Problem 1. Velocity dispersion in a planetesimal disk

Consider first only a single particle with eccentricity e and semi-major axis a. In the epicyclic approximation

$$r \approx a(1 + e\sin(nt + \alpha))$$

and radial velocity component

$$u \approx ev_c \cos(nt + \alpha)$$

with $v_c = na$ is the velocity of a particle in a circular orbit. The expectation values can be estimated by integrating over all possible angles α assuming that a location in the disk contains all possible phases. This allows us to estimate $\langle u \rangle = 0$ and

$$\langle u^2 \rangle \sim \frac{1}{2\pi}\int_0^{2\pi} a^2 n^2 e^2 \cos^2 \alpha d\alpha \sim \frac{v_c^2 e^2}{2}$$

If we have an eccentricity dispersion σ_e^2 then we expect that

$$\langle u^2 \rangle \sim \frac{v_c^2 \sigma_e^2}{2}$$

Write the angular momentum $L = r^2\dot{\theta}$. To first order $L = na^2$ is independent of the eccentricity. To first order

$$\dot{\theta} = n(1 - 2e\sin(nt + \alpha))$$

using our expression for r.

When we consider v_θ we must be careful to specify our radial location. We compute v_θ at r_0 but orbits could have mean radius r_g. We write this $r_0 = r_g + dr$. Note that $\dot{\theta} = n_g(1 - 2dr/r_g)$. At r_0

$$v_\theta(r_0) = r_0\dot{\theta}(a_g, \alpha, e) = r_0 n_g(1 - 2dr/r_g)$$

We expand

$$n_g = n_0 + \frac{\partial n}{\partial a}(r_0)(r_g - r_0) = n_0 - \frac{\partial n}{\partial a}dr$$

Insert this into the previous equation

$$v_\theta(r_0) = r_0 n_0 \left(1 - \frac{dr}{r_0}\left(\frac{n_{,a} r_0}{n_0} + 2\right)\right)$$

We now compute the dispersion

$$\langle v_\theta^2 \rangle - \bar{v}_\theta^2 = r_0 n_0 \left(\frac{n_{,a} r_0}{n_0} + 2\right)^2 \frac{e^2}{2}$$

where the factor of $1/2$ comes from the expectation value of $\cos^2$. For the Keplerian system $n \propto a^{-3/2}$ so that $n_{,a} a/n = -3/2$ and

$$\sigma^2_{v_\theta} = \frac{v_c^2 \sigma_e^2}{8}$$

A similar calculation is done in section 3.2 by B+T.

We can now answer the questions.

a,b) As both quantities depend on dr with expectation value of 0, $\langle v_r \rangle = \langle v_\theta \rangle = 0$. Asymmetric drift ($\langle v_\theta - v_c \rangle$) requires a second order estimate.

c) The dispersions $\sigma^2_{v_r} \approx \frac{v_c^2 \sigma_e^2}{2} = 4\sigma^2_{v_\theta}$.

Caveats: We did not take into account the density gradient and this was done in a first order epicyclic approximation. Jean's equations could be used to find relations between moments taking into account density gradients for an isotropic disk.

Problem 2. Unperturbed Keplerian Hamiltonians

a) We know that the angular rotation rate is set by the mean motion n. As all orbital elements except for the mean anomaly are conserved we know that the Hamiltonian only depends on that conjugate to M or Λ. We require that $\frac{\partial H}{\partial \Lambda} = n = \sqrt{\frac{GM_*}{a^3}}$. We also require that $H = -\frac{GM_*}{2a}$.

Let $a(\Lambda)$. The previous two conditions imply that

$$\frac{GM_*}{2a^2} \frac{da}{d\Lambda} = \sqrt{\frac{GM_*}{a^3}}$$

or

$$\frac{\sqrt{GM_*}}{2a^{1/2}} da = d\Lambda$$

We can integrate this finding $\Lambda = \sqrt{GM_* a} + \text{constant}$. Plugging this back into H, we find the required form for H. The constant is arbitrary.

b) We have two conditions

$$\frac{\partial H_0}{\partial \Lambda_i} = \sqrt{\frac{GM_*}{a_i^3}}$$

and

$$H_0 = \sum_i -\frac{GM_* m_i}{2a_i}$$

Taking the derivative of H with respect to Λ_i

$$\frac{GM_* m_i}{2a_i^2} \frac{da_i}{d\Lambda_i} = \sqrt{\frac{GM_*}{a_i^3}}$$

and

$$da_i a_i^{-1/2} (GM)^{1/2} \frac{m_i}{2} = d\Lambda_i$$

Consequently we expect

$$\Lambda_i = m_i \sqrt{GM_* a_i}$$

Subbing this back into H_0 we find

$$H_0 = \sum_i -\frac{(GM_*)^2 m_i^3}{2\Lambda_i^2} \tag{1}$$

as desired.

Problem 3. Deriving Lagrange's Equations from Hamilton's equations

Lagrange's Planetary Equations are given on page 251,252 by M+D. Let's look at them when the inclination $I = 0$

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \epsilon} \\
\frac{d\epsilon}{dt} &= -\frac{2}{na}\frac{\partial \mathcal{R}}{\partial a} + \frac{\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial e} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial \varpi} \\
\frac{d\varpi}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial e}
\end{aligned}$$

where $\lambda = nt + \epsilon$.

The relevant Poincare momentum and coordinates (see page 60 or equation 2.179 by M+D) are

$$\begin{aligned}
\lambda &= M + \omega + \Omega & \Lambda &= \sqrt{a} \\
\gamma &= -\omega - \Omega = -\varpi & \Gamma &= \sqrt{a}(1-\sqrt{1-e^2})
\end{aligned}$$

I have set $GM = 1$.

The Hamiltonian would be $H = H_0 - \mathcal{R}$ with $H_0 = -1/(2\Lambda^2)$. Hamilton's equations are

$$\begin{aligned}
-\frac{\partial \mathcal{R}}{\partial \Gamma} &= \frac{d\gamma}{dt} = -\frac{d\varpi}{dt} \\
\frac{\partial \mathcal{R}}{\partial \gamma} &= \frac{d\Gamma}{dt} = -\frac{\partial \mathcal{R}}{\partial \varpi} \\
n - \frac{\partial \mathcal{R}}{\partial \Lambda} &= \frac{d\lambda}{dt} = n + \frac{d\epsilon}{dt} \\
\frac{\partial \mathcal{R}}{\partial \lambda} &= \frac{d\Lambda}{dt} = \frac{\partial \mathcal{R}}{\partial \epsilon}
\end{aligned}$$

We need to write out some time derivatives.

$$\begin{aligned}\dot\Lambda &= \frac{1}{2}\dot a a^{-1/2}\\ \dot\Gamma &= \frac{1}{2}\dot a a^{-1/2}(1-\sqrt{1-e^2}) - \frac{e\dot e a^{1/2}}{\sqrt{1-e^2}}\end{aligned}$$

Using the first of these and one of Hamilton's equations we find

$$\dot a = 2a^{1/2}\frac{\partial \mathcal{R}}{\partial \epsilon} = \frac{2}{na}\frac{\partial \mathcal{R}}{\partial \epsilon}$$

where it is useful to use relations $na = a^{-1/2}$ and $na^2 = a^{1/2}$. This is the first of Lagrange's equations. We can insert the above into our expression for $\dot\Gamma$

$$\dot\Gamma = (1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{e\dot e a^{1/2}}{\sqrt{1-e^2}}$$

Using another of Lagrange's equations

$$-\frac{\partial \mathcal{R}}{\partial \varpi} = (1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{e\dot e a^{1/2}}{\sqrt{1-e^2}}$$

Solving for $\dot e$ (and using $na^2 = a^{1/2}$)

$$\dot e = -\frac{\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2})\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial \varpi}$$

This is consistent with another of Lagrange's equations.

The other equations we need to compute a Jacobian matrix (note we could have started out by doing this!)

$$\begin{vmatrix}\frac{\partial\Lambda}{\partial a} & \frac{\partial\Lambda}{\partial e}\\ \frac{\partial\Gamma}{\partial a} & \frac{\partial\Gamma}{\partial e}\end{vmatrix} = \begin{vmatrix}\frac{1}{2a^{1/2}} & 0\\ \frac{1}{2a^{1/2}}(1-\sqrt{1-e^2}) & ea^{1/2}(1-e^2)^{-1/2}\end{vmatrix}$$

Inverting the Jacobian matrix

$$\begin{vmatrix}\frac{\partial a}{\partial\Lambda} & \frac{\partial a}{\partial\Gamma}\\ \frac{\partial e}{\partial\Lambda} & \frac{\partial e}{\partial\Gamma}\end{vmatrix} = \begin{vmatrix}2a^{1/2} & 0\\ \frac{-(1-\sqrt{1-e^2})\sqrt{1-e^2}}{ea^{1/2}} & \frac{\sqrt{1-e^2}}{ea^{1/2}}\end{vmatrix}$$

This can be checked with the following inversion

$$\begin{aligned}e &= \sqrt{1-\left(1-\frac{\Gamma}{\Lambda}\right)^2}\\ a &= \Lambda^2\end{aligned}$$

Let's go back to the Hamilton's equation that gave us $\dot\epsilon$

$$\begin{aligned} -\dot\epsilon &= \frac{\partial \mathcal{R}}{\partial \Lambda} = \frac{\partial \mathcal{R}}{\partial a}\frac{\partial a}{\partial \Lambda} + \frac{\partial \mathcal{R}}{\partial e}\frac{\partial e}{\partial \Lambda} \\ &= \frac{\partial \mathcal{R}}{\partial a} 2a^{1/2} + \frac{\partial \mathcal{R}}{\partial e}\frac{(-1)(1-\sqrt{1-e^2})}{ea^{1/2}} \\ \dot\epsilon &= -\frac{2}{na}\frac{\partial \mathcal{R}}{\partial a} + \frac{(1-\sqrt{1-e^2})}{ena^2}\frac{\partial \mathcal{R}}{\partial e} \end{aligned}$$

and consistent with the Lagrange's equation for $\dot\epsilon$.

Going back to the Hamilton's equation that gave us $\dot\varpi$

$$\begin{aligned} \dot\varpi &= \frac{\partial \mathcal{R}}{\partial \Gamma} = \frac{\partial \mathcal{R}}{\partial a}\frac{\partial a}{\partial \Gamma} + \frac{\partial \mathcal{R}}{\partial e}\frac{\partial e}{\partial \Gamma} \\ &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathcal{R}}{\partial e} \end{aligned}$$

Consistent with the other Lagrange's equation.

Problem 4. Relations between Laplace Coefficients

a) Divide the integral up into equal sized regions where $\cos jx$ is positive and where it is negative. It is possible to show that the denominator is always larger where the cosine is negative than the equivalent region where the cosine is positive. This implies that the total must always be positive. The denominator is always larger than $(1-\alpha)^{2s}$. Fourier coefficients must be smaller than the integral of the max of the function over the interval.

b)Using the integral form

$$\begin{aligned} b_s^j(\alpha) &= \frac{1}{\pi}\int_0^{2\pi} \frac{\cos jx \; dx}{(1+\alpha^2 - 2\alpha\cos x)^s} = \frac{1}{\pi}\int \frac{\cos jx (1+\alpha^2 - 2\alpha \cos x)}{(1+\alpha^2-2\alpha\cos x)^{s+1}} \\ &= (1+\alpha^2) b_{s+1}^j - \alpha \frac{1}{\pi}\int_0^{2\pi} \frac{2\cos jx \cos x \; dx}{(1+\alpha^2-2\alpha\cos x)^{s+1}} \end{aligned}$$

Using a trig identity

$$2\cos jx \cos x = \cos((j+1)x) + \cos((j-1)x)$$

we find

$$b_s^j = (1+\alpha^2) b_{s+1}^j - \alpha\left(b_{s+1}^{j+1} + b_{s+1}^{j-1}\right)$$

c)

$$\left(1 + \frac{\alpha}{s} D\right) b_s^j = (1-\alpha^2) b_{s+1}^j$$

follows immediately from the two relations.

d) Assuming $\lim_{\alpha\to 1} b^j_{1/2}(\alpha) = -\frac{\pi}{2}\ln(1-\alpha)$ and using

$$\lim_{\alpha\to 1} b^j_{s+1}(\alpha) \approx \lim_{\alpha\to 1} \frac{1}{2s(1-\alpha)} D b^j_s(\alpha)$$

recursively I find for $s > 1/2$

$$b^{(j)}_s(\alpha) \sim (1-\alpha)^{1-2s} \frac{2}{\pi} \frac{(2s-3)!!}{(2s-2)!!}$$

where the double factorial contains odd or even numbers only.

e)

$$b^j_s(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos(jx)dx}{(1+\alpha^2-2\alpha\cos x)^s}$$

Let $u = (1+\alpha^2-2\alpha\cos x)^{-s}$ and $dv = \cos(jx)dx$. This means that $du = -2\alpha s \sin x(1+\alpha^2-2\alpha\cos x)^{-(s+1)}$ and $v = j^{-1}\sin(jx)$.

$$b^j_s = \frac{1}{\pi} \left. \frac{j^{-1}\sin(jx)}{(1+\alpha^2-2\alpha\cos x)^s}\right|_0^{2\pi} + \frac{1}{\pi}\int_0^{2\pi} \frac{2\alpha s}{j} \frac{\sin x \sin(jx) dx}{(1+\alpha^2-2\alpha\cos x)^{s+1}}$$

We can use the trig identity $\sin x \sin(jx) = \frac{1}{2}\left(\cos((j-1)x) - \cos((j+1)x)\right)$ to find

$$b^j_s = \frac{\alpha s}{j}\left(b^{j-1}_{s+1} - b^{j+1}_{s+1}\right)$$

Problem 5. Asymptotic Limits for Laplace Coefficients

a) On the unit circle $z = e^{i\phi}$. As $\cos\phi = \frac{1}{2}\left(e^{i\phi}+e^{-i\phi}\right)$ we can write

$$f(\phi) = (1+\alpha^2 - \alpha(e^{i\phi}+e^{-i\phi}))^{-s}$$

It follows that $f(z)$ is equivalent to $f(\phi)$ on $|z| = 1$.

b) We can write a Fourier coefficient of $f(\phi)$ as

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(n\phi) f(\phi) d\phi$$

using the normalization $\frac{1}{\pi}\int_0^{2\pi}\cos(n\phi)\cos(m\phi)d\phi = \delta_{mn}$. We can then sum

$$f(\phi) = \sum_{n=0}^{\infty} a_n \cos(n\phi)$$

This implies that $a_n = b^n_s(\alpha)$.

c) By using the quadratic formulae we can show that the roots of $z(1+\alpha^2-\alpha(z+z^{-1}))$ are at α and α^{-1}. Consequently $f(z)$ has no discontinuities within the annulus $\alpha < |z| < \alpha^{-1}$. In fact

$$f(z) = \left(-\frac{\alpha}{z}(z-\alpha)(z-\alpha^{-1})\right)^{-s}$$

d)

$$f(\phi) = \sum_{n=0}^{\infty} a_n \cos(n\phi) = \sum_{n=0}^{\infty} b_s^n(\alpha)\cos(n\phi) = \sum_{n=0}^{\infty} b_s^n(\alpha)\frac{1}{2}\left(e^{in\phi} + e^{-in\phi}\right)$$

so that

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2} b_s^n(\alpha) z^n$$

e) The above is a series. Because the function is analytic in the annulus, the series much converge at all points within the annulus. For a series to converge the terms must decrease in size. The Cauchy root test for convergence states that the limit (large n) of the absolute value of the terms $|b_s^n(\alpha)z^n|$ must be less than 1. Applying the condition near the pole at α^{-1} we find that for large n

$$|b_s^n(\alpha)| \lesssim \alpha^n$$

where the above should really be described as a lim sup type of thing in the limit of large n.

f) the exponential follows by

$$\alpha^n = \exp(n \ln \alpha) = \exp(n \ln(1-\delta)) \sim \exp(-n\delta)$$

Problem 6. Computing a Precession Rate

Following the recipe on page 249 of M+D.

Our chosen argument does not involve mean longitudes (as we are considering secular effects) and is $j(\lambda' - \lambda)$ with $j = 0$. We want the term that depends on e^2 (or e'^2) so that can take derivatives with respect to Γ to calculate the precession rate. Looking at tables B.1 and B.3 we find that we want the f_2 function. Zero-th order argument for $f_2 = A_j/2$ (see Table B.3 M+D). The argument we want is $j(\lambda - \lambda')$ in table B.1 M+D with $j = 0$. We are interested in terms proportional to $e^2 + e'^2$ or f_2 in Table B.3. This gives

$$f_2(\alpha, j = 0) = \frac{1}{8}[2\alpha D + \alpha^2 D^2] A_0(\alpha)$$

with $A_j = b_{1/2}^{(j)}(\alpha)$ and $D = \frac{d}{d\alpha}$. There is a handy relation (M+D equation 7.4)

$$2\alpha \frac{db_{1/2}^{(0)}}{d\alpha} + \alpha^2 \frac{d^2 b_{1/2}^{(0)}}{d\alpha^2} = \alpha b_{3/2}^{(1)}$$

So

$$f_2 = \frac{1}{8}\alpha b_{3/2}^{(1)}(\alpha)$$

and the term in the direct part of the disturbing function

$$\mathcal{R}_D = f_2(e^2 + e'^2)$$

The disturbing function must be multiplied by other stuff that depends on whether the object is internal or external to the planet. See equations 6.44, 6.45 (M+D). External to the planet

$$\mathcal{R} = \frac{Gm_p}{a_p} f_2(\alpha) e^2, \qquad \alpha = \frac{a_p}{a}$$

Internal to the planet

$$\mathcal{R} = \frac{Gm_p}{a} f_2(\alpha) e^2, \qquad \alpha = \frac{a}{a_p}$$

The Poincaré variable $\Gamma = \sqrt{GM_* a}(1 - \sqrt{1-e^2})$ and for small eccentricity $\Gamma \sim \sqrt{GM_* a} e^2/2$. Inserting this into our disturbing function for external particles

$$\mathcal{R} = \frac{Gm_p}{a_p} f_2 \frac{2\Gamma}{\sqrt{GM_* a}} = \sqrt{\frac{GM_*}{a_p^3}} \frac{m_p}{M_*} \left(\frac{a_p}{a}\right)^{1/2} 2 f_2 \Gamma$$

and internal

$$\mathcal{R} = \sqrt{\frac{GM_*}{a_p^3}} \frac{m_p}{M_*} \left(\frac{a_p}{a}\right)^{3/2} 2 f_2 \Gamma$$

The previous two equations can be used to compute the precession rates by taking the derivative with respect to Γ. In units of $n_p = \sqrt{GM_*/a_p^3}$

$$\begin{aligned}
\dot{\varpi}_E &= n_p \mu \alpha^{1/2} 2 f_2(\alpha) = \frac{1}{4} n_p \mu \alpha^{3/2} b_{3/2}^{(1)}(\alpha) \\
\dot{\varpi}_I &= n_p \mu \alpha^{-3/2} 2 f_2(\alpha) = \frac{1}{4} n_p \mu \alpha^{-1/2} b_{3/2}^{(1)}(\alpha)
\end{aligned}$$

where $\mu = m_p/M_*$ is the mass ratio.

We may want to write precession rates in units of n

$$\begin{aligned}
\dot{\varpi}_E &= \frac{1}{4} n \mu b_{3/2}^{(1)}(\alpha) \\
\dot{\varpi}_I &= \frac{1}{4} n \alpha \mu b_{3/2}^{(1)}(\alpha) \qquad (2)
\end{aligned}$$

We explain the signs. The Hamiltonian $H = H_0 - \mathcal{R}$. Hamilton's equations give

$$\frac{\partial H}{\partial \Gamma} = -\frac{\partial R}{\partial \Gamma} = \dot{\gamma}$$

but the Poincaré coordinate $\gamma = -\varpi$. The precession is always positive (with rotation).

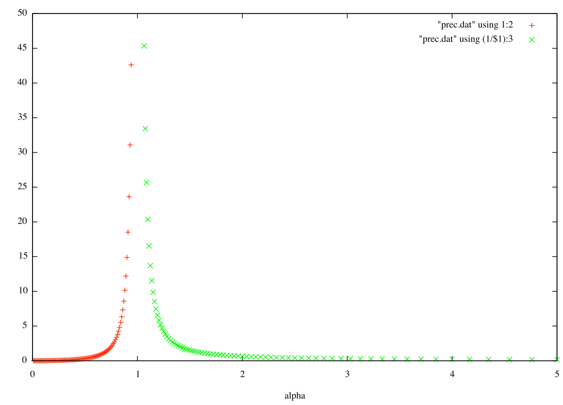

FIGURE 1. Precession rate in units of the object's mean motion. The y value shown should be multiplied by the planet's mass ratio. Computed using equations 2.

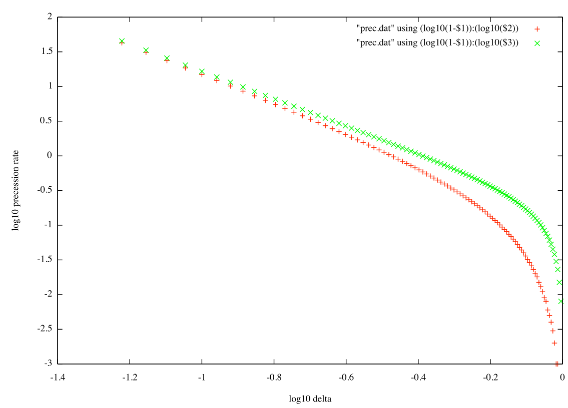

FIGURE 2. Precession rate in units of the object's mean motion on a log plot. The asymptotic limit is pretty good for $1-\alpha = \delta < 0.3$. The precession rate goes as $\mu\delta^{-2}$ in the asymptotic limit

Problem 7. Secular Perturbations to first order

Problem 7.1 (page 318 M+D) considers two planets with mass ratio q and ratio of semi-major axes α

$$\begin{aligned}\beta &\equiv b^{(2)}_{3/2}(\alpha)/b^{(1)}_{3/2}(\alpha)\\ q &\equiv \frac{m_2}{m_1}\\ \sigma &\equiv \frac{1}{4}n_1\mu_1\alpha b^{(1)}_{3/2}(\alpha)\end{aligned}$$

Here m_1, m_2 are masses of inner and outer planets and $\mu_1 \approx \frac{m_1}{M_*}$. First order secular perturbation theory describes two eccentricity eigenvector frequencies

$$g_\pm = \frac{\sigma}{2}\left[q\alpha + \alpha^{3/2} \pm \sqrt{(q\alpha - \alpha^{3/2})^2 + 4q\alpha^{5/2}\beta^2}\right]$$

Note the ratio of mean motions $\nu = n_2/n_1 = \alpha^{3/2}$. The eigenvectors can be written as

$$(e_{1f}, e_{2f}) \qquad (e_{1s}, e_{2s})$$

Where each component refers to the amplitude for a planet and I have taken $+$ to be fast and $-$ to be slow.

The ratio of the eigenvector amplitudes (again from the problem)

$$\left(\frac{e_2}{e_1}\right)_\pm = \frac{\alpha^{3/2}\beta}{\alpha^{3/2} - g_\pm/\sigma}$$

Some useful limits: As $q \to 0$, the frequencies $g_f \to \sigma\alpha^{3/2}$, $g_s \to 0$. As $\alpha \to 1$, $\beta \to 1$.

The condition that the outer planet was initially in a circular orbit is $e_{2f} - e_{2s} = 0.0$. The maximum eccentricity experienced by the outer planet would be $e_{2f} + e_{2s}$.

Problem 8. Precession rates for inclined circular orbits

a) The torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = -\mathbf{r} \times \nabla\Phi$. Φ only varies radially and along θ. At low inclination

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta} \sim \frac{\partial\Phi}{\partial z} \sim \frac{1}{r}\frac{\partial\Phi}{\partial\mu}$$

The radial derivative will give no torque, so we only need keep track of the z derivative of Φ. The sign convention is such that $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\hat{\boldsymbol{\phi}}$. Hence $\tau = \frac{\partial\Phi}{\partial\mu}\hat{\boldsymbol{\phi}}$.

b,c) Writing out the components

$$\boldsymbol{\tau} \approx (na)^2 J_2 \left(\frac{R_p}{a}\right)^2 3\sin f \sin i \times \begin{pmatrix} -\sin\Omega\cos f - \cos\Omega\sin f\cos i \\ \cos\Omega\cos f - \sin\Omega\sin f\cos i \\ 0 \end{pmatrix}$$

where we used $GM/r = (na)^2$. Only terms proportional to $\sin^2 f$ will not average to zero

$$\langle \boldsymbol{\tau} \rangle = -(na)^2 J_2 \left(\frac{R_p}{a}\right)^2 \frac{3}{2} \sin i \cos i (\cos\Omega, \sin\Omega, 0)$$

Taking

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = h(\sin\Omega \sin i, -\cos\Omega \sin i, \cos i)$$

with $h = na^2$ and

$$\dot{\mathbf{h}} = h \sin i \dot{\Omega} (\cos\Omega, \sin\Omega, 0)$$

we find

$$\frac{\dot{\Omega}}{n} = -\frac{3}{2} J_2 \left(\frac{R_p}{a}\right)^2$$

The precession is retrograde.

d) We assume the star has orientation $\mathbf{R} = D(\cos(n_* t), \sin(n_* t), 0)$ where n_* is the mean motion of the planet in orbit and the particle has position $\mathbf{r} = (x, y, z)$ so that

$$\mu' = \hat{\mathbf{r}} \cdot \hat{\mathbf{R}} = \frac{1}{r} \left(x \cos(n_* t) + y \sin(n_* t) \right)$$

$$P_2(\mu') = \frac{1}{2} \left(3\frac{x^2}{r^2} \cos^2(n_\odot t) + 3\frac{y^2}{r^2} \sin^2(n_* t) + \frac{6xy}{r^2} \sin(n_\odot t) \cos(n_\odot t) - 1 \right)$$

$$\langle P_2(\mu') \rangle = \frac{1}{2} \left(\frac{3}{2} \frac{(x^2 + y^2)}{r^2} - 1 \right) = \frac{1}{4} \left(1 - \frac{3z^2}{r^2} \right)$$

Consequently

$$\langle \Phi_T \rangle = -\frac{GM_* r^2}{D^3} \frac{1}{4} \left(1 - 3\mu_E^2 \right)$$

where $\mu_E = \mathbf{r} \cdot \hat{\mathbf{E}}$ and $\hat{\mathbf{E}}$ is the northern ecliptic pole.

By analogy with b,c we find that

$$\langle \boldsymbol{\tau} \rangle = -\frac{GM_* a^2}{D^3} \frac{3}{4} \sin i \cos i (\cos\Omega, \sin\Omega, 0)$$

and

$$\frac{\dot{\Omega}}{n} = -\frac{1}{n^2 a^2} \frac{GM_* a^2}{D^3} \frac{3}{4} = -\frac{3}{4} \left(\frac{n_*}{n}\right)^2 = -\frac{3}{4} \left(\frac{M_*}{M_p}\right) \left(\frac{a}{D}\right)^3$$

Problem 9. Asymptotic limits for Laplace coefficients in the limit $\alpha \to 1$

Useful relations for Laplace coefficients from Brouwer and Clemens are

$$b_s^{(j)} = \frac{j-1}{j-s} (\alpha + \alpha^{-1}) b_s^{(j-1)} - \frac{j+s-2}{j-s} b_s^{(j-2)} \tag{3}$$

$$b_{s+1}^{(j)} = \frac{(j+s)(1+\alpha^2) b_s^{(j)} - 2(j-s+1)\alpha b_s^{(j+1)}}{s(1-\alpha^2)^2} \tag{4}$$

By putting $j \to -j$

$$b_{s+1}^{(j)} = \frac{(s-j)(1+\alpha^2)b_s^{(j)} + 2(j+s-1)\alpha b_s^{(j-1)}}{s(1-\alpha^2)^2} \tag{5}$$

They also show that

$$b_{1/2}^{(0)} = \frac{4}{\pi}K(\alpha)$$

$$b_{1/2}^{(1)} = \frac{4}{\pi}\frac{K(\alpha)-E(\alpha)}{\alpha}$$

where The complete elliptic integral of the first kind $K(k)$ is

$$K(k) \equiv \int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}d\theta = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}dt$$

and the complete elliptic integral of the second kind $E(k)$ is

$$E(k) = \int_0^{\pi/2}(1-k^2\sin^2\theta)^{1/2}d\theta = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{1/2}dt$$

Assymptotic limits for the complete elliptic integral in the limit $k \to 1$

$$K(k) \sim -\frac{\log(1-k)}{2} + ...$$

$$\lim_{k\to 1} E(k) = 1$$

Consequently in the limit $\alpha \to 1$

$$b_{1/2}^0(\alpha) \sim -\frac{2}{\pi}\log(1-\alpha)$$

$$b_{1/2}^1(\alpha) \sim b_{1/2}^0$$

a) Show using equation 3 that in the asymptotic limit $\alpha \to 1$

$$b_s^j(\alpha) \approx b_{1/2}^{(0)}(\alpha) \sim -\frac{2}{\pi}\log(1-\alpha)$$

a)-sol) Suppose that $b_{1/2}^{(j-1)} = b_{1/2}^{(j-2)}$ then equation 3 becomes

$$b_{1/2}^{(j)} \quad \approx \quad \left[\frac{j-1}{j-s}(\alpha+\alpha^{-1}) - \frac{j+s-2}{j-s}\right] b_s^{(j-1)}$$

In the limit $\alpha \to 1$ the factor $\alpha + \alpha^{-1} \sim 2$ so

$$\begin{aligned} b_{1/2}^{(j)} &\approx \left[\frac{2j-2-(j+s-2)}{j-s}\right] b_s^{(j-1)} \\ &\approx b_s^{(j-1)} \end{aligned}$$

We already know that $b_{1/2}^{(0)} \sim b_{1/2}^{(1)}$ in this limit. Hence all $b_{1/2}^{(j)}$ have the same asymptotic limit. This is perhaps not surprising as for $\alpha \to 1$ we are reaching the limit for the function $(1-\alpha z)^s(1-\alpha z^{-1})^s$ to be analytic and the series expansion does not converge.

For $s = 1/2$ by using equation 3 iteratively, it is possible to write all the Laplace coefficients as sums of complete elliptical integrals, $K(\alpha)$ and $E(\alpha)$. For all j each coefficient decreases in size; $b_s^{(j+1)} > b_s^{(j)}$. In the limit of $\alpha \to 1$ the difference $b_{1/2}^{(j)} - b_{1/2}^{(j+1)}$ is a constant of order 1.

b) Show using equations 4, 5 that

$$b_{3/2}^{(0)} = \frac{4}{\pi}\frac{1}{(1-\alpha^2)^2}\left[2E(\alpha) - K(\alpha)(1-\alpha^2)\right]$$

and that

$$b_{3/2}^{(1)} = \frac{4}{\pi}\frac{1}{\alpha(1-\alpha^2)^2}\left[E(\alpha)(1+\alpha^2) - K(\alpha)(1-\alpha)^2\right]$$

Find asymptotic limits for $\alpha \to 1$ for $b_{3/2}^{(j)}(\alpha)$.

b) sol) straightforward plugging in gives you these relations. For both asymptotic limits the $K(\alpha)$ term is smaller giving

$$b_{3/2}^{(0)}(\alpha) \sim \frac{8}{\pi}(1-\alpha^2)^{-2} \sim \frac{2}{\pi}(1-\alpha)^{-2}$$

$$b_{3/2}^{(1)}(\alpha) \sim \frac{2}{\pi}(1-\alpha)^{-2}$$

As the two are the same, all subsequent j values will give the same asymptotic limit as in a).

c) Show that the asymptotic limit for $\alpha \to 1$

$$b_s^{(0)}(\alpha) \sim (1-\alpha)^{2s-1}\frac{2}{\pi}\frac{(2s-3)!!}{(2s-2)!!}$$

for $s > 1/2$.

c-sol) Using equation 4 and assuming the asymptotic limits $\alpha \to 1$ give $b_s^{(0)} \sim b_s^{(1)}$ we find

$$b_{s+1}^{(0)} \approx \frac{(2s-1)}{2s(1-\alpha^2)^2} b_s^{(0)}$$

which can only be used for $s > 1/2$. Evaluating for $s = 3/2, 5/2$

$$\begin{aligned} b_{5/2}^{(0)} &\approx \frac{2}{3}(1-\alpha)^{-2} b_{3/2}^{(0)} \\ b_{7/2}^{(0)} &\approx \frac{4}{5}(1-\alpha)^{-2} b_{5/2}^{(0)} \\ b_{9/2}^{(0)} &\approx \frac{6}{7}(1-\alpha)^{-2} b_{5/2}^{(0)} \end{aligned}$$

This gives

$$b_s^{(0)}(\alpha) \sim (1-\alpha)^{1-2s}\frac{2}{\pi}\frac{(2s-3)!!}{(2s-2)!!}$$

for $s > 1/2$.