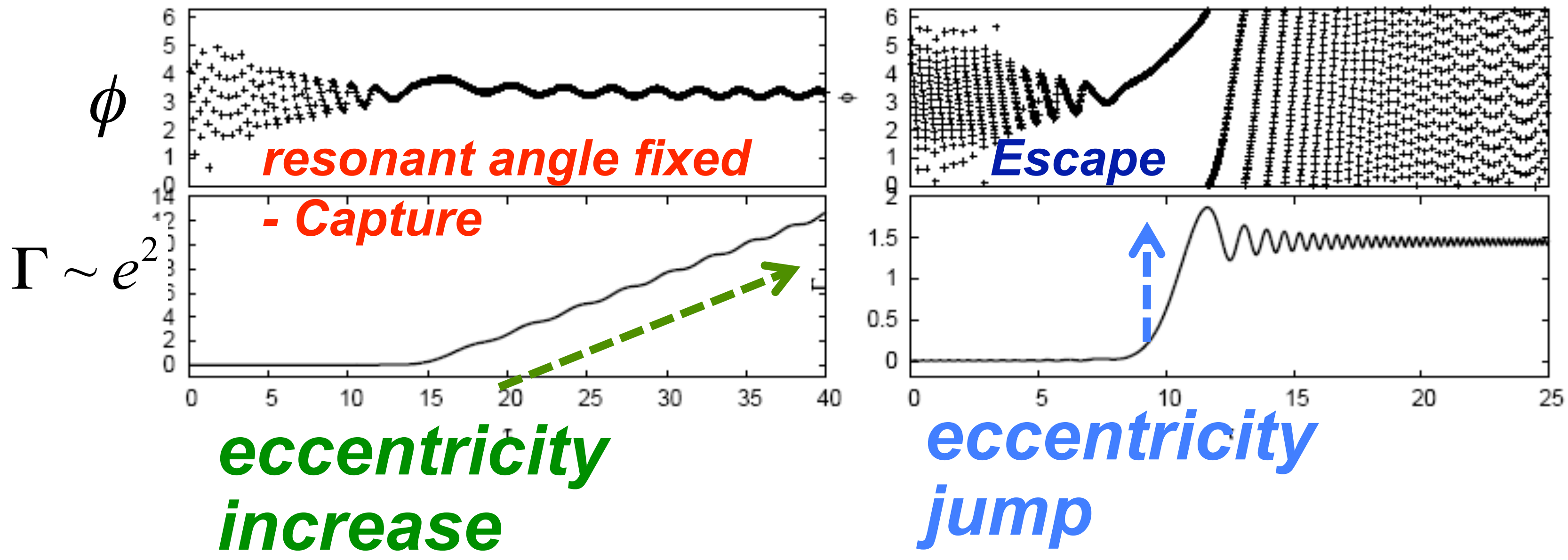
- Libration timescale, expand about fixed point $\ddot{\theta} = -\epsilon a \theta$

Libration period $\boxed{\frac{2\pi}{\sqrt{a\epsilon}}}$



note similarity between these expressions and those derived via dimensional analysis

Behavior of a drifting resonant system



We would like to know the capture probability as a function of:

- *initial conditions*
- *migration or drift rate*
- *resonance properties*

Adiabatic Invariants

- If motion is slow one can average over phase angle
- Action variable is conserved
- Volume in phase space is conserved

$$H(p,q,t) = \frac{p^2}{2} + \lambda(t)\frac{q^2}{2}$$

drifting harmonic oscillator

$$H(t) = I\omega(t) \qquad \omega(t) = \sqrt{\lambda(t)}$$

as long as

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \sqrt{\lambda}$$

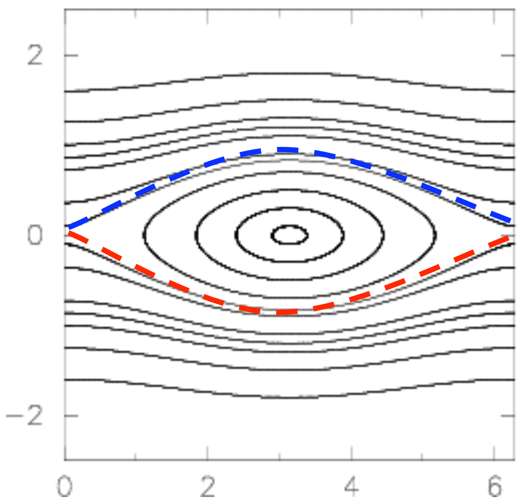
Adiabatic Capture Theory for Integrable Drifting Resonances

V_+ = Rate of Volume swept by upper separatrix

V_- = Rate of Volume swept by lower separatrix

$V_+ - V_-$ = Rate of Growth of Volume in resonance

$$P_c = \frac{V_+ - V_-}{V_+}$$ = Probability of Capture



Theory introduced by Yoder and Henrard, special treatment of separatrix that has an infinite period

Applied to mean motion resonances by Borderies and Goldreich, Malhotra, Peale …

Adiabatic limit for drifting pendulum

$$H(p,\theta) = a\frac{p^2}{2} + bp + \epsilon\cos\theta$$

- Drift rate should be slow
- Time to drift the resonant width should be much longer than libration timescale $\frac{\sqrt{\epsilon a}}{\dot{b}} \gg \frac{1}{\sqrt{\epsilon a}}$

Via dimensional analysis. Units of *db/dt* are s^{-2}. Our only unit of time is $1/\sqrt{\epsilon a}$

→ We are in the adiabatic limit if $\boxed{\dot{b} \ll \epsilon a}$

Note there is no capture if dε/dt=0 as the volume in the resonance does not grow

Keplerian Hamiltonian

- Unperturbed, in the plane

$$H_0(\Lambda, \lambda; \Gamma, \gamma) = -\frac{1}{2\Lambda^2}$$

- Poincaré coordinates

$$\begin{aligned} \Lambda &= \sqrt{a} \\ \Gamma &\approx \sqrt{a}e^2/2 \end{aligned}$$

λ mean longitude

-γ=ϖ longitude of perihelion

- Hamiltonian only depends on Λ. Everything is conserved except λ which advances with

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \Lambda} = n = a^{-3/2}$$

Expand Near Resonance

$$\begin{aligned} H_0 &= -\frac{1}{2\Lambda^2} \\ &\approx \frac{1}{\Lambda_0^3}(\Lambda - \Lambda_0) - \frac{3}{2\Lambda_0^4}(\Lambda - \Lambda_0)^2 + \text{constant} \\ &= bp + ap^2 + \text{constant} \end{aligned}$$

Must keep distance to resonance (b) and take expansion to at least second order in momentum otherwise our Hamiltonian won't look like a pendulum and so won't be able to differentiate between libration and oscillation

Note: we have the first part of our pendulum Hamiltonian

Our perturbation

- *R*, The disturbing function $$H = H_0 + R$$

$$R = \frac{\mu}{a_p} \sum_m W_m(r/a_p) \cos(m(\theta - n_p t))$$

- μ ratio of planet mass to stellar mass
- a_p n_p Planet semi-major axis and mean motion
- W_m Laplace coefficient – comes from integral of $GM_* m_p/|r-r_p|$ over orbit
- Radius r will be expanded in terms of eccentricity

$$\Gamma/\Lambda \sim e^2/2$$

giving us terms like $\Gamma^{1/2} \cos(j\lambda - (j-1)\lambda_p - \varpi)$

- *Note: we are getting cosine terms like our pendulum*

Full Hamiltonian

- Putting together H_0 (unperturbed but expanded near resonance) and R (planetary perturbation)
- We ignore all cosine terms with rapidly varying angles

  → We get something that looks our pendulum Hamiltonian
- Examples of expansion near resonance
  - Murray & Dermott, *Solar System Dynamics,* section 8.8
  - Holman, M. & Murray, N. 1996 AJ, 112, 1278
  - Quillen 2006, MNRAS, 365, 1367

Hamiltonian

$$K(\Lambda,\psi;\Gamma,-\varpi) = a\Lambda^2 + b\Lambda + c\Gamma$$

$$-\sum_{p=0}^{k} \delta_{k,p}\Gamma^{(k-p)/2}\cos(\psi - (k-p)\varpi - p\varpi_p)$$

Mean motion resonances can be written

$$H(\Gamma,\phi) = \frac{a\Gamma^2}{k^2} + \frac{b\Gamma}{k} + \delta_{k,0}\Gamma^{k/2}\cos(k\phi)$$

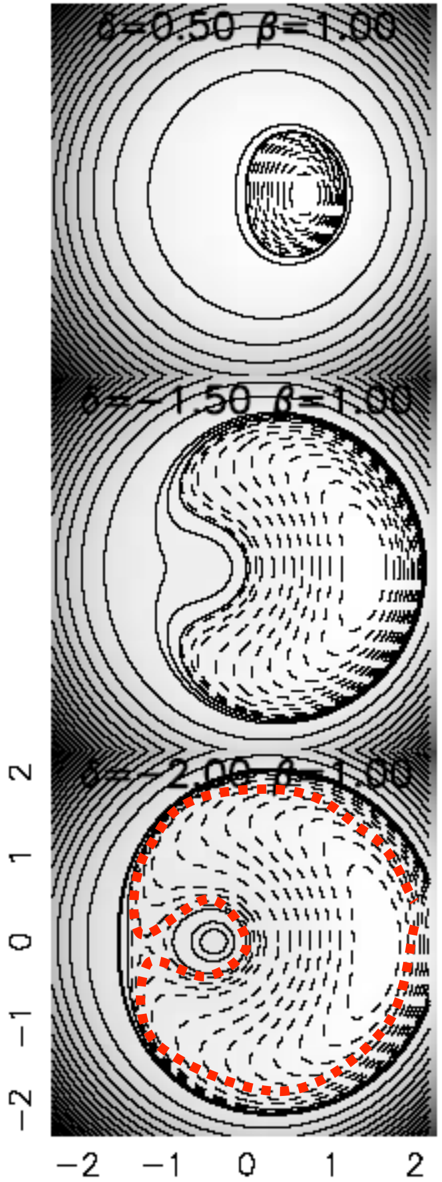
k corresponds to order 2:1 3:2 (first order)

Coefficients depend on time in
drifting/migrating systems

b sets the distance to the resonance

$\frac{db}{dt}$ gives the drift rate

$$\psi = j\lambda_1 - (j-k)\lambda_p$$



Canonical Transformation and reducing the dimension

$$H(\mathbf{p},\mathbf{q}) = \sum_i (a_i p_i^2 + b_i p_i) + f(\mathbf{p})\cos(\mathbf{k}\cdot\mathbf{q})$$

$$F_2(\mathbf{P},\mathbf{q}) = P_1 \sum_{i=1}^{N} k_i q_i + \sum_{j=2}^{N} P_j q_j$$ generating function

$$\frac{\partial F_2}{\partial q_1} = P_1 k_1 = p_1 \qquad \frac{\partial F_2}{\partial P_1} = \mathbf{k}\cdot\mathbf{q} = \phi$$

$$\frac{\partial F_2}{\partial q_{j\neq 1}} = P_1 k_j + P_j = p_j \qquad \frac{\partial F_2}{\partial P_{j\neq 1}} = q_j$$

$$K(P_1,\phi;P_j,q_j) = \sum_{j=0}^{N} a_j k_j^2 P_1^2 + (\mathbf{k}\cdot\mathbf{b})P_1$$
$$+ f(\mathbf{P})\cos\phi + \text{terms lacking } P_1$$

New Hamiltonian only depends on one coordinate φ, so all action variables are conserved except P_1

Andoyer Hamiltonian (see appendix C of book by Ferraz-Mello)

- *k* is order of resonance or power of eccentricity in low eccentricity expansion

$$H(p,\phi) = ap^2 + bp + \epsilon p^{k/2}\cos(k\phi)$$

- If *p* is large then we can think of

$$\sqrt{\epsilon p^{k/2}/a}$$

  as a resonance width, and *p* as varying about the initial value
- The resonance will not grow much during drift → unlikely to capture

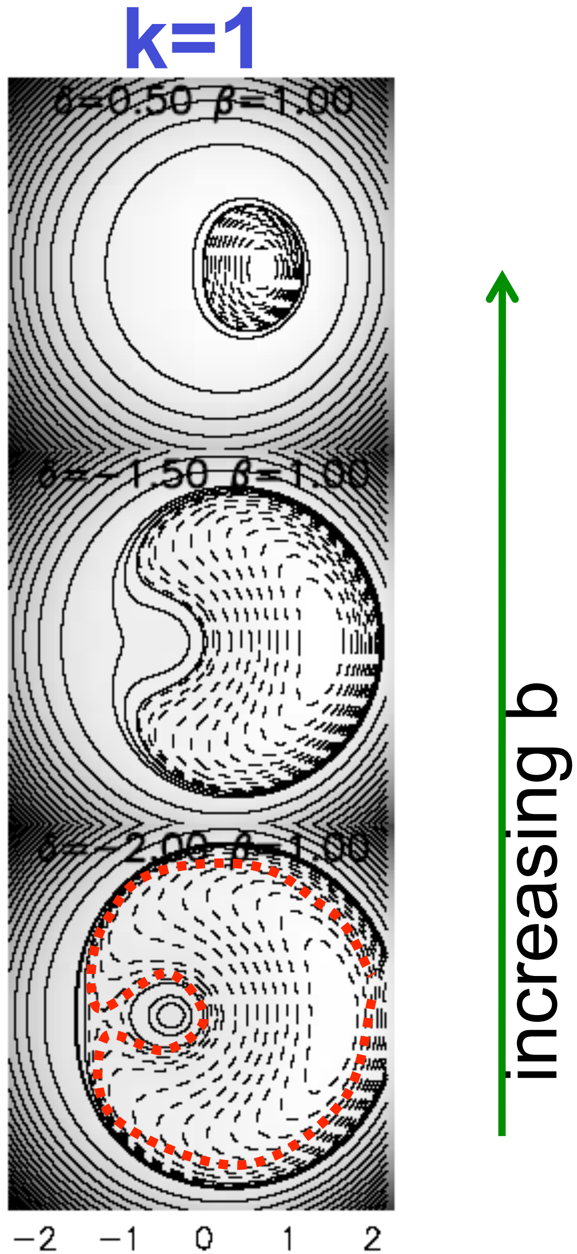
→ Low eccentricity formulism is pretty good for resonance capture prediction

Structure of Resonance

$$x = \sqrt{2p}\cos\phi$$

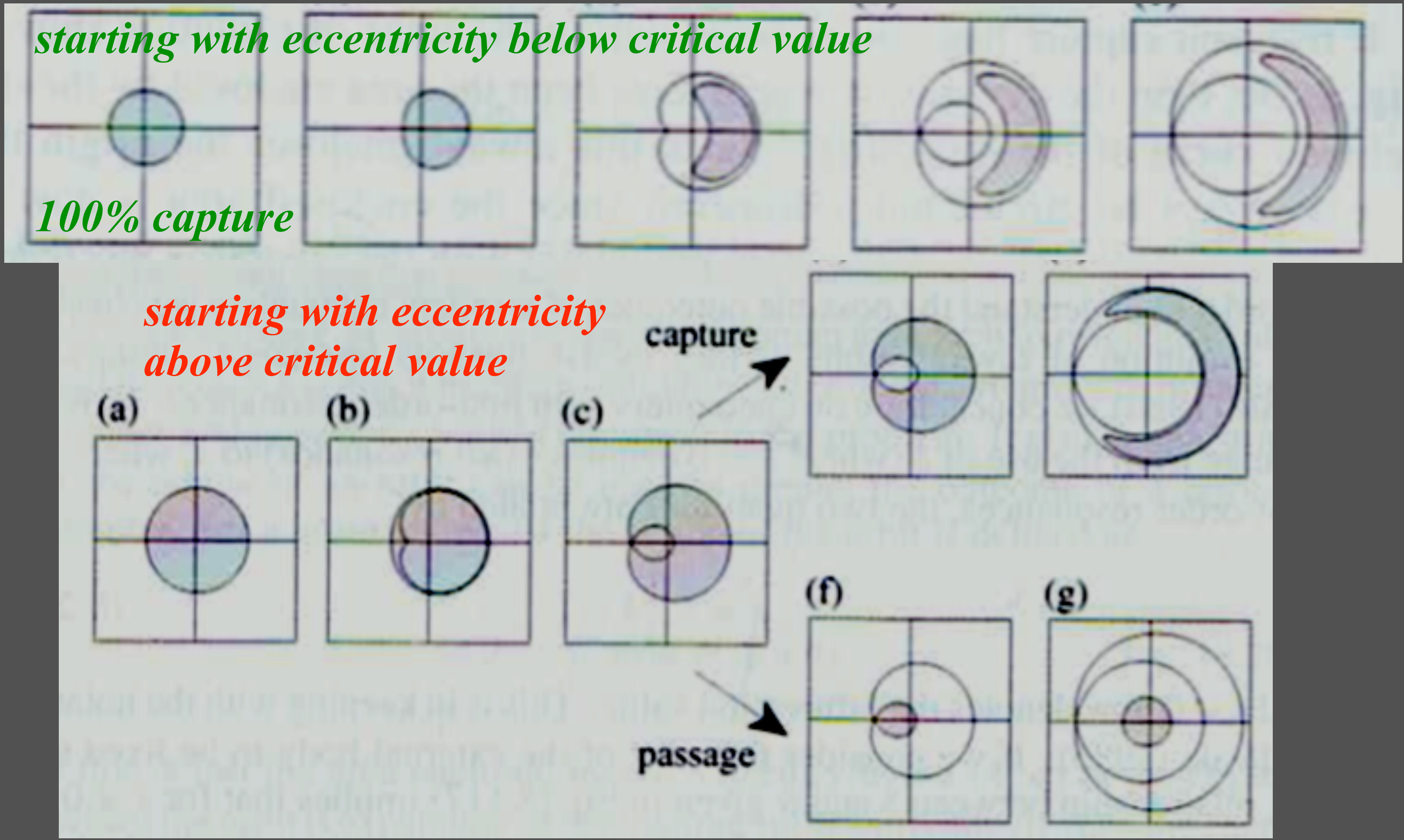
$$y = \sqrt{2p}\sin\phi$$

- Radius on plot is ~ eccentricity
- When b~0 there are NO circular orbits
- Capture means lifting into one of the islands where the angle becomes fixed (does not circulate)
- Capture in only one direction of drift



For Adiabatic drift:

Computing probability involves computing volumes when separatrix appears

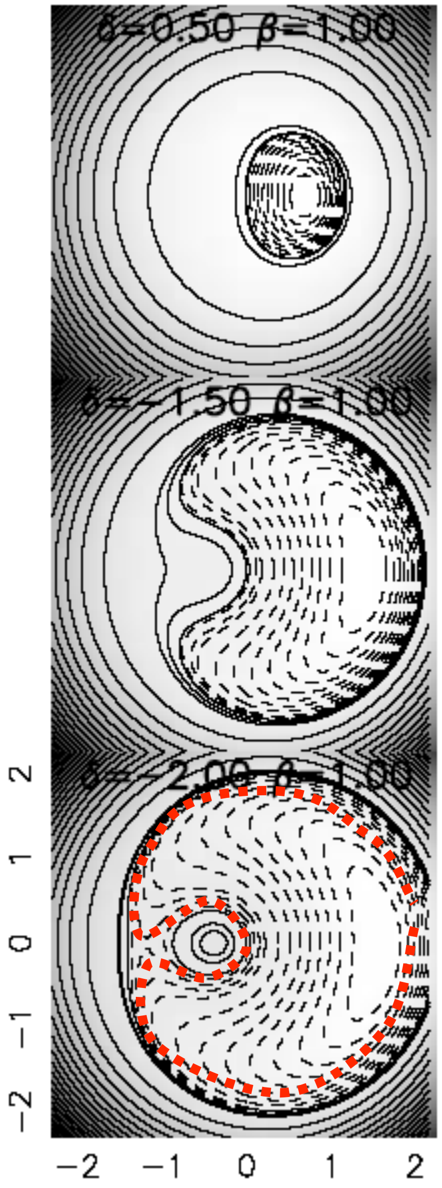


Resonance Capture in the Adiabatic Limit

- Application to tidally drifting satellite systems by Borderies, Malhotra, Peale, Dermott, based on analytical studies of general Hamiltonian systems by Henrard and Yoder.
- Capture probabilities are predicted as a function of resonance order and coefficients via integration of volume
- Capture probability depends

on initial particle eccentricity.

-- Below a critical eccentricity capture is ensured.

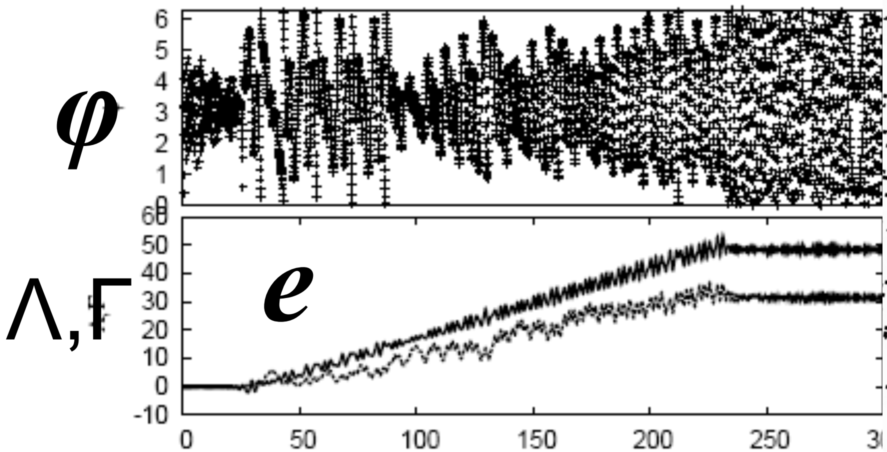


Limitations of Adiabatic theory

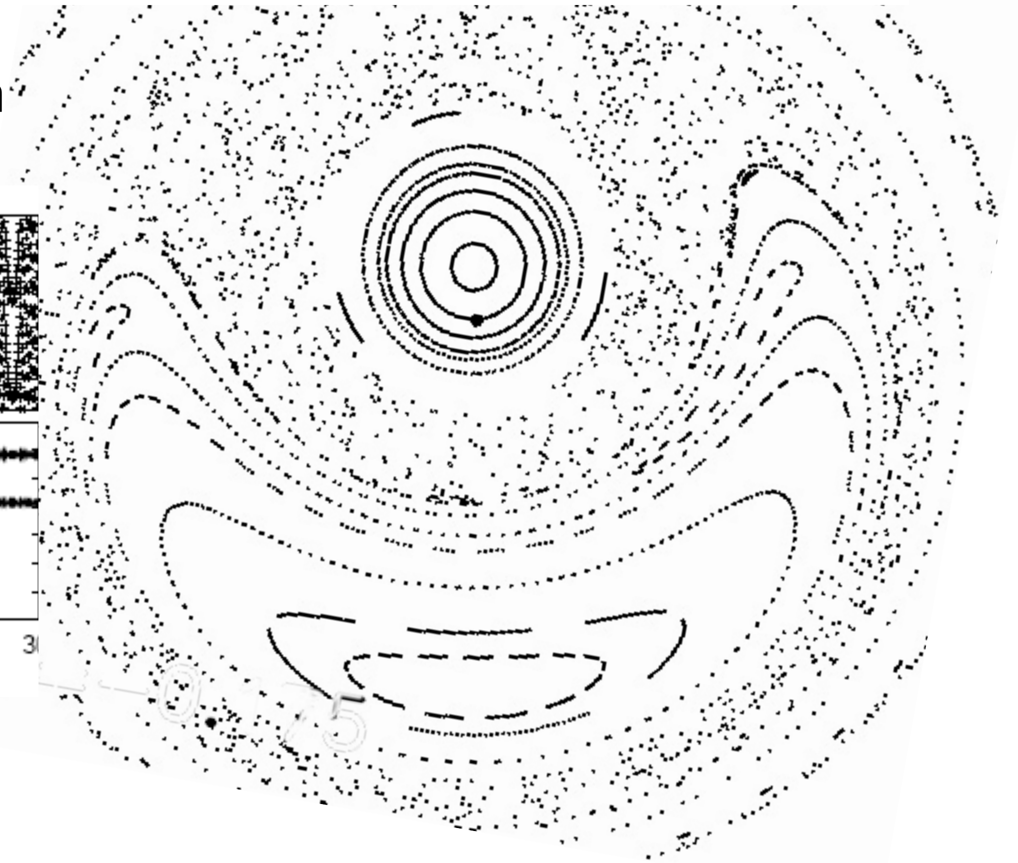
$$K(\Lambda,\psi;\Gamma,\gamma) = a\Lambda^2 + b\Lambda + c\Gamma \quad \propto \mu e^p$$

$$-\sum_{p=0}^{k} \delta_{k,p}\Gamma^{(k-p)/2}\cos(\psi - (k-p)\varpi - p\varpi_p)$$

- At fast drift rates resonances can fail to capture -- the non-adiabatic regime.
- Subterms in resonances can cause chaotic motion



temporary capture in a chaotic system



Chaotic Tidal Evolution of Uranian Satellites Tittemore & Wisdom

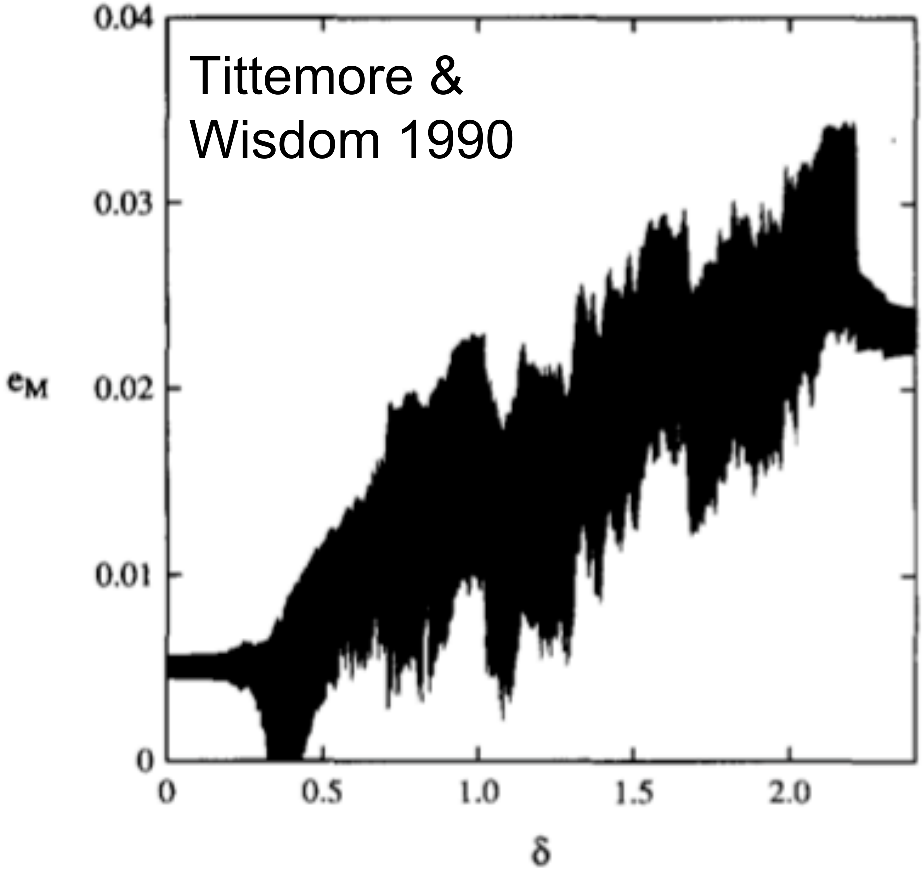


FIG. 11. Eccentricity variations of Miranda for a trajectory evolved through the 3 : 1 Miranda–Umbriel eccentricity resonances, perturbed by the secular variations of Ariel. The maximum and minimum eccentricities are plotted in intervals of $\Delta\delta = 0.0033$. The eccentricity variations are more irregular than in the unperturbed case.

Dimensional analysis on the Andoyer Hamiltonian

- We only have two important parameters if we ignore distance to resonance

$$H(p, \phi) = ap^2 + bp + \epsilon p^{k/2} \cos(k\phi)$$

- *a* dimension cm^{-2}
- ε dimension cm^{2-k} s$^{-2-k/2}$
- Only one way to form a timescale and one way to make a momentum sizescale.
- The square of the timescale will tell us if we are in the adiabatic limit
- The momentum sizescale will tell us if we are near the resonance (and set critical eccentricity ensuring capture in adiabatic limit)

Rescaling

By rescaling momentum and time

$$\bar{\Gamma} = \left|\frac{\delta_{k,0} k^2}{a}\right|^{-2/(4-k)} \Gamma$$

This factor sets dependence on initial eccentricity

$$\tau = \delta_{k,0}^{2/(4-k)} \left|\frac{a}{k^2}\right|^{(2-k)/(4-k)} t$$

This factor sets dependence on drift rate

The Hamiltonian $H(\Gamma,\phi) = \frac{a\Gamma^2}{k^2} + \frac{b\Gamma}{k} + \delta_{k,0}\Gamma^{k/2}\cos(k\phi)$

can then be written

$$K(\bar{\Gamma},\phi) = \bar{\Gamma}^2 + \bar{b}\bar{\Gamma} + \bar{\Gamma}^{k/2}\cos k\phi$$

All k-order resonances now look the same → opportunity to develop a general framework

Rescaling

$$\bar{\Gamma} = \left| \frac{\delta_{k,0} k^2}{a} \right|^{-2/(4-k)} \Gamma$$

$$\tau = \delta_{k,0}^{2/(4-k)} \left| \frac{a}{k^2} \right|^{(2-k)/(4-k)} t$$

$$K = \bar{\Gamma}^2 + \bar{b}\bar{\Gamma} + \bar{\Gamma}^{k/2} \cos k\phi$$

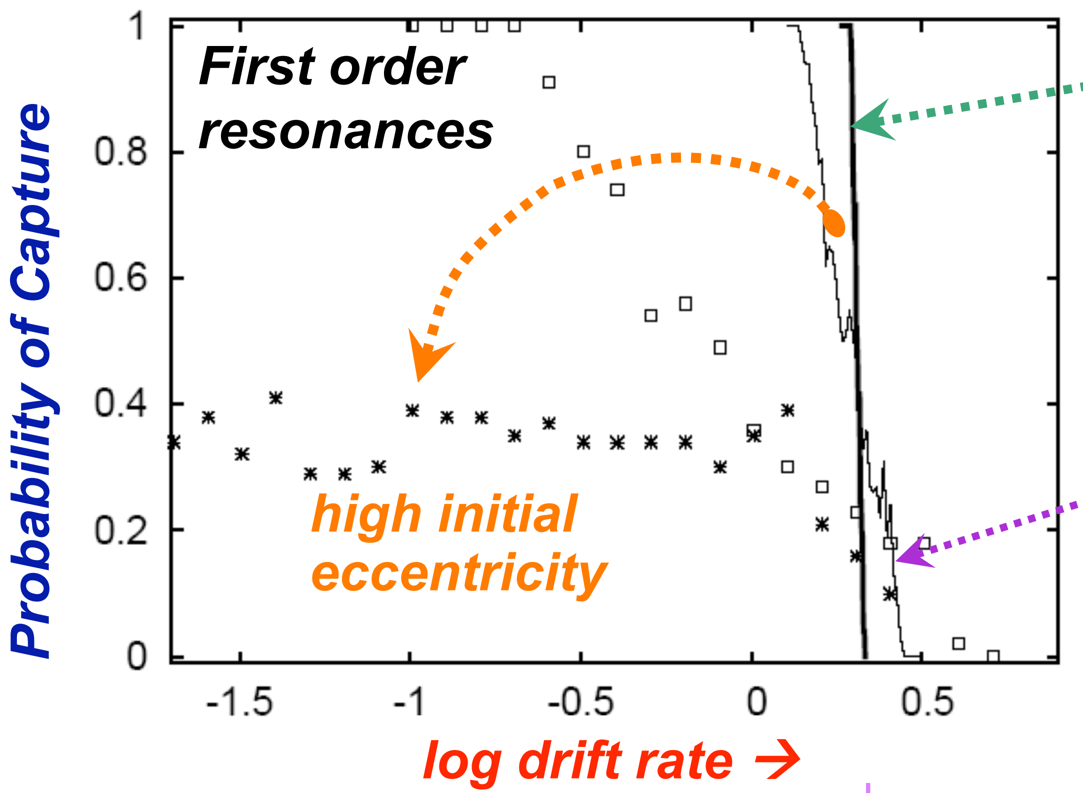
Drift rates have units t^{-2}

First order resonances have critical drift rates that scale with planet mass to the power of -4/3. Second order to the power of -2.

Confirming and going beyond previous theory by Friedland and numerical work by Ida, Wyatt, Chiang..

Numerical Integration of rescaled systems

Capture probability as a function of drift rate and initial eccentricity



At low initial eccentricity the transition is very sharp

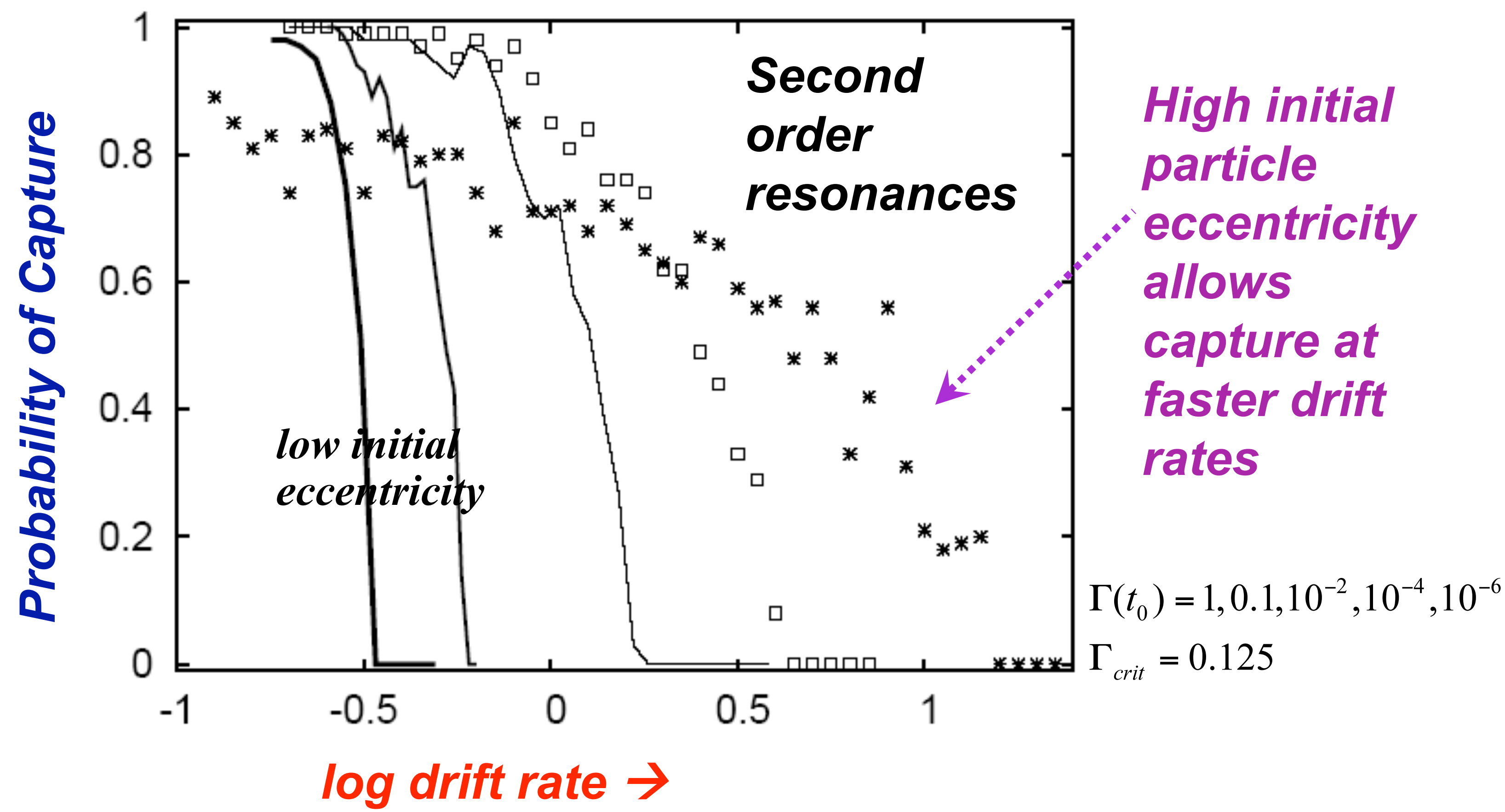
Critical drift rate– above this capture is not possible

$\Gamma(t_0) = 2.2, 1, 0.1, 10^{-4}$

$\Gamma_{crit} = 1.5$

Transition between "low" initial eccentricity and "high" initial eccentric is set by the critical eccentricity below which capture is ensured in the adiabatic limit (depends on rescaling of momentum). **See Alex Mustil on sensitivity to initial conditions.**

Capture probability as a function of initial eccentricity and drift rate

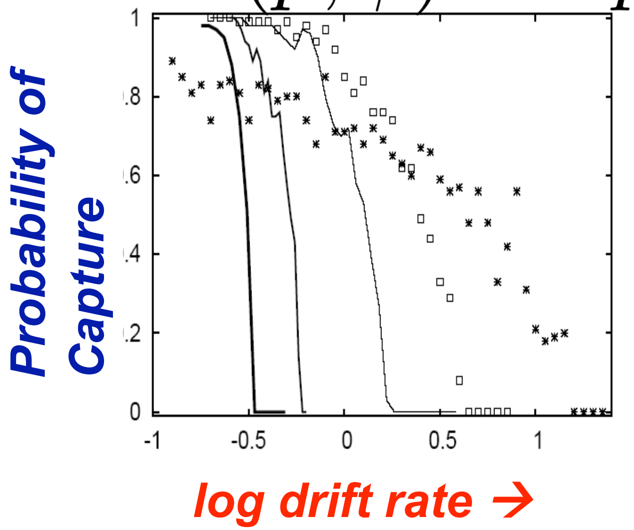


Sensitivity to Initial Eccentricity

At high eccentricity initial the resonance doesn't grow much but libration takes place on a faster timescale

$$t \sim 1/\sqrt{\epsilon p^{k/2} a}$$

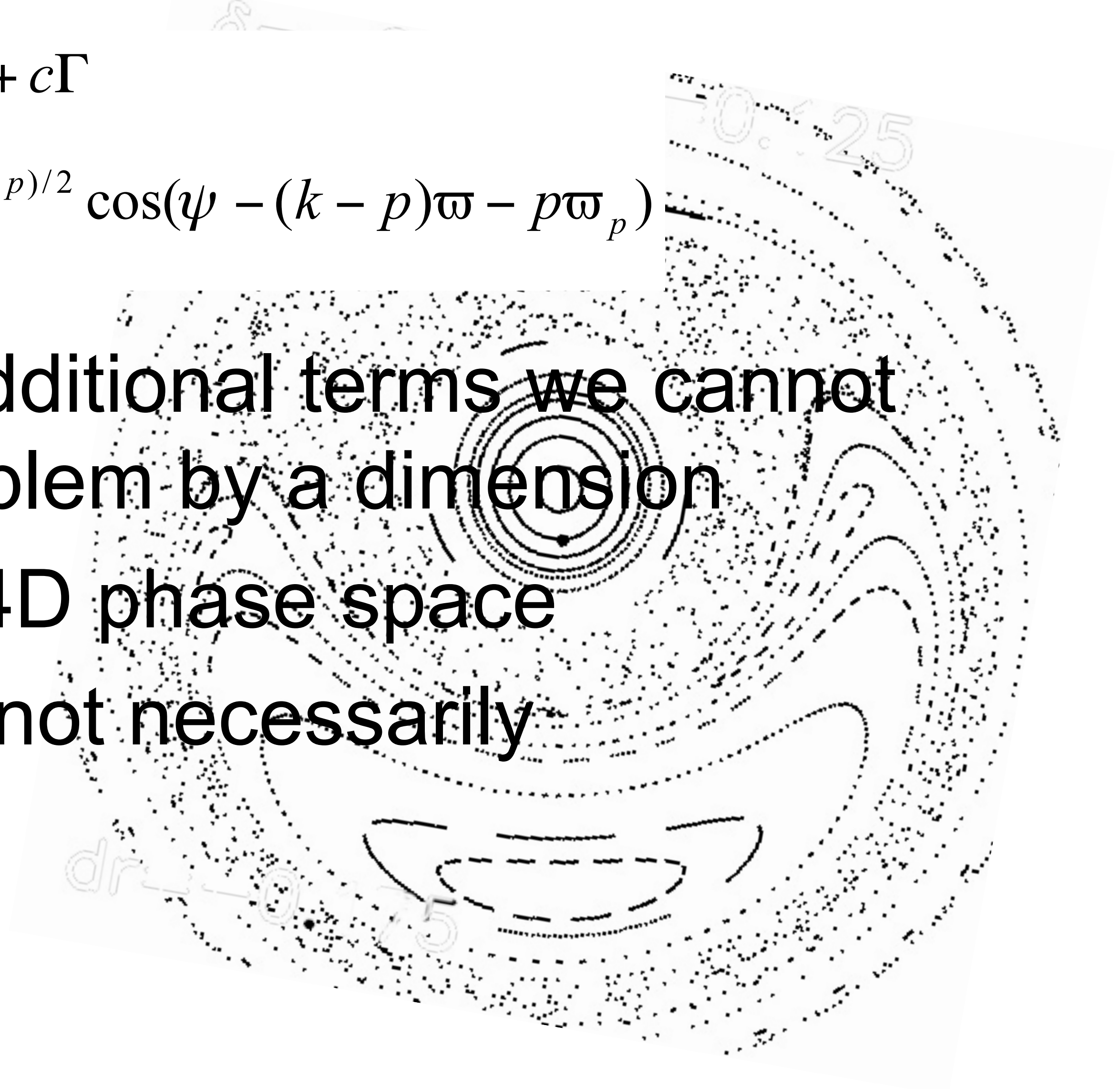
$$H(p,\phi) = ap^2 + bp + \epsilon p^{k/2}\cos(k\phi)$$



Weak resonances can capture at low probability in high eccentricity regime

Adding additional terms

$$K(\Lambda,\psi;\Gamma,\gamma) = a\Lambda^2 + b\Lambda + c\Gamma$$
$$- \sum_{p=0}^{k} \delta_{k,p} \Gamma^{(k-p)/2} \cos(\psi - (k-p)\varpi - p\varpi_p)$$



- If we include additional terms we cannot reduce the problem by a dimension
- In the plane a 4D phase space
- This system is not necessarily integrable

Role of corotation resonance in preventing capture

drift

separation set by secular precession rate

$$K(\overline{\Lambda},\psi;\overline{\Gamma},\gamma) = \overline{\Lambda}^2 + \overline{b}\overline{\Lambda} + \overline{c}\overline{\Gamma}$$

corotation resonance

$$-\overline{\Gamma}^{1/2}\cos(\psi - \varpi) + \overline{\varepsilon}\cos(\psi - \varpi_p)$$

proportional to planet eccentricity

- Above in rescaled units
- There is another libration timescale in the problem set by $1/\sqrt{\bar{\epsilon}}$
- Corotation resonance cannot grow during drift so it cannot capture. But it can be fast enough to prevent capture.

When there are two resonant terms

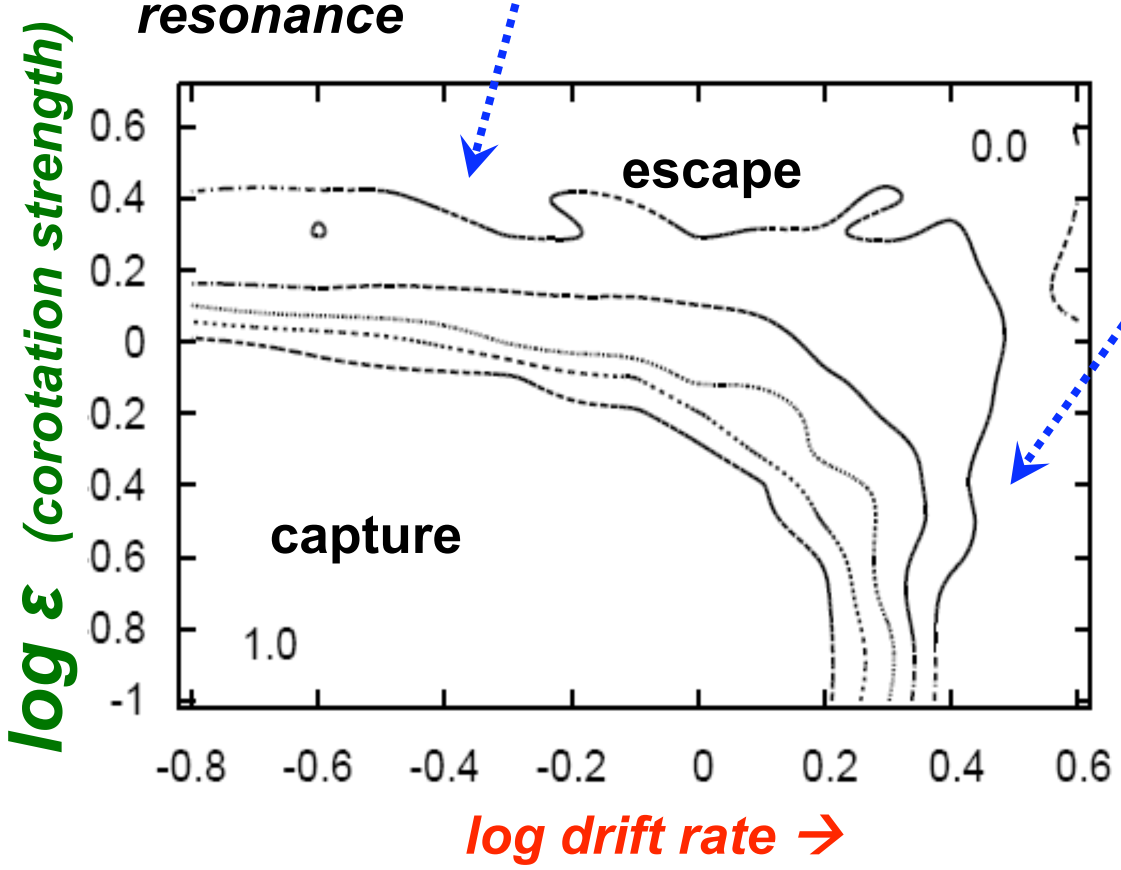
drift

separation set by secular precession

$$K(\bar{\Lambda},\psi;\bar{\Gamma},\gamma) = \bar{\Lambda}^2 + \bar{b}\bar{\Lambda} + \bar{c}\bar{\Gamma} - \bar{\Gamma}^{1/2}\cos(\psi - \varpi) + \bar{\varepsilon}\cos(\psi - \varpi_p)$$

depends on planet eccentricity

Capture is prevented by corotation resonance



Capture is prevented by a high drift rate

Comment on Temporary Capture

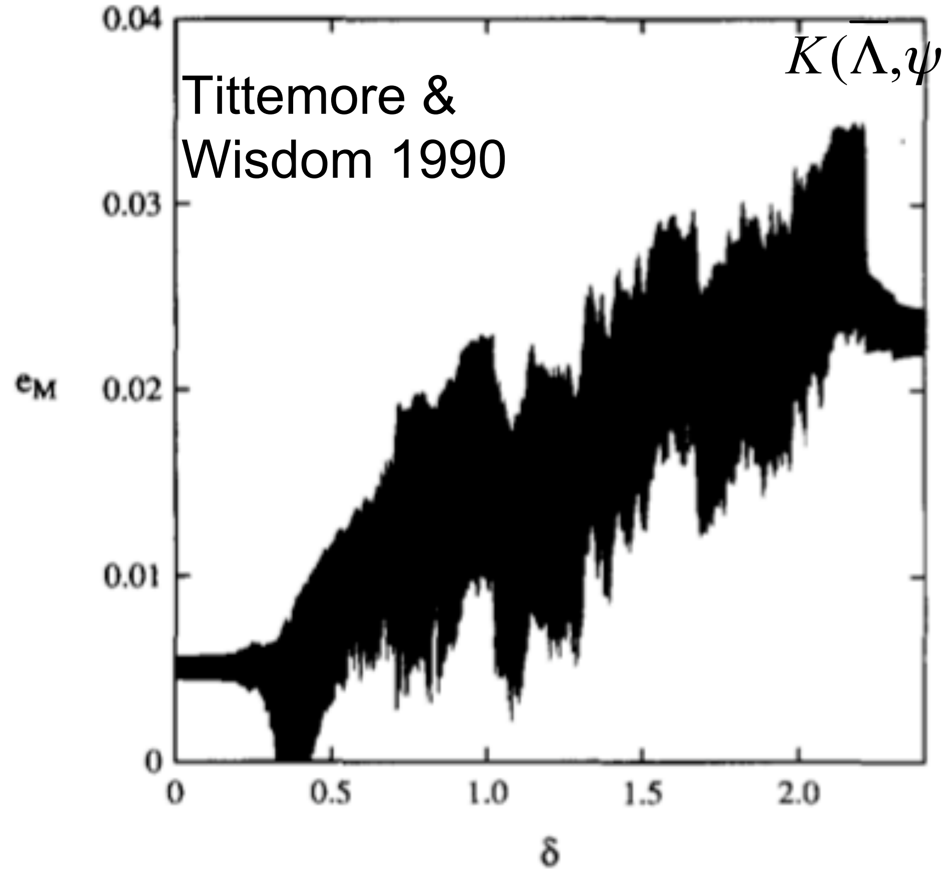


FIG. 11. Eccentricity variations of Miranda for a trajectory evolved through the 3 : 1 Miranda–Umbriel

$$K(\bar{\Lambda},\psi;\bar{\Gamma},\gamma) = \bar{\Lambda}^2 + \bar{b}\bar{\Lambda} + \bar{c}\bar{\Gamma} - \bar{\Gamma}^{1/2}\cos(\psi-\varpi) + \bar{\varepsilon}\cos(\psi-\varpi_p)$$

~ Slow evolution in one degree of freedom pulls other out of resonance?

Limits

Critical drift rates defining the adiabatic limit for each resonance can be computed from resonance strengths.

$$\dot{n}_{p,crit} = 2(j-1)^{-1} \left|\delta_{1,0}\right|^{4/3} \left|a\right|^{2/3} \quad \text{for } k = 1$$

$$= 0.5(j-2)^{-2} \delta_{2,0}^{2} \quad \text{for } k = 2$$

where $\delta_{1,0} = \mu\alpha^{5/4} f_{31}$ (+ possibly an indirect term)

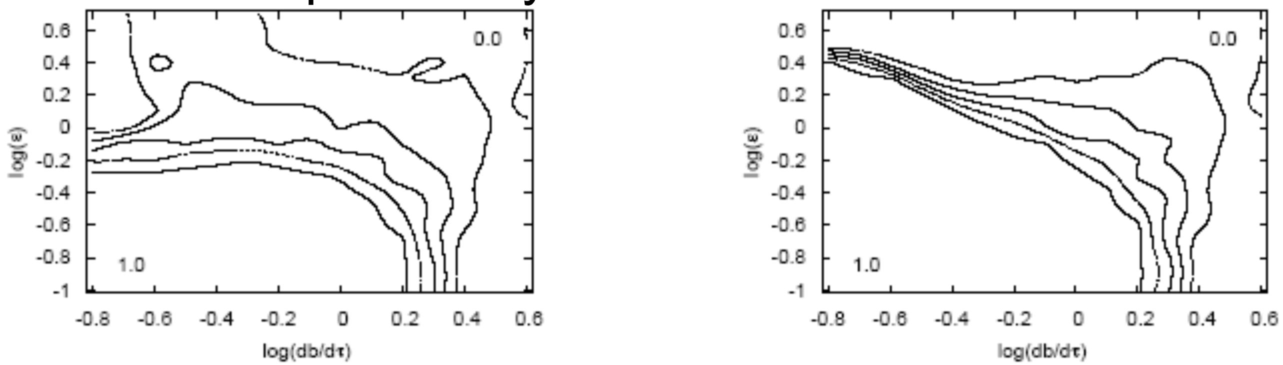
$$a = \tfrac{3}{2} j^2 \alpha^2$$

Other uses for dimensional analysis

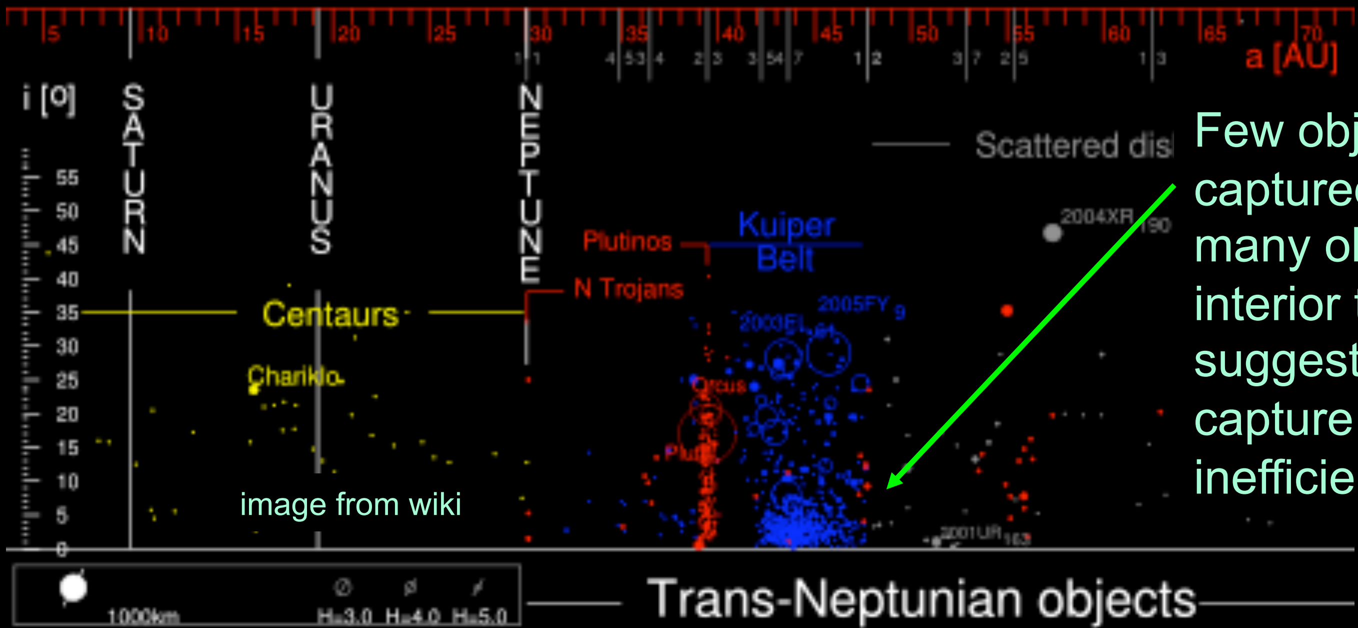
- If resonance fails to capture, there is jump in eccentricity. (e.g. eccentricity increases caused by divergent migrating systems when they go through resonance). Can be predicted dimensionally or using toy model and rescaling.
- Resonant width at low eccentricity can be estimated by estimating the range of b where the resonance is important. Jack Wisdom's $\mu^{2/3}$ for first order resonances.
- Perturbations required to push a particle out of resonance can be estimated

Trends

- Higher order (and weaker) resonances require slower drift rates for capture to take place. Dependence on planet mass is stronger for second order resonances.
- If the resonance is second order, then a higher initial particle eccentricity will allow capture at faster drift rates – but not at high probability.
- If the planet is eccentric the corotation resonance can prevent capture
- Resonance subterm separation (set by the secular precession frequencies) does affect the capture probabilities – mostly at low drift rates – the probability contours are skewed.



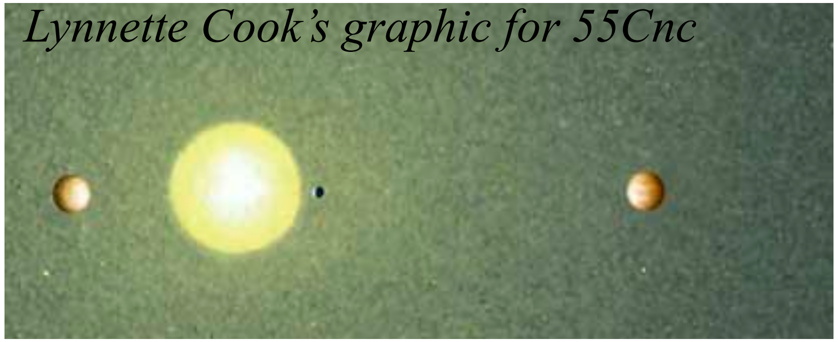
Applications to Neptune's migration



Few objects captured in 2:1 and many objects interior to 2:1 suggesting that capture was inefficient

- Neptune's eccentricity is ~ high enough to prevent capture of TwoTinos. 2:1 resonance is weak because of indirect terms, but corotation term not necessarily weak.
- Critical drift rate predicted is more or less consistent with previous numerical work.

Application to multiple planet systems



$$\text{drift timescale } \tau \equiv \frac{a_p}{\dot{a}_p}$$

For capture:

$$\tau_{2:1} > 0.4\mu^{4/3}P$$

$$\tau_{3:1} > 15\mu^2 P$$

A migrating extrasolar planet can easily be captured into the 2:1 resonance with another large planet but must be moving fairly slowly to capture into the 3:1 resonance. ~Consistent with timescales chosen by Kley for hydro simulations.

Super earth's require slow drift to capture each other.

This work has not yet been extended to do 2 massive bodies, but could be. Resonances look the same but coefficients must be recomputed and role of multiple subterms considered.

Final Comments

- Formalism is general – if you look up resonance coefficients, critical drift rates for non adiabatic limit can be estimated for any resonance : (*Quillen 2006 MNRAS 365, 1367*) ***See my website for errata!*** ***There is a new one recently found by Alex Mustill***
- This "*theory*" has not been checked numerically. Values for critical drift rate predicted via rescaling have not been verified to factors of a few.
- We have progress in predicting the probability of resonance capture, however we lack good theory for predicting lifetimes
- No understanding of k=>4
- Evolution during temporary capture can be complex
- Evolution following capture can be complex
- Extension to 2 massive bodies not yet done