Problem 1. On Hyperbolic orbits

a) Finding semi-major axis a:
Total energy must take into account the center of mass motion.

$$E = \frac{GMm}{2a} + \frac{(M+m)V_{com}^2}{2}$$

The mass m starts with velocity v_∞, the other one with zero velocity. This means

$$E = \frac{mv_\infty^2}{2}$$

and

$$(m+M)V_{com} = mv_\infty$$

Putting together our relations for Energy and subbing in for V_{com} we can solve for a

$$a = \frac{G(M+m)}{v_\infty^2}$$

Finding eccentricity e:
Using the equation of motion we found that the orbit is described by

$$r = \frac{p}{1 + e\cos f} \tag{1}$$

with

$$p = \frac{h^2}{G(M+m)}$$

and angular momentum $h = bv_\infty$. The parameter p and the orbit equation also imply that

$$p = a(e^2 - 1)$$

Using our expression for a, h and p we can solve for e^2 finding

$$e^2 = 1 + \frac{b^2 v_\infty^4}{G^2(M+m)^2}$$

Finding pericenter q:
The orbit equation implies that the closest approach has

$$q = a(e-1) = \frac{G(M+m)}{v_\infty^2}\left[\left(1 + \frac{b^2 v_\infty^4}{G^2(M+m)^2}\right)^{1/2} - 1\right]$$

b) Showing the relation for e:

Using $q = a(e-1)$ and $p = a(e^2-1)$ and $v_0 = \sqrt{2G(M+m)/q}$ we find that

$$e + 1 = \frac{p}{q} = \frac{pv_0^2}{2G(M+m)} = \frac{b^2 v_0^2 v_\infty^2}{2G^2(M+m)^2}$$

and then

$$e - 1 = \frac{p}{a(e+1)} = \frac{2v_\infty^2}{v_0^2} \tag{2}$$

and so

$$e = 1 + 2v_\infty^2/v_0^2$$

where v_0 is the escape velocity at q.

c) Using the equation describing the orbit (equation 1) we find that $r \to \infty$ when

$$\sec f_0 = -e$$

where f_0 is the true anomaly for infinite r. Relating deflection angle, ψ to this true anomaly gives

$$\psi = 2f_0 - \pi \qquad f_0 = (\psi + \pi)/2$$

Subbing into our relation for f_0 we find

$$\cos\left(\frac{\pi}{2} + \frac{\psi}{2}\right) = -e^{-1}$$

equivalent to

$$\sin\left(\frac{\psi}{2}\right) = e^{-1}$$

d) Jupiter, with mass $M_p = 1898.6 \times 10^{24}$kg, and radius $R = 71,398$ km. The escape velocity at the surface $v_0 = 60$ km/s. Plugging in $v_\infty = 10$ km/s I find an eccentricity of $e = 1.06$ and a maximum grazing deflection angle of about 141 degrees.

Problem 2. Impulse approximation and velocity kick due to a passing planet

Consider a particle in a planetary ring with semi-major axis a that is in a circular orbit about a planet. There is a nearby moon, also in a circular orbit. The ratio of the moon to planet mass is μ. The difference between moon and particle semi-major axes is da.

a) The close passage would give a velocity kick towards the object. This would give a radial velocity v_r. We need to know the difference in velocity between the moon and particle. This depends on the difference in angular rotation rate (if they have the same angular rotation rate then they are fixed with respect to one another). Let n be the mean motion of the moon with mass m and semi-major axis a from the planet of mass M. The mean motion of the ring particle would be $n_r = n + \frac{dn}{da}da$. The mean motion $n(a) = (GM)^{1/2}a^{-3/2}$ so $dn/da = -\frac{3}{2}\frac{n}{a}$. This means that $n_r = n\left(1 - \frac{3}{2}\frac{da}{a}\right)$. The relative velocity between planet and ring particle is then $v_t = -\frac{3}{2}n\ da$.

We can how use the impulse approximation giving a radial velocity

$$dv = v_r \sim \frac{2Gm}{v_t da} \sim \frac{Gm}{n da^2}$$

Let us divide this by the velocity of the moon or $\sqrt{GM/a} = na$ finding

$$\frac{v_r}{na} \sim \mu \left(\frac{a}{da}\right)^2$$

where $\mu = m/M$.

b) After the close approach the orbit has its highest v_r which from the orbit we would expect has a value e times the circular velocity. Thus the above expression also gives the eccentricity following the encounter. The above expression is consistent with that estimated by Murray and Dermott.

c) The impulse approximation is not expected to be very good since the velocity difference between objects is not high.

d) Using our formulas for a hyperbolic orbit

$$\begin{aligned}
\Delta v_{M,\perp} &= \frac{2mbV_0^3}{G(M+m)}\frac{1}{C} \\
\Delta v_{M,\parallel} &= \frac{2mV_0}{G(M+m)}\frac{1}{C} \\
C &= 1 + \frac{b^2V_0^4}{G^2(M+m)^2}
\end{aligned}$$

The impulse approximation is okay when $C > 1$ or when

$$\frac{b^2V_0^4}{G^2(M+m)} > 1$$

This condition can be rewritten as

$$\frac{V_0}{v_b} > 1$$

where $v_b = \sqrt{G(M+m)/b}$ is approximately the escape velocity at b.

Above considers an encounter between m and M where we are considering an encounter between a massless particle and a moon of mass m.

Using our expression for $v_t \sim n\ da$ for V_0 and impact parameter $b = da$ and $v_b \sim \sqrt{Gm/da}$ the condition becomes

$$\frac{v_t}{v_b} \sim \left(\frac{M}{m}\right)^{1/2}\left(\frac{da}{a}\right)^{3/2} > 1$$

As long as this condition is satisfied the impulse approximation is okay.

Another way to rewrite the condition in a more physically meaningful way is to compare the timescale of the close approach with the orbital timescale. The timescale of the close approach is $(da)^{3/2}/(Gm)^{1/2}$ and the orbital timescale is $a^{3/2}/(GM)^{1/2}$ and the ratio the same as the ratio of v_t/v_b given above.

Problem 3. Tidal forces

Problem 4. Estimates of Gravitational Heating Rates

a) Assume we are in a dispersion dominated regime and consider the diffusion coefficient from perpendicular velocity perturbations.

First let us estimate the Coulomb log, $\ln \Lambda$

$$\Lambda = \frac{b_{max} V_0^3}{G(M+m)}$$

For the maximum impact parameter, the galactocentric radius is an upper limit. The important mass in the problem is that of molecular clouds or $10^6 M_\odot$. The relevant velocity probably the velocity dispersion or 10km/s. Altogether this gives $\ln \Lambda \sim 19$.

Now onto the diffusion coefficient

$$D(\Delta v_\perp^2) \sim \frac{4\sqrt{2}\pi G^2 \rho m_a \ln \Lambda}{\sigma} \left(\frac{\mathrm{erf}(X) - G(X)}{X} \right)$$

Assume that the function of $X = v/\sigma$ is of order 1. Here σ is the velocity dispersion. Here $\rho = n_a m_a$ is the mass density of molecular clouds.

Assume the molecular gas has a thickness of 100 pc. The mass surface density of molecular clouds is of order $\Sigma \sim 1 M_\odot$ pc^{-2}. The number density (numbers of clouds per unit volume) is then approximately $n_a \sim \Sigma/h/m_a$ where $h = 100$pc and m_a is a million solar masses. This gives a number density of $n_a \sim 10^{-8}$ pc^{-3} $\sim 3.7 \times 10^{-64}$cm^{-3}. Inserting all values into our expression for $D(\Delta v_\perp^2)$ I estimate 0.002 in cgs. So what are the units of this? It's a diffusion coefficient in velocity so it is in units of v^2/t or cm^2/s^3. Let us check that this is correct. It is. Let us convert the units to that it is in km/s per Myr. This corresponds to multiplying by 3×10^{13} (for the Myr) and dividing by 10^{10} for the (km/s)^2 or altogether multiply by 300. This gives a diffusion coefficient of order 1 (km/s)^2 Myr^{-1}.

b)

c)

Problem 5. Disruption of Binary Planetesimals by a close approach to a planet

Many massive Kuiper Belt objects are nearly equal mass binaries. For example, Pluto and Charon are separated by about 2×10^9 cm. Their masses are 13 and 2 $\times 10^{24}$g. Their densities are about 1.8 g/cm^3.

a) The Hill radius of Pluto, but at 5 AU where Jupiter is located, is $r_H \sim 0.003$ AU or 0.5 million km. At a location of Neptune the Hill radius is $r_H \sim 0.02$ AU or 3 million km. Objects can remain bound to Pluto as long as they are within this separation.

b) Expanding the potential about the larger body we estimate that the the smaller body is no longer bound to the larger one when

$$r \lesssim d \left(\frac{M_p}{M_{Pluto}} \right)^{1/3}$$

where d is the separation of the binary and r the distance to the planet. Inserting distance an mass we estimate a distance of 2 million km.

c) When the objects are separated they gain a component of velocity with respect to their original center of mass from rotation in the bound system. The velocity they gain depends on the mass of the object, effectively the lower mass object is flung out faster than the larger one. For one object to become bound energy must be removed from it and transferred to the other one. The one that is flung out with respect to the center of mass can be flung out in a direction that slows it down with respect to the planet, reducing its energy and angular momentum with respect to the planet. This means that the lower mass object is more likely to become the one that is bound.

Problem 6. Orbital timescales in the outer parts of a Hill sphere

The orbital period of the planet $P = 2\pi \frac{a_p^{3/2}}{(GM_*)^{1/2}}$. The orbital period of the satellite about the planet $P_s = 2\pi \frac{a_s^{3/2}}{(GM_p)^{1/2}}$. The ratio of these

$$\frac{P_s}{P} = \left(\frac{a_s}{a_p} \right)^{3/2} \left(\frac{M_*}{M_p} \right)^{1/2}$$

Remember that $r_H = a_p \left(\frac{M_p}{3M_*} \right)^{1/3}$. If we sub in for the mass ratio

$$\frac{P_s}{P} \sim \left(\frac{a_s}{a_p} \right)^{3/2} \left(\frac{a_p}{r_H} \right)^{3/2} 3^{-1/2} \sim \left(\frac{a_s}{r_H} \right)^{3/2}$$

This radio as $a_s/r_H \to 1$ is of order 1. Collision timescales in the outer parts of Hill radii can be similar to orbital timescales.