PROBLEM SET #3 **AST570**

Problem 1. First order canonical transformations

a) Consider a Hamiltonian with a time dependent perturbation

$$H(I, \theta, t) = I\omega + \epsilon I^{1/2} \cos(\Omega_p t)$$

Find new variables J, ϕ such that the Hamiltonian becomes

$$K(J, \phi) = J\omega$$

and so is in action angle variables.

Hint: Try a generating function in the form

$$S_2(\theta, J) = \theta J + f(J) g(\Omega_p t)$$

with functions $f()$ and $g()$ to be determined.

b) Consider a Hamiltonian with a small perturbation term

$$H(I, \theta) = g(I) + \epsilon h(I) \cos\theta$$

where ϵ is small. Using a generating function in the form

$$S_2(\theta, J) = \theta J + \epsilon f(\theta, J)$$

to show that the Hamiltonian can be put via canonical transformation into a form $K(J, \phi) = g(J) + O(\epsilon^2)$... that is to first order in action angle variables.

Problem 2. Fixed points for a First order Mean Motion resonance

Consider the Hamiltonian for a first order resonance

$$H(\Gamma, \phi) = a\Gamma^2 + \delta\Gamma + \epsilon\Gamma^{1/2} \cos\phi$$

see Murray and Dermott section 8.8.

a) Perform a canonical transformation to new variables (x, y)

$$\begin{aligned} x &= \sqrt{2\Gamma} \cos\phi \\ y &= \sqrt{2\Gamma} \sin\phi \end{aligned}$$

b) Using Hamilton's equations set to zero

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0$$

find relations satisfied by the fixed points.

c) Why are these fixed points called periodic orbits?

d) For what value of δ are there 3 fixed points instead of 1? How does the sign of ϵ and a affect your answer?

e) If the Hamiltonian has units of $(L/T)^2$ (square of length/time) what units do the coefficients have?

f) Construct a parameter with units of frequency using the coefficients a and ϵ.

Problem 3. First order resonances in the limit of high j

First order mean motion resonances depend on coefficients f_{27}^j and f_{31}^j according to Table B.4 in M+D where

$$
\begin{aligned}
f_{27}^j(\alpha) &= \frac{1}{2}\left[-2j - \alpha D\right] b_{1/2}^j(\alpha) \\
f_{31}^j(\alpha) &= \frac{1}{2}\left[-1 + 2j + \alpha D\right] b_{1/2}^{j-1}(\alpha)
\end{aligned}
$$

Consider $\alpha = \frac{j-1}{j} = 1 - j^{-1}$.

Estimate how $f_{27}(1 - j^{-1})$ and $f_{31}(1 - j^{-1})$ depend on j for large j. You can use asymptotic limits estimated in the previous problem set.

Problem 4. Chaotic Zone Width and 2/7-th Law

Consider a planet in a circular orbit with semi major axis $a_p = 1$. Let $GM_* = 1$ and the ratio of the planet to stellar mass $\mu = m_p/M_*$.

a) Estimate the width of a $j : j+1$ mean motion resonance with the planet as a function of j and μ the reduced planet mass.

A Hamiltonian model for a first order mean motion resonance is

$$H(J, \phi) = aJ^2 + bJ + \epsilon J^{1/2} \cos\phi$$

with $a \sim j^2$, and $b \sim j dn$ where dn is the distance to resonance in terms of the mean motion. The perturbation strength $\epsilon \sim \mu j$ in the high j limit. The derivation of the Hamiltonian model (and coefficients) can be seen in a number of works (also Quillen 2006).

Here the coefficient b sets the distance to resonance. You can think of this as the range of mean motions over which the resonance is important.

b) To first order estimate the distance between resonance centers as a function of j. Consider a resonant condition $\frac{j}{j+1} = \frac{a}{a_p}^{\frac{3}{2}}$. How far apart in a is the $j : j+1$ resonance compared to the $j+1 : j+2$ resonance as a function of j in the limit of high j?

c) Show that the resonances overlap at a distance $da \sim \mu^{2/7} a_p$ from the planet.

The original work on the 2/7 law was by Jack Wisdom (1980).

Problem 5. Canonical Transformation to a Rotating Coordinate System

Consider the following Hamiltonian that has been used to represent a fourth order epicyclic approximation near a Lindblad resonance

$$H(I_1, \theta_1; I_2, \theta_2; t) = \Omega I_2 + \kappa I_1 + a I_1^2 + b I_2^2 + c I_1 I_2 + \epsilon I_1^{1/2} \cos(\theta_1 - m(\theta_2 - \Omega_p t))$$

where m is an integer and Ω_p a pattern speed. Here Ω and κ are the angular rotation rate and epicyclic frequency.

Consider the following generating function

$$F_2 = [\theta_1 - m(\theta_2 - \Omega_p t)] J_1 + \theta_2 J_2$$

a) Show that $I_2 + m I_1$ is a conserved quantity.
b) Find the form of the Hamiltonian in new coordinates.
c) Explain how $[(c - 2bm)(I_2 + m I_1) + \kappa - m(\Omega - \Omega_p)]$ can be considered the distance to the resonance.

Problem 6. Eccentricity increase caused by a jump across a mean motion resonance

Assume a planetesimal is in a circular orbit exterior to a planet that is slowly migrating inwards. The planet has a mass ratio (with the star) of μ. The planetesimal crosses the 3:2 mean motion resonance with the planet. Assume that the planetesimal initially has a zero eccentricity.
a)Show that the change $\delta e^2 \propto \mu^{2/3}$ so that the final eccentricity $e \propto \mu^{1/3}$.

b) How large a perturbation is required to remove the planetesimal from resonance?

Coefficients can be found in the paper *Reducing the probability of Capture into Resonance* by Quillen 2006 or Mustill & Wyatt 2011, 2011, MNRAS, 413, 554; http://arxiv.org/abs/1012.3079

Problem 7. Phase Lag in a drifting system captured into resonance

Consider the Hamiltonian

$$H(p, \phi) = ap^2 + bp + \epsilon p^{1/2} \cos \phi$$

that is often used to describe a first order mean motion resonance or Lindblad resonance. Consider an adiabatically drifting system with distance to resonance, $b(t)$, slowly variation. Above consider coefficients a, ϵ as constants.

Assume that the system captures into resonance. In resonance ϕ remains nearly constant, librating about either $\phi_o = 0$ or π depending upon the signs of a and ϵ.

a) Using Hamilton's equation for $\frac{\partial H}{\partial p}$ and assuming that $\dot{\phi}$ averages to zero, show that as p grows that

$$p \sim -\frac{b}{2a} \qquad \text{and} \qquad \dot{p} \sim -\frac{\dot{b}}{2a}$$

This is relevant to how eccentricity can increase for a particle or object captured into resonance.

b) Show that there is a phase lag

$$\delta\phi \sim \frac{-\dot{b}}{\epsilon\sqrt{-2ab}} \cos\phi_o$$

This is relevant to an asymmetry in the Earth's resonant dust ring that was detected by observations from infrared satellites such as COBE.

Problem 8. Periods for Conjunctions

a) Consider a mean motion resonance $jn \sim (j+k)n_p$ where n and n_p are the mean motions of object and planet, respectively and j, k are positive integers and $k < j$. Assume both are on circular orbits. What is the time period between conjunctions? Write your answer in terms of the planet's rotation period.

b) Consider a Laplace resonance with argument $\phi = p\lambda_1 - (p+q)\lambda_2 + q\lambda_3$ where $\lambda_1, \lambda_2, \lambda_3$ are longitudes of three different bodies, $\dot{\phi} \sim 0$ and q, p are positive integers. The mean motions of the three bodies are n_1, n_2, n_3 with $n_1 > n_2 > n_3$ interior to exterior. What is the time period between identical configurations? Assume that p and q contain no common factors greater than unity.

c) Find p, q for the 1:2:4 Laplace resonance between the Galilean moons Io, Europa and Ganymede. What is the time period between identical configurations for this system?

Problem 9. Planets migrating in disks

Consider two planets embedded in a gas disk and trapped into a $j : j+k$ mean motion resonance. Let the outer one be migrating inwards because it is driving spiral density waves into the outer disk. Let the innermost planet be more massive than the outer one. Let the dissipation forces act only on the outer planet with eccentricity damping timescale $\tau_e = e_o/\dot{e}_o$ and $\tau_a = a_o/\dot{a}_o$ with e_o, a_o the eccentricity and semi-major axis of the outer planet.

The Tisserand relation

$$C = \frac{1}{2\alpha} + \sqrt{\alpha(1 - e^2} \cos I$$

is approximately conserved in resonance. Here $\alpha = a_i/a_o$ is the ratio of the two planet's semi-major axes with a_o is the outer and a_i is the inner planet's semi-major axis. However the dissipation forces do not conserve the Tisserand relation.

In resonance $\alpha = a_i/a_o = [j/(j+k)]^{2/3}$ depends only on the integer ratios for the resonance. In resonance α must remain fixed even when the system is drifting.

a) What rate do you expect the eccentricity of the outer planet to increase as a function of migration rate?

As the resonance approximately conserves the Tisserand relation we can write

$$\dot{C} = \frac{\partial C}{\partial a_o} \dot{a}_{o,dis} + \frac{\partial C}{\partial e_o} \dot{e}_{o,disp}$$

which we can write

$$\dot{C} = \frac{\partial C}{\partial a_o} \frac{a_o}{\tau_a} + \frac{\partial C}{\partial e_o} \frac{e_o}{\tau_e}$$

b) Compute the limiting eccentricity e_{lim} where $\dot{C} = 0$. Show that if τ_e is long (weak eccentricity damping) the limiting eccentricity is larger, than if τ_e is small.