PROBLEM SET #2 **AST570**

Problem 1. Velocity dispersion in a planetesimal disk

Consider a distribution of particles in a disk with an approximately Gaussian distribution of eccentricities with mean $\langle e \rangle = e_0$ and dispersion $\langle (e - e_0)^2 \rangle = \sigma_e^2$. Consider the velocity distribution measured at a particle radius.

a) What is the mean of the radial velocity of the distribution? You can assume that σ, e_0 are not large.
b) What is the mean of the tangential velocity component?
c) What are the dispersions of both components?

This problem can be done with a first order epicyclic approximation, neglecting the density gradient in the disk. For the Keplerian system the ratio of the standard deviations

$$\frac{\sigma_{v_r}}{\sigma_{v_\theta}} = \frac{1}{2}$$

Problem 2. Unperturbed Keplerian Hamiltonians

a) Show that the Hamiltonian for the pure Keplerian system (massless particle in orbit about the Sun) can be written

$$H_0 = -\frac{\mu^2}{2\Lambda^2}$$

for $\mu \sim GM_*$. Write the momentum Λ in terms of orbital elements.

b) Consider N bodies in motion about a star with mass M_*. Construct a Hamiltonian for the Keplerian system neglecting interactions between bodies except for those between the bodies and the star. Also neglect the motion of the star. The energy for this situation is

$$E = -\sum_i \frac{GM_* m_i}{2a_i}$$

Show that the Hamiltonian lacking body-body interactions can be written

$$H_0 = -\mu^2 \sum_{i=1}^{N} \frac{m_i^3}{2\Lambda_i^2}$$

with $\Lambda_i = m_i \sqrt{GM_* a_i}$.

Problem 3. Deriving Lagrange's equations from Hamilton's equations

Using Poincaré momenta and coordinates and the Hamiltonian

$$H = -\frac{\mu^2}{2\Lambda^2} - \mathcal{R}$$

where $\mathcal{R}$ is the disturbing function, derive some of Lagrange's planetary equations of motion using Hamilton's equations.

To simplify things by reducing the dimension of the problem assume the inclination $I = 0$.

The fastest way to do this problem is to compute the Jacobian matrix for derivatives of the Poincaré coordinates in terms of the orbital elements.

Problem 4. Relations between LaPlace coefficients

The Laplace coefficient is a Fourier component of a function that depends on α

$$b_s^j(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos jx \; dx}{(1+\alpha^2 - 2\alpha\cos x)^s}$$

a) Show that all Laplace coefficients are positive. Show that $b_s^j(\alpha) < 2(1-\alpha)^{-2s}$.

b) Show that

$$b_s^j(\alpha) = (1+\alpha^2)b_{s+1}^j - \alpha\left(b_{s+1}^{j+1} + b_{s+1}^{j-1}\right)$$

A similar relation using $D = d/d\alpha$ is

$$Db_s^j(\alpha) = -2\alpha s b_{s+1}^j(\alpha) + s\left(b_{s+1}^{j+1}(\alpha) + b_{s+1}^{j-1}(\alpha)\right)$$

c) Using the two relations show that

$$\left(1 + \frac{\alpha}{s}D\right) b_s^j = (1-\alpha^2) b_{s+1}^j$$

In the limit $\alpha \to 1$ this relation implies that

$$\lim_{\alpha\to 1} b_{s+1}^j(\alpha) \approx \lim_{\alpha\to 1} \frac{1}{2s(1-\alpha)} Db_s^j(\alpha)$$

d) The asymptotic limit $\lim_{\alpha\to 1} b_{1/2}^j(\alpha) = -\frac{\pi}{2}\ln(1-\alpha)$. Find a formula for $\lim_{\alpha\to 1} b_{s+i}^j(\alpha)$.

e) Integrating the formula for b_s^j by parts show that

$$b_s^j = \frac{\alpha s}{j}\left(b_{s+1}^{j-1} - b_{s+1}^{j+1}\right)$$

Problem 5. Asymptotic limits for Laplace coefficients for high j

The Laplace coefficient

$$b_s^j(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos jx \; dx}{(1+\alpha^2 - 2\alpha\cos x)^s}$$

In this problem we will consider

$$\lim_{j\to\infty} b_s^j(\alpha)$$

a) Consider the function

$$f(\phi) = (1+\alpha^2 - 2\alpha\cos\phi)^{-s}$$

Show that this function is equivalent to

$$f(z) = (1+\alpha^2 - \alpha(z+z^{-1}))^{-s}$$

on the unit circle for z a complex number.

b) Show that the Laplace coefficients are Fourier coefficients of the function $f(\phi)$.

c) Show that the function $f(z)$ can be analytically continued in the annulus $\alpha < |z| < \alpha^{-1}$.

d) Show that

$$f(z) = \frac{1}{2}\sum_{n=-\infty}^{\infty} b_s^n(\alpha) z^n$$

e) Using the Cauchy root test for convergence at the edge of the annulus show that at large n

$$|b_s^n(\alpha)| \lesssim \alpha^n$$

f) In the limit of $\alpha \to 1$ or $\delta = 1-\alpha \to 0$ show that the Laplace coefficients decay exponentially

$$|b_s^n(\alpha)| \lesssim e^{-n\delta}$$

The opposite limit, $\alpha \to 0$, can be done with expansion in terms of α as α is then a small parameter. The series form (M+D equation 6.68)

$$\begin{aligned}\frac{1}{2}b_s^j(\alpha) &= \frac{s(s+1)....(s+j-1)}{j!}\alpha^j \\ &\times \left[1 + \frac{s(s+j)}{1(j+1)}\alpha^2 + \frac{s(s+1)(s+j)(s+j+1)}{1\cdot 2(j+1)(j+2)}\alpha^4 + ...\right]\end{aligned}$$

Problem 6. Using a Laplace coefficient to compute a Precession rate

Consider a system consistent of a single planet of mass m and a single massless particle in orbit about a star of mass M_*. The planet's semi-major axis is a_p and that of the particle a. Let $\mu = m/M_* \ll 1$. Assume that the particle and planet's eccentricities and inclinations are low. Numerically compute the rate of precession, $\dot{\varpi}$, of the angle of peri-astron for the particle as a function of semi-major axis a. Numerically compute the precession rate for particles both internal and external to the planet. Give your answer in units of $\mu\sqrt{GM_*/a_p^3} = \mu n_p$ where n_p is the planet's mean motion.

At what distance from the planet is the asymptotic limit of $\sim \mu(1-\alpha)^2$ a good approximation?

You can follow the recipe on page 249 of M+D.

Problem 7. Secular Perturbations to first order

Consider the planet Fomalhaut B which is in proximity to an eccentric disk. Suppose it is not isolated and there are additional planets in the system.

a) First consider a two planet system. Can any limits or constraints on an additional planet can be placed from linear secular perturbation theory.

b) Consider the possibility that the disk itself is massive. What limits on the planet's orbit can be made assuming that the planet's orbit never crosses that of the disk edge.

c) Assume that the system initially had 2 planets and a disk all in circular orbits. Following a perturbation to the innermost planet we reach the current configuration. What types of systems and perturbations could be consistent with the current observations?

It may be useful to study problem 7.1 by Murray and Dermott.

Problem 8. Precession rates for inclined circular orbits

Consider a gravitational potential due to an oblate planet of mass M_p

$$\Phi(r,\mu) = -\frac{GM_p}{r}\left(1 - J_2\left(\frac{R_p}{r}\right)^2 P_2(\mu)\right)$$

Here $\mu = \cos\theta$ and θ is a spherical coordinate angle and $\theta = 0$ in the equatorial plane of the planet. The coefficient J_2 is the first term in a multipole expansion and $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$ is a Legendre polynomial. Here R_p is the radius of the planet.

Consider a satellite in a low inclination circular orbit; almost in the equatorial plane of the planet.

a) Show that the torque on the satellite is approximately

$$\boldsymbol{\tau} \sim \frac{\partial\Phi}{\partial z} r\hat{\boldsymbol{\phi}} \sim \frac{\partial\Phi}{\partial\mu}\hat{\boldsymbol{\phi}}$$

where $\hat{\boldsymbol{\phi}}$ is a unit vector in the equatorial plane pointing in the direction of rotation that is perpendicular to $\mathbf{r}$, the position vector of the satellite with respect to the center of the planet. We can write $\hat{\boldsymbol{\phi}} = (-y, x, 0)/\sqrt{x^2 + y^2}$ with z along the spin axis of the planet.

For true anomaly f, longitude of ascending node, Ω, and inclination, i, the position of the satellite can be written (orbital elements for the satellite in orbit about the planet)

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \cos\Omega\cos f - \sin\Omega\sin f\cos i \\ \sin\Omega\cos f + \cos\Omega\sin f\cos i \\ \sin f\sin i \end{pmatrix}$$

(see M+D equation 2.122 with ω ignored). Here

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = h(\sin\Omega\sin i, -\cos\Omega\sin i, \cos i)$$

is the angular momentum vector. For a circular orbit $h = na^2$ with n and a the mean motion and semi-major axis respectively. For an orbit that is precessing at constant inclination the torque

$$\tau = \dot{\mathbf{h}} = h\sin i\dot{\Omega}(\cos\Omega, \sin\Omega, 0)$$

b) By averaging over the true anomaly show that the torque on the satellite (averaged over time or position in orbit) is

$$\langle\tau\rangle = -\frac{3}{2}(na)^2 J_2 \left(\frac{R_p}{a}\right)^2 \sin i\cos i(\cos\Omega, \sin\Omega, 0)$$

c) Show that the precession rate

$$\frac{\dot{\Omega}}{n} \approx -\frac{3}{2}J_2\left(\frac{R_p}{a}\right)^2$$

where the semi-major axis $a = r$ for a circular orbit.

Is precession prograde or retrograde?

We now ignore J_2 due to the oblateness of the planet and consider the effect of a stellar tide. Consider a gravitational potential due to a stellar tide near a planet

$$\Phi_T(r, \mu') = -\frac{GM_*}{D}\left(\frac{r}{D}\right)^2 P_2(\mu')$$

Here $\mu' = \hat{\mathbf{r}} \cdot \hat{\mathbf{R}} = \cos\theta$ where $\theta = 0$ in the direction toward the star that has mass M_* and position vector $\mathbf{R}$ between star and planet. Here $|\mathbf{R}| = D$ is the distance of the planet to the star and we assume that the planet is in a circular orbit around the star. Here $\mathbf{r}$ is the position of the satellite with respect to the planet center.

d) Show that the tidal gravitational potential on the satellite at position $\mathbf{r}$ averaged over the orbit of the planet in orbit around the star (that means averaging over different planet/star orientations) is

$$\langle \Phi_T \rangle = -\frac{GM_* r^2}{D^3}\frac{1}{4}\left(1 - 3\mu_E^2\right)$$

where $\mu_E = \mathbf{r}\cdot\hat{\mathbf{E}}$ and $\hat{\mathbf{E}}$ is the northern ecliptic pole.

e) Show that the precession rate for a low inclination circular orbit is

$$\frac{\dot{\Omega}}{n} = -\frac{3}{4}\left(\frac{n_*}{n}\right) = -\frac{3}{4}\left(\frac{M_*}{M_p}\right)\left(\frac{a}{D}\right)^3$$

(see problem 6.6 by M+D). Note here low inclination is with respect to the ecliptic rather than the equatorial plane of the planet.

It may help to assume that the star has orientation $\mathbf{R} = D(\cos(n_* t), \sin(n_* t), 0)$ where n_* is the mean motion of the planet in orbit.

Problem 9. Asymptotic limits for Laplace coefficients in the limit $\alpha \to 1$

Useful relations for Laplace coefficients from Brouwer and Clemens 1961 are

$$b_s^{(j)} = \frac{j-1}{j-s}(\alpha + \alpha^{-1})b_s^{(j-1)} - \frac{j+s-2}{j-s}b_s^{(j-2)} \tag{1}$$

$$b_{s+1}^{(j)} = \frac{(j+s)(1+\alpha^2)b_s^{(j)} - 2(j-s+1)\alpha b_s^{(j+1)}}{s(1-\alpha^2)^2} \tag{2}$$

By putting $j \to -j$

$$b_{s+1}^{(j)} = \frac{(s-j)(1+\alpha^2)b_s^{(j)} + 2(j+s-1)\alpha b_s^{(j-1)}}{s(1-\alpha^2)^2} \tag{3}$$

They also show that

$$b_{1/2}^{(0)} = \frac{4}{\pi}K(\alpha)$$

$$b_{1/2}^{(1)} = \frac{4}{\pi}\frac{K(\alpha) - E(\alpha)}{\alpha}$$

where the complete elliptic integral of the first kind, $K(k)$, is

$$K(k) \equiv \int_0^{\pi/2}(1 - k^2\sin^2\theta)^{-1/2}d\theta = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}dt$$

and the complete elliptic integral of the second kind, $E(k)$, is

$$E(k) \equiv \int_0^{\pi/2}(1 - k^2\sin^2\theta)^{1/2}d\theta = \int_0^1 (1-t^2)^{-1/2}(1-k^2t^2)^{1/2}dt$$

Assymptotic limits for the complete elliptic integral in the limit $k \to 1$

$$K(k) \sim -\frac{\log(1-k)}{2} + ...$$
$$\lim_{k \to 1} E(k) = 1$$

Consequently in the limit $\alpha \to 1$

$$b^0_{1/2}(\alpha) \sim -\frac{2}{\pi} \log(1-\alpha)$$
$$b^1_{1/2}(\alpha) \sim b^0_{1/2}$$

a) Show using equation 1 that in the asymptotic limit $\alpha \to 1$

$$b^j_s(\alpha) \approx b^{(0)}_{1/2}(\alpha) \sim -\frac{2}{\pi} \log(1-\alpha)$$

b) Show using equations 2, 3 that

$$b^{(0)}_{3/2} = \frac{4}{\pi} \frac{1}{(1-\alpha^2)^2} \left[2E(\alpha) - K(\alpha)(1-\alpha^2)\right]$$

and that

$$b^{(1)}_{3/2} = \frac{4}{\pi} \frac{1}{\alpha(1-\alpha^2)^2} \left[E(\alpha)(1+\alpha^2) - K(\alpha)(1-\alpha)^2\right]$$

Find asymptotic limits for $\alpha \to 1$ for $b^{(j)}_{3/2}(\alpha)$.

c) Show that the asymptotic limit for $\alpha \to 1$

$$b^{(0)}_s(\alpha) \sim (1-\alpha)^{2s-1} \frac{2}{\pi} \frac{(2s-3)!!}{(2s-2)!!}$$

for $s > 1/2$.