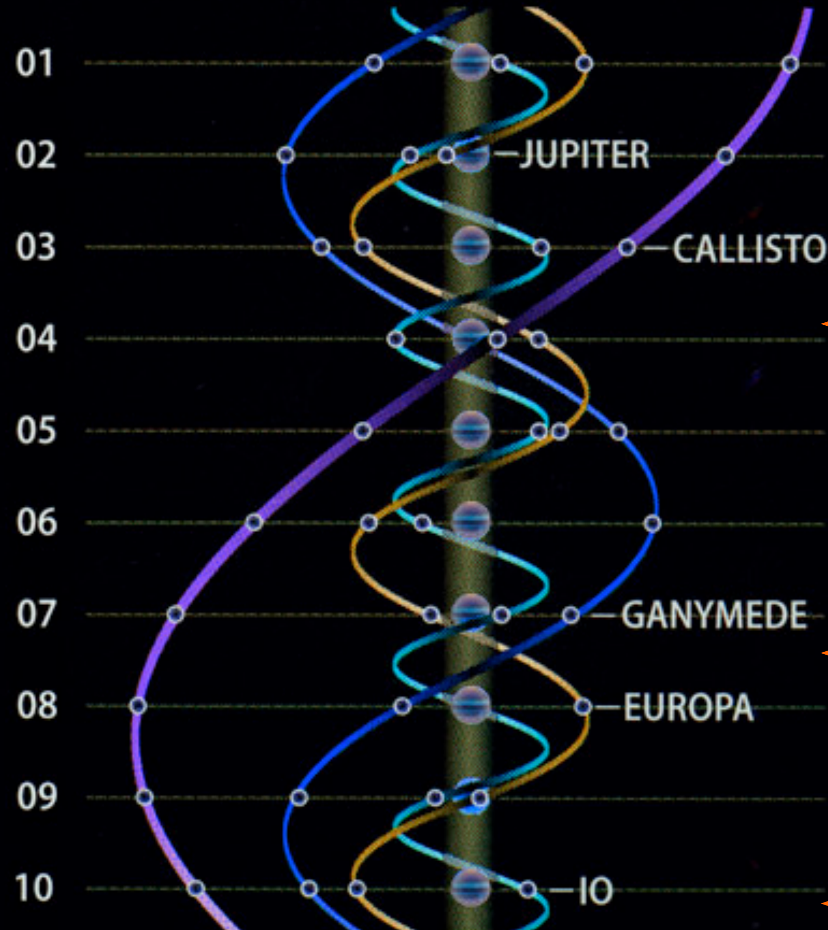


Orbital trajectories of Jupiter's four largest moons over a period of 10 Earth days, illustrating the 4:2:1 resonance among the orbits of the inner three

Mean Motion Resonances

- Hamilton-Jacobi equation and Hamiltonian formulation for 2-body problem in 2 dimensions
- Cannonical transformation to heliocentric coordinates for N-body problem
- Symplectic integrators
- Canonical transformation with resonant angle
- Mean motion resonances

Integrable motion

- Integrable: n degree of freedom Hamiltonian has n conserved quantities
- Examples:
  - Hamiltonian is a function of coordinates only
  - Hamiltonian is a function of momenta only – in this case we can call the momenta action variables and we can say we have transformed to action angle variables
  - Hamiltonian has 1 degree of freedom and is time independent. H(x,v) gives level contours and motion is along level contours.
- Arnold-Louiville theorem - integrable implies that the Hamiltonian can be transformed to depend only on actions

Hamilton Jacobi equation

- If the coordinate *q* does not appear in the Hamiltonian then the corresponding momentum *p* is constant
- Try to find a Hamiltonian that vanishes altogether, then everything is conserved
- Generating function $S_2(q_1, q_2, \ldots; P_1, P_2, \ldots, t)$ function of old coordinates and new momenta which we would like conserved
- New Hamiltonian

$$K = H(q_1, q_2, ..; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, ..., t) + \frac{\partial S}{\partial t} = 0$$

Finding conserved momenta 2-body problem

- Our new momenta are conserved, constants of integration, P_i

$$Q_i = \frac{\partial S}{\partial P_i} \qquad\qquad p_i = \frac{\partial S}{\partial q_i}$$

- 2 body in 2 dimensions

$$H(r, \theta; p_r, L) = \frac{p_r^2}{2} + \frac{L^2}{2r^2} - \frac{\mu}{r}$$

- Hamilton Jacobi equation

old coordinates

new momenta

$$H\left(r, \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}\right) + \frac{\partial S}{\partial t} = 0$$

- Assume separable

$$S = S_r(r) + S_\theta(\theta) + S_t(t)$$

- Substitute momenta in to Hamilton Jacobi equation

$$\frac{1}{2}\left(\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\theta}{\partial \theta}\right)^2\right) - \frac{\mu}{r} = -\frac{\partial S}{\partial t}$$

- Since separable $\frac{\partial S_\theta}{\partial \theta} = \alpha_2 \quad \frac{\partial S_t}{\partial t} = -\alpha_1$

Because we choose *S* separable these derivatives must be constant,
as they are constant we also make them be our new momenta

- Hamilton Jacobi equation gives

$$\left(\frac{dS_r}{dr}\right)^2 = 2\left(\alpha_1 + \frac{\mu}{r}\right) - \frac{\alpha_2^2}{r^2}$$

$$S = -\alpha_1 t + \alpha_2 \theta + \int^r dr \sqrt{2(\alpha_1 + \frac{\mu}{r}) - \frac{\alpha_2^2}{r^2}}$$

$$S = -\alpha_1 t + \alpha_2 \theta + \int^r dr \sqrt{2(\alpha_1 + \frac{\mu}{r}) - \frac{\alpha_2^2}{r^2}}$$

S depends on new monenta so $\alpha_1 \alpha_2$ are new momenta

$$\frac{\partial S}{\partial \theta} = \alpha_2 = L = P_2 \qquad P_1 = \alpha_1 \qquad = \theta$$

$$\boxed{P_2 = L = \sqrt{\mu a(1-e^2)}}$$

using expression for L

$$K = H + \frac{\partial S}{\partial t} = H - \alpha_1 = 0$$

$$\boxed{P_1 = -\frac{\mu}{2a}}$$

is energy

New coordinates

$$\frac{\partial S}{\partial \alpha_1} = Q_1 \qquad \frac{\partial S}{\partial \alpha_2} = Q_2$$

$$Q_1 = -t + \int^r dr \left[2(\alpha_1 + \frac{\mu}{r}) - \frac{\alpha_2^2}{r^2}\right]^{-1/2}$$

$$Q_1 = -t + \int^r dr \left[2(\alpha_1 + \frac{\mu}{r}) - \frac{\alpha_2^2}{r^2} \right]^{-1/2}$$

- Sub in for constants and use

$$r = a(1 - e\cos E), \qquad dr = ae\sin E dE$$

$$\begin{aligned} Q_1 &= -t + \frac{a^{3/2}}{\sqrt{\mu}} \int (1 - e\cos E) dE \\ &= -t + \frac{a^{3/2}}{\sqrt{\mu}} (E - e\sin E) \\ &= -t + \frac{M}{n} \end{aligned}$$

$$Q_1 = -\tau$$

Q_1 is time of perihelion

(Hamilton Jacobi equation and 2 body problem continued)

- With a similar integral we can show that Q_2 is angle of perihelion

$$Q_2 = \theta - f = \omega \qquad Q_1 = -\tau$$

$$P_2 = L = \sqrt{\mu a(1-e^2)} \qquad P_1 = -\frac{\mu}{2a} \qquad K = 0$$

- 3D problem done similarly
- These coordinates not necessarily ideal. Add and subtract them to find the Delaunay, modified Delauney and Poincaré coordinates

Cannonical transformations

Different approaches

1. Choose desirable generating functions
2. Solve integrals resulting from Hamilton-Jacobi equation
3. Choose new coordinates and momenta and show they satisfy Poisson brackets
4. Use expansions (e.g., Birkhoff normal form) (can lead to problems with small divisors)

Example

(continuing two body problem)

$$H = 0, \qquad q_1 = -\tau, \qquad p_1 = -\frac{\mu}{2a}$$

Desirable to have a third angle as a coordinate

The mean anomaly

M = n(t-τ)

$$F_2 = (t-\tau)g(P_1)$$

$$\frac{\partial F_2}{\partial t} = K = g(P_1)$$

$$\frac{\partial F_2}{\partial -\tau} = g(P_1) = p_1 = -\frac{\mu}{2a} \qquad \frac{dg}{da} = \frac{\mu}{2a^2}$$

$$\frac{\partial F_2}{\partial P_1} = g'(P_1)(t-\tau) = M$$

$$n = \mu^{1/2} a^{-3/2} = g'(P_1) \qquad \frac{dg}{dP_1} = \frac{dg}{da}\frac{da}{dP_1}$$

$$\frac{dP_1}{da} = \frac{\mu^{1/2}}{2a^{1/2}} \qquad P_1 = \sqrt{\mu a}$$

as expected

$$K = g(P_1) = -\frac{\mu}{2a} = -\frac{\mu^2}{2P_1^2}$$

Harmonic oscillator

$$H = \frac{p^2}{2} + K\frac{q^2}{2}$$

$$H(I,\phi) = I\kappa$$

$$\begin{aligned} q &= \sqrt{\frac{2I}{\kappa}}\sin\phi \\ p &= \sqrt{2I\kappa}\cos\phi \end{aligned}$$

$$\kappa = \sqrt{K}$$

- Use Poisson bracket to check that these variables are cannonical

$$\{x,y\} = \frac{\partial x}{\partial\phi}\frac{\partial y}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial y}{\partial\phi}$$

$$\begin{aligned} \{q,p\} &= \sqrt{\frac{2I}{\kappa}}\cos\phi\frac{\sqrt{2\kappa}}{2\sqrt{I}}\cos\phi + \frac{\sqrt{2/\kappa}}{2\sqrt{I}}\sin\phi\sqrt{2I\kappa}\sin\phi \\ &= 1 \end{aligned}$$

you don't get 1 here
unless there is the factor
of 2 in the variable

Harmonic oscillator

$$F_2 = \frac{p^2}{2}\tan\phi$$

Using a generating function

$$\frac{\partial F_2}{\partial \phi} = \frac{p^2}{2}\sec^2\phi = I$$

$$\frac{\partial F_2}{\partial p} = p\tan\phi = q$$

$$\frac{p}{q} = \cot\phi$$

$$p^2(1+\tan^2\phi) = 2I$$

$$2I = p^2 + q^2$$

$$\begin{aligned} p &= \sqrt{2I}\cos\phi \\ q &= \sqrt{2I}\sin\phi \end{aligned}$$

Heliocentric coordinates

$$H = \sum_i \frac{p_i^2}{2m_i} - \sum_{i \neq j} \frac{G m_i m_j}{2|r_i - r_j|}$$

N-bodies in inertial frame

$$F_2(\mathbf{r}_i, \mathbf{P}_i) = \sum_{i>0} (\mathbf{r}_i - \mathbf{r}_0) \cdot \mathbf{P}_i + \mathbf{r}_0 \cdot \mathbf{P}_0$$

generating function

$$\frac{\partial F_2}{\partial \mathbf{P}_i} = (\mathbf{r}_i - \mathbf{r}_0) = \mathbf{Q}_i \qquad i \neq 0 \qquad \frac{\partial F_2}{\partial \mathbf{P}_0} = \mathbf{r}_0$$

$$\frac{\partial F_2}{\partial \mathbf{r_i}} = \mathbf{P}_i = \mathbf{p}_i \qquad i \neq 0 \qquad \frac{\partial F_2}{\partial \mathbf{r}_0} = \mathbf{P}_0 - \sum_{i>0} \mathbf{P}_i = \mathbf{p}_0$$

$$H = \sum_{i>0} \left(\frac{P_i^2}{2m_i} - \frac{G m_i m_0}{|Q_i|} \right) - \sum_{i \neq j \;\; i,j>0} \frac{G m_i m_j}{2|r_i - r_j|} + \frac{1}{2m_0} (P_0 - \sum_{i>0} P_i)^2$$

new Hamiltonian

Democratic heliocentric coordinate system

$$\frac{\partial F_2}{\partial \mathbf{r}_0} = \mathbf{P}_0 - \sum_{i>0} \mathbf{P}_i = \mathbf{p}_0 \qquad \mathbf{P}_0 = \sum \mathbf{p}_i$$

barycenter so can be set to zero
This is equivalent to using momenta that are in center of mass coordinate system

- If $P_0=0$ Hamiltonian becomes

$$H = \sum_{i>0} \left(\frac{p_i^2}{2m_i} - \frac{G m_i m_0}{|Q_i|} \right) - \sum_{i \neq j \;\; i,j>0} \frac{G m_i m_j}{2|r_i - r_j|} +$$

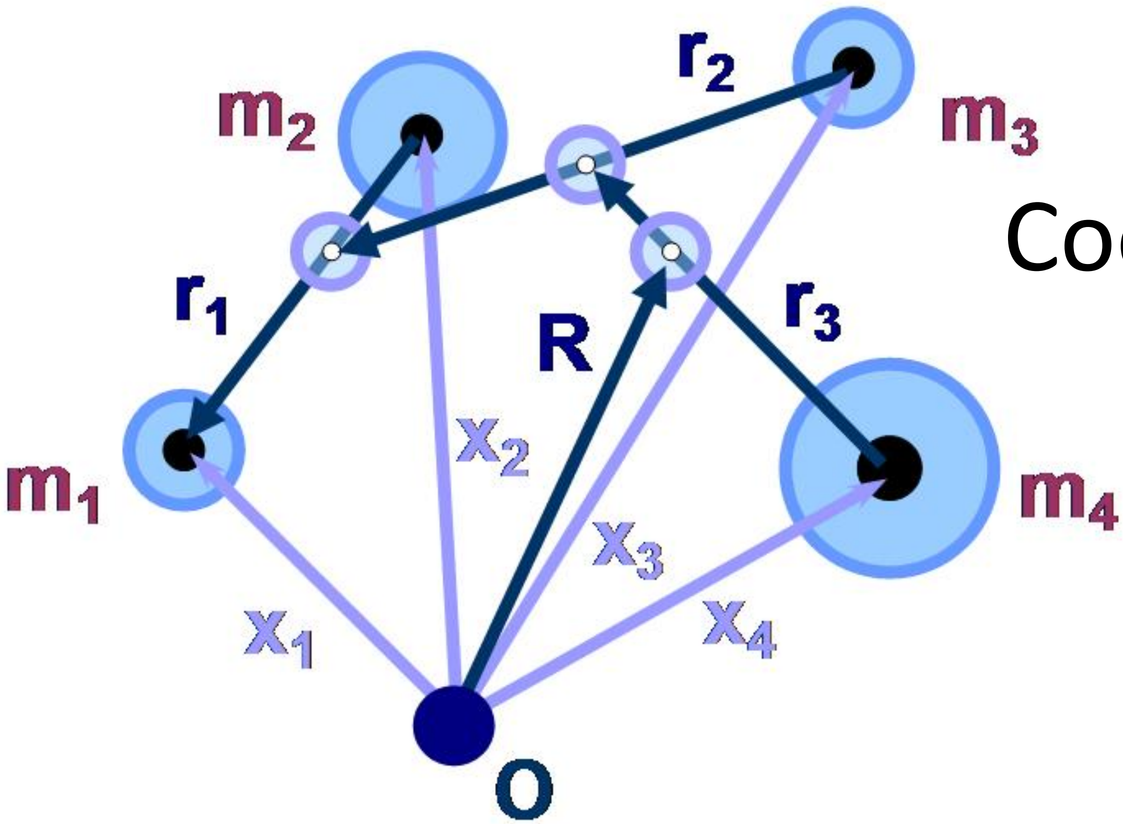
$$\frac{1}{2m_0} (\sum_{i>0} P_i)^2$$

Keplerian term
Interaction term
Drift term

Democratic heliocentric coordinates

- Hamiltonian is nicely separable in
  - heliocentric coordinates and
  - barycentric momenta

$$H = \sum_{i>0} \left(\frac{P_i^2}{2m_i} - \frac{G m_i m_0}{|Q_i|} \right) - \sum_{i \neq j \;\; i,j>0} \frac{G m_i m_j}{2|r_i - r_j|} + \frac{1}{2m_0} (\sum_{i>0} P_i)^2$$



Jacobi Coordinates

To add a body: work with respect to center of mass of all previous bodies.
Coordinate system requires a tree to define.

$$H = \sum_{i>0} \left(\frac{P_i^2}{2m_i} - \frac{G m_i m_0}{|Q_i|} \right) - \sum_{i \neq j \;\; i,j>0} \frac{G m_i m_j}{2|r_i - r_j|}$$

Symplectic Integrators

- Wisdom and Holman used a Hamiltonian in Jacobi coordinates for the N-body system that also separated into Keplerian and Interaction terms.
- Hamiltonian approximated by

$$H = H_{kep} + \delta(\Omega t) H_{int}(\mathbf{Q})$$

  - Integrated all bodies with f,g functions for dt = 1/Ω (that's H_{kep})
  - Velocities given a kick caused by Interactions
  - Integrate bodies again with f,g functions dt = 1/Ω
- Integrator is symplectic but integrates a Hamiltonian that approximates the real one.
- The integrator has bounded energy error and allows very large step sizes

Second order Symplectic integrator

$$\frac{dz}{dt} = \frac{\partial z}{\partial q}\dot{q} + \frac{\partial z}{\partial p}\dot{p} = \frac{\partial z}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial z}{\partial p}\frac{\partial H}{\partial q}$$

$$\frac{dz}{dt} = \{z, H\} = D_H z$$

Poisson bracket with H gives evolution

$$z(\tau) = \exp(\tau D_H) z(0)$$

$$z(\tau) = \exp[\tau(D_{H_0} + D_{H_1})] z(0) \qquad H = H_0 + H_1$$

$$\exp[\tau(A+B)] = \prod_{i=1}^{n} \exp(c_i \tau A) \exp(d_i \tau B) + o(\tau^{n+1})$$

Find coefficients so this is true to whatever order you desire
(see Yoshida review)

Second order Symplectic integrator

Expand to second order

$$\exp\frac{\tau}{2}A\ \ \exp\tau B\ \ \exp\frac{\tau}{2}A$$

$$\sim\ (1+\frac{\tau}{2}A+\frac{\tau^2}{8}A^2)(1+\tau B+\frac{\tau^2}{2}B^2)(1+\frac{\tau}{2}A+\frac{\tau^2}{8}A^2)$$

$$= 1+\tau(A+B)+\frac{\tau^2}{2}(A^2+B^2)+\frac{\tau^2}{2}(AB+BA)+o(\tau^3)$$

$$=\exp[\tau(A+B)]+o(\tau^3)$$

Second order for N-body

$$\exp\frac{\tau}{2}D_{kep}\ \exp\frac{\tau}{2}D_{drift}\ \exp\tau D_{int}\ \exp\frac{\tau}{2}D_{drift}\ \exp\frac{\tau}{2}D_{kep}$$

- Evolve Kepsteps via f,g functions
- Drift step lets positions change as the Hamiltonian term only depends on momenta
- Interaction steps only vary velocities as they only depend on coordinates
- Reverse order.
- Central step chosen to be most computationally intensive

  – This is known as the democratic heliocentric second order integrator (Duncan, Levison & Lee 1998)

Symplectic integrators- Harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2)$$

- The exact solution is

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}$$

- To first order in τ

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

- However $H' = \frac{1}{2}(p'^2 + q'^2) = \frac{1}{2}(1+\tau^2)(p^2+q^2)$

  and energy increases with time
- A symplectic scheme can be constructed with

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1-\tau^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

More on symplectic integrators

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1-\tau^2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

- This is symplectic because the det is 1 and so volume is conserved
- There is a conserved quantity but it's not the original Hamiltonian

$$\bar{H} = \frac{1}{2}(p^2 + q^2) + \frac{\tau}{2} pq$$

- You can show this by computing this quantity for p',q'
- The difference between new and old Hamiltonian depends on timestep!
- Integrator is no longer symplectic if timestep varied.

Approximating interactions with Periodic delta functions

- Interaction terms are used to perturb the system every time step
- Class of integrators where fast moving terms are replaced with periodic delta function terms.

$$H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p}, \mathbf{q}) + H_{int}(\mathbf{q})$$

- Note interaction term only depends on coordinates.
- Here is an example: $H(I, \phi) = I\omega + A\cos\phi$
- We add extra terms to interaction term

$$\begin{aligned} H'_{int} &= A \sum_{n=-\infty}^{\infty} \cos(\phi + 2\pi n t) \\ &= A\cos\phi \; 2\pi \sum_n \delta(t = 2\pi n) \end{aligned}$$

periodic function

Integrating the approximate Hamiltonian

$$H' = I\omega + 2\pi A \cos\phi \; \delta(t = 2\pi n)$$

- For *t≠2πn* $$\frac{\partial H'}{\partial I} = \dot{\phi} = \omega$$
- For *t=0,2π,4π…* $$\frac{\partial H'}{\partial \phi} = -2\pi A \sin\phi \; \delta(t = 2\pi n) = -\dot{I}$$
- Integrate over the delta function $$\Delta I = 2\pi A \sin\phi$$
- Procedure: integrate unperturbed Hamiltonian between delta function spikes. At each t=*0,2π,4π…* update momenta. These are the velocity kicks.

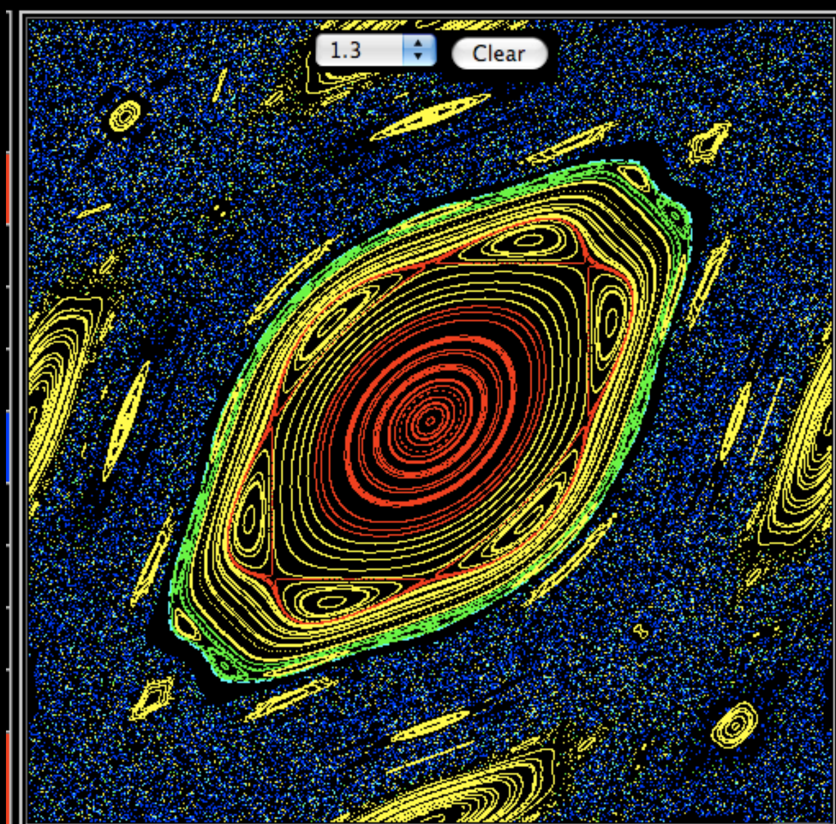
Justifications

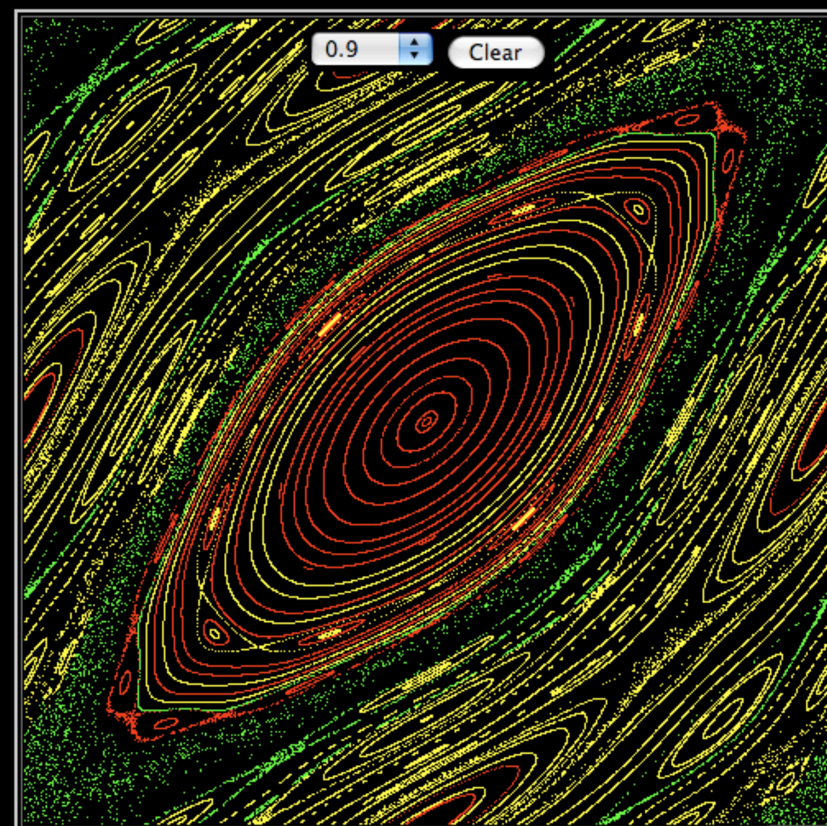
- Often the Hamiltonian has many cosine terms. As long as frequencies are not commensurate, a perturbative transformation can remove non-resonant terms. Dynamics is only weakly sinusoidally varied by these terms. → Cosine terms that are not commensurate are ignored.
- We can add in cosine terms without significantly changing the dynamics.
- The approximate Hamiltonian is conserved exactly.
- Sizes of errors can be quantified. Errors are bounded as there is a conserved quantity.
- You can't change the step size as this would change the approximate Hamiltonian integrated. If you change the step size, the bounded property of errors is lost.
- Wisdom, Holman, and collaborators

Similarity to Standard map

$$
\begin{aligned}
I_{n+1} &= I_n + K \sin \theta_n \\
\theta_{n+1} &= \theta_n + I_{n+1}
\end{aligned}
$$

- I, θ mod 2π
- If constant K is small there is no chaos.
- Width of chaotic zone depends on K
- Above a certain value of K -> global chaos





Regularization

Given

- Hamiltonian H(p,q)
- Initial conditions $H(p_0,q_0,t=0)=H_0$

Let $K(p,q) = f(q)(H - H_0)$

$$f(q) = \frac{dt}{d\tau}$$

$$\frac{\partial K}{\partial q} = f'(q)(H - H_0) + f(q)\frac{\partial H}{\partial q} = -f(q)\frac{dp}{dt}$$

$$\frac{\partial K}{\partial p} = f(q)\frac{\partial H}{\partial p} = f(q)\frac{dq}{dt}$$

$$\frac{\partial K}{\partial q} = -\frac{dp}{d\tau}$$

$$\frac{\partial K}{\partial p} = \frac{dq}{d\tau}$$

We have a new Hamiltonian system but time can be re-scaled

Symplectic integrators that require equal timesteps can be constructed with the new time variable

`Extended phase space' using time to deal with initial conditions issue

General view of resonance

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}, \boldsymbol{\theta}) \qquad \boldsymbol{\omega} = \nabla_I H_0(\mathbf{I})$$

vector of integers **k** such that $\mathbf{k} \cdot \boldsymbol{\omega} \sim 0$

Contrast with Periodic orbits

a period T such that for every frequency ω_i

$\left| \frac{T\omega_i}{2\pi} \right|$ is close to an integer Z_i

T is a multiple of the period of oscillation for every angle

Small divisor problem

$$H(\mathbf{I},\boldsymbol{\theta}) = H_0(\mathbf{I}) + \epsilon H_1(\boldsymbol{\theta})$$ $$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} = \nabla_I H_0(\mathbf{I})$$ frequencies

$$H_1(\boldsymbol{\theta}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \cos(\mathbf{k}\cdot\boldsymbol{\theta})$$ expand perturbation in Fourier series

We would like to find new variables similar to the old variables

$$\boldsymbol{\theta} = \boldsymbol{\theta}' + \text{perturbation}$$
$$\mathbf{I} = \mathbf{I}' + \text{perturbation}$$

try this $$F_2 = \mathbf{J}\cdot\boldsymbol{\theta} + \sum_{\mathbf{k}} c_k \sin(\mathbf{k}\cdot\boldsymbol{\theta})$$ and try to find nice values for c_k

$$\nabla_J F_2 = \boldsymbol{\theta} = \boldsymbol{\theta}'$$
$$\nabla_\theta F_2 = \mathbf{J} + \sum_{\mathbf{k}} c_k \mathbf{k} \cos(\mathbf{k}\cdot\boldsymbol{\theta}) = \mathbf{I}$$

Canonical transformation

Insert this back into Hamiltonian

$$K(\mathbf{J},\boldsymbol{\theta}') = H_0(\mathbf{J}) + \sum_{\mathbf{k}} c_k \nabla_I H_0(J)\cdot\mathbf{k}\cos(\mathbf{k}\cdot\boldsymbol{\theta}) + \epsilon\sum_{\mathbf{k}} a_k \cos(\mathbf{k}\cdot\boldsymbol{\theta}) + \ldots$$
$$= H_0(\mathbf{J}) + \sum_{\mathbf{k}} (c_k \boldsymbol{\omega}\cdot\mathbf{k} + \epsilon a_k)\cos(\mathbf{k}\cdot\boldsymbol{\theta}) + \ldots$$

Choose $$c_{\mathbf{k}} = -\frac{\epsilon a_{\mathbf{k}}}{\boldsymbol{\omega}\cdot\mathbf{k}}$$ These can be small!

Small divisor problem continued

$$K(\mathbf{J},\boldsymbol{\theta}) = H_0(J) + \epsilon^2 ...$$

first order perturbations removed

$$\mathbf{I} = \mathbf{J} - \sum_{\mathbf{k}} \frac{\epsilon a_{\mathbf{k}} \mathbf{k}}{\boldsymbol{\omega}\cdot\mathbf{k}} \cos(\mathbf{k}\cdot\boldsymbol{\theta})$$

Nearing a resonance, the momentum goes to infinity
We will see that the infinity is not real, but a result of our assumption that new and old variables are similar

If there are no small divisors when removing first order perturbations, there may be small divisors when attempting to remove higher order perturbations from Hamiltonian via canonical transformation

Using the resonant angle

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta})$$

$$F_2 = \mathbf{k} \cdot \boldsymbol{\theta} J_0 + \sum_{i>0} J_i \theta_i$$ generating function

$$\frac{\partial F_2}{\partial \theta_i} = k_i J_0 + J_i = I_i \qquad \frac{\partial F_2}{\partial J_i} = \theta_i = \theta_i' \qquad i \neq 0$$

$$\frac{\partial F_2}{\partial \theta_0} = k_0 J_0 = I_0 \qquad \frac{\partial F_2}{\partial J_0} = \mathbf{k} \cdot \boldsymbol{\theta} = \phi$$ resonant angle is a new coordinate

$$K(J_0, \phi; J_i, \theta_i) = H_0(k_0 J_0, k_i J_0 + J_i) + \epsilon \cos \phi$$

n-1 conserved quantities J_i because Hamiltonian lacks associated angles
Can expand H_0 in orders of J_0

$$K(J_0, \phi; J_i, \theta_i) = a J_0^2 + b J_0 + \epsilon \cos \phi + \text{constants} + \text{higher order terms}$$

Resonant angle

$$K(J_0, \phi; J_i, \theta_i) = H_0(k_0 J_0, k_i J_0 + J_i) + \epsilon \cos\phi$$

Simple 2D Hamiltonian. There is no infinity in the problem.
Dynamics is similar to that of the pendulum.
We did not assume that all new angles were similar to old angles in the transformation.
The infinite response previously seen near resonance was caused by the choice of coordinate system

Above we considered only a single cosine term
When there is more than one cosine term, then dynamics can be more complicated

Adopt a condition that the system is always sufficiently far from resonance, allowing perturbation theory to be done at all orders (Kolmogorov approach)
Consider proximity of resonances (Chirikov approach)

Expanding about a mean motion resonance (Celestial mechanics)

- j:j-k resonance exterior to a planet

$$\Lambda = \sqrt{\mu a}, \qquad \Gamma = \sqrt{\mu a}(1 - \sqrt{1 - e^2})$$

$$H_0(\Lambda, \lambda; \Gamma, \gamma) = -\frac{1}{2\Lambda^2}$$

unperturbed Hamiltonian units μ=1

- generating function

$$F_2 = I(j\lambda - (j-k)\lambda_p)$$

mean longitude of planet

- new momenta $I = \Lambda/j, \qquad \psi = j\lambda - (j-k)\lambda_p$

ψ is resonant angle

$$H_0'(I, \psi; \Gamma, \gamma) = \frac{-1}{2j^2 I^2} - (j-k) I n_p.$$

Expand around resonance

$$H_0'(I,\psi;\Gamma,\gamma) = \frac{-1}{2j^2I^2} - (j-k)In_p. \qquad L \equiv I - I_0$$

$$K_0(L,\psi;\Gamma,\gamma) = \text{constant} +$$
$$\left[-(j-k)n_p + j^{-2}I_0^{-3}\right]L - \frac{3L^2}{2j^2I_0^4}$$

$$\boxed{K_0(L,\psi;\Gamma,\gamma) = aL^2 + bL + \text{constant}}$$

with coefficients

$$a = -\frac{3}{2}j^2\alpha^2$$
$$b = -(j-k)n_p + jn_0.$$

Adding in resonant terms from the disturbing function

$$K(L,\psi;\Gamma,\gamma) = aL^2 + bL + c\Gamma$$

secular term

$$+\sum_{p=0}^{k} \delta_{k,p}\Gamma^{(k-p)/2}\cos\left(\psi - (k-p)\varpi - p\varpi_p\right)$$

$$c = -\frac{\mu}{4}\alpha^{5/2} b_{3/2}^{(1)}(\alpha)$$

$$\delta_{1,0} = -\mu\sqrt{2}\alpha^{5/4} f_{31}$$

$$\delta_{1,1} = -\mu e_p \alpha f_{27}$$

$$f_{27} = \frac{1}{2}[-2j - \alpha D] b_{1/2}^{(j)}$$

$$f_{31} = \frac{1}{2}[-1 + 2j + \alpha D] b_{1/2}^{(j-1)}$$

have opposite signs

First order resonances

sets distance to resonance

$$K(L,\psi;\Gamma,\gamma) = aL^2 + bL + c\Gamma + \delta_{1,0}\Gamma^{1/2}\cos(\psi - \varpi)$$

$$+\delta_{1,1}\cos(\psi - \varpi_p)$$

corotation term proportional to planet eccentricity usually dropped but can be a source of resonance overlap + chaos (Holman, Murray papers in 96)

One last C-transformation and going down a dimension

$$\begin{aligned} K(L,\psi;\Gamma,\gamma) &= aL^2 + bL + c\Gamma \\ &\quad +\delta_{1,0}\Gamma^{1/2}\cos(\psi-\varpi) \end{aligned}$$

$$F_2 = (\psi - \varpi)J_1 + \psi J_2 \qquad \begin{aligned} J_1 &= \Gamma, & \phi &= \psi - \varpi \\ J_1 + J_2 &= L & \theta &= \psi \end{aligned}$$

$$K = a'\Gamma^2 + b'\Gamma + bJ_2 + \delta\Gamma^{1/2}\cos\phi$$

New Hamiltonian has no second angle so J_2 is conserved
We can ignore it in dynamics
It may be useful to remember J_2 later to relate changes in eccentricity to changes in semi-major axis in the resonance

$$J_2 = L - \Gamma = \frac{\Lambda}{j} - I_0 - \Gamma$$

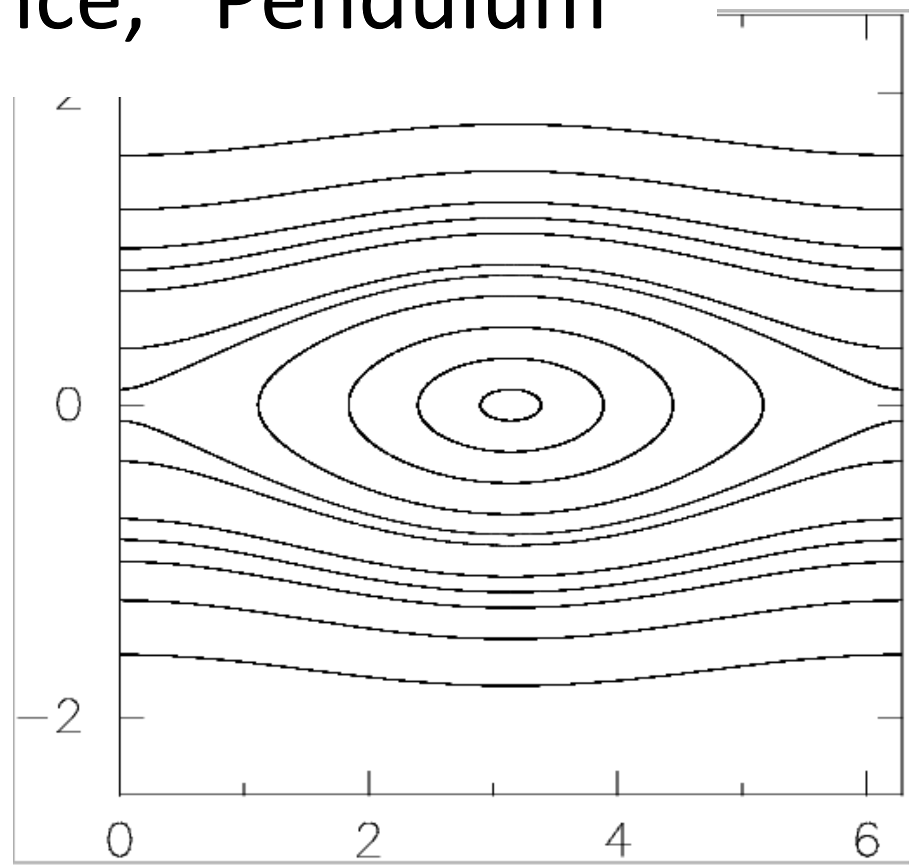
Distance to Resonance, Pendulum

$$H(p,\phi) = \frac{p^2}{2} + bp + \epsilon\cos\phi$$

$$\frac{\partial H}{\partial p} = 0 \qquad p + b = 0$$

$$\frac{\partial H}{\partial \phi} = 0 \qquad \epsilon\sin\phi = 0$$

fixed point is shifted



$$F_3 = -(p+b)Q$$

$$P = p + b \qquad K = \frac{P^2}{2} + \epsilon\cos\phi + \text{constant}$$

Hamiltonian transformed to a pendulum Hamiltonian but with momentum shifted

Dimensional analysis

$$H = a\Gamma^2 + \delta\Gamma^{1/2}\cos\phi$$

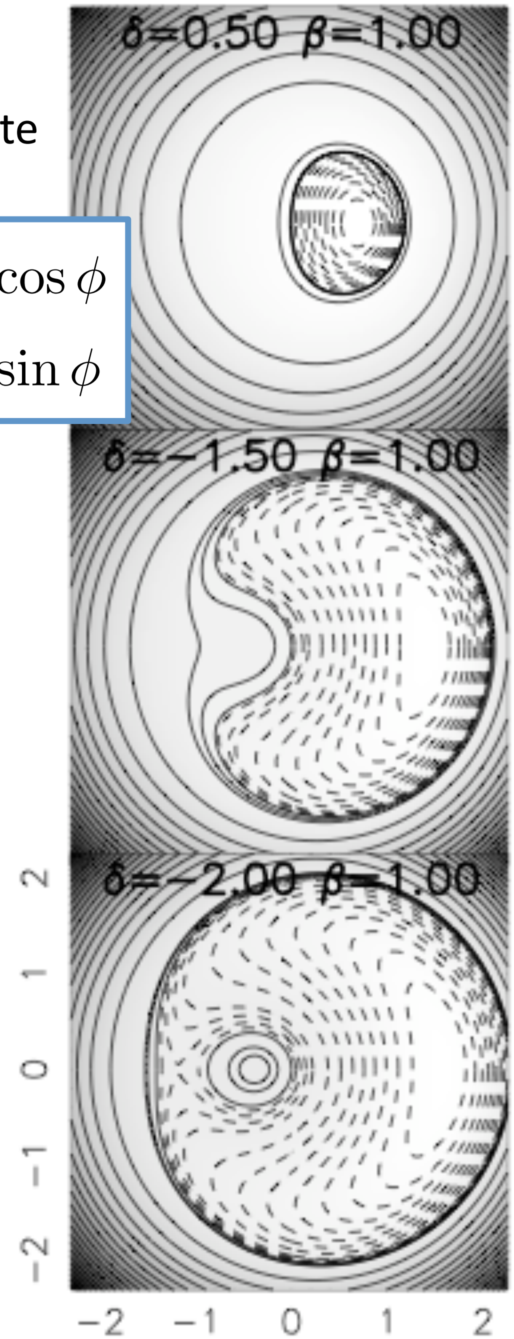
- Units of H cm^2/s^2 Units of Γ cm^2/s
- Units of a cm^{-2} Units of δ cm $s^{-3/2}$
- $\delta^2 \times a$ units s^{-3}
  - Unit of time $a^{-1/3}\,\delta^{-2/3}$ sets typical libration period
- $\delta^{2/3}$ units $cm^{2/3}\,s^{-1}$
  - Unit of momentum $\delta^{2/3}a^{-2/3}$ sets resonant widths
- δ is proportional to Planet mass
  - Resonant widths depend on planet mass to the 2/3 power.
- Migration is a variation in mean motion so has units s^{-2}
  - Relevant to figure out adiabatic limit

As distance to resonance varied

$$K = a'\Gamma^2 + b'\Gamma + \delta\Gamma^{1/2}\cos\phi$$

coordinate system

$$x = \sqrt{2\Gamma}\cos\phi$$
$$y = \sqrt{2\Gamma}\sin\phi$$



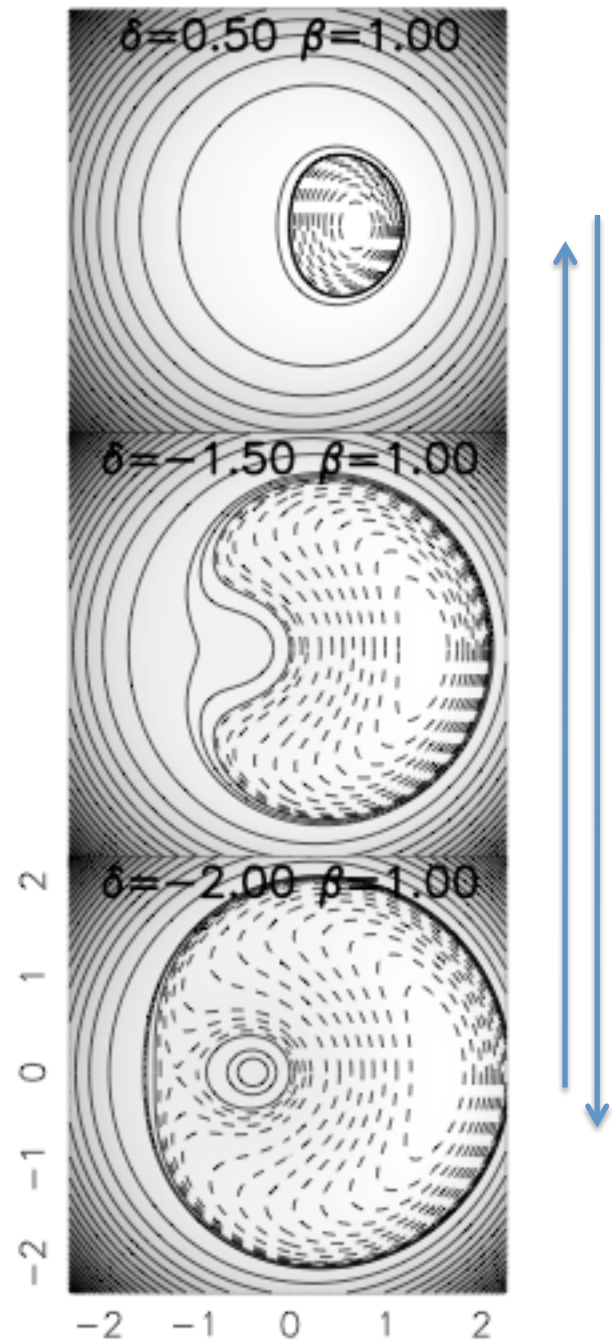
- Only one side has a separatrix
- On one side it looks like a harmonic oscillator
- 3 fixed points but only 2 stable
- Without circulating about origin means in resonance
- Can think of drifting problems as having time dependent b

Drifting systems

- Migrating planets
- Dust under radiation forces (though dissipative dynamics is NOT Hamiltonian)
- Satellite systems with tidal dissipation

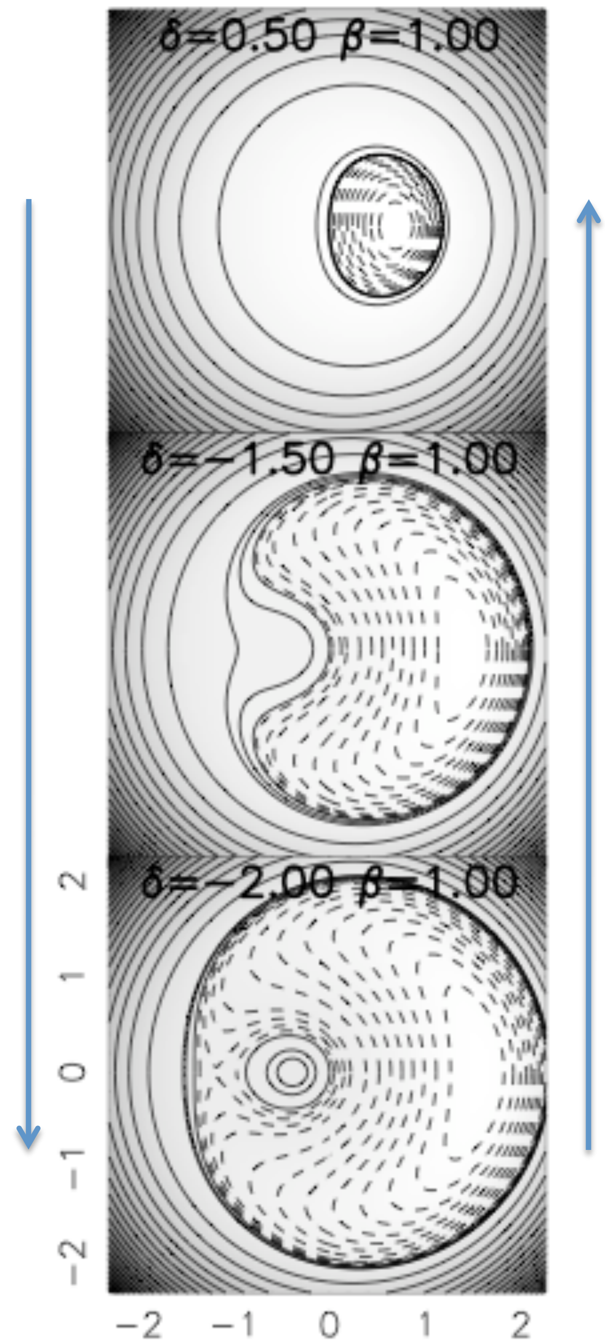
$$K = a'\Gamma^2 + b'\Gamma + \delta\Gamma^{1/2}\cos\phi$$

$$\frac{db}{dt} \neq 0$$



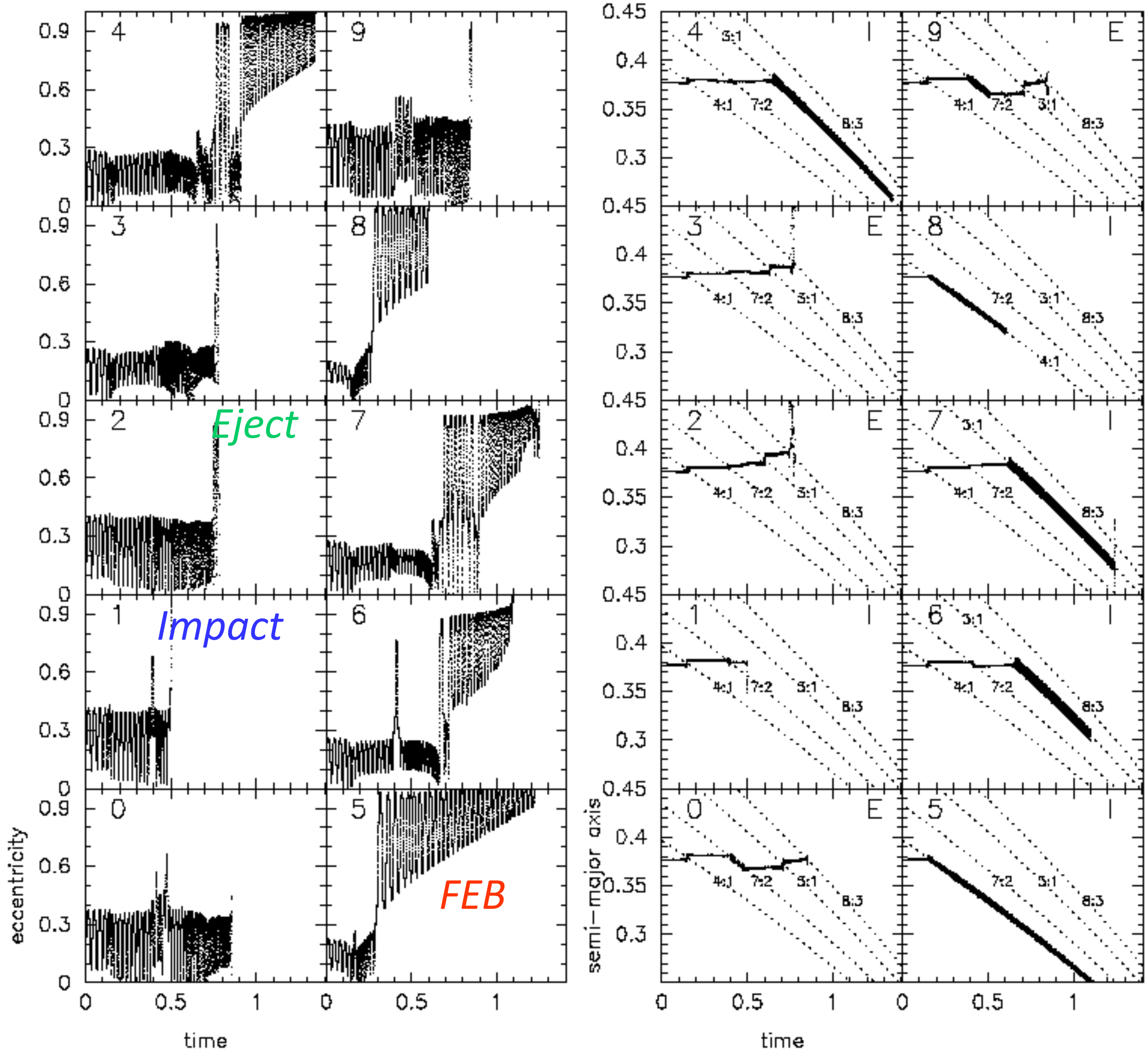
Drifting systems

Particle can be pumped to high eccentricity and remain in librating island,
resonance capture possible



Particle must jump to other side,
no capture possible

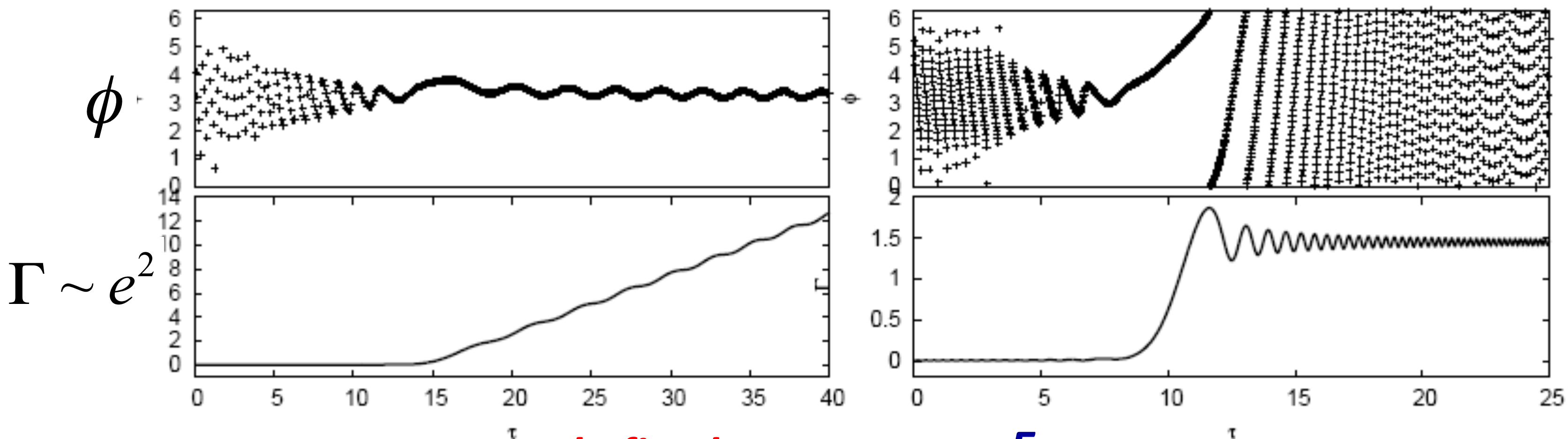
Numerical integration of a planet migrating inwards



Production of Star Grazing and Star-Impacting Planetesimals via Planetary Orbital Migration

Quillen & Holman 2000, but also see Quillen 2002 on the Hyades metallicity scatter

Integration with toy model



resonant angle fixed --
Capture

Escape

Failure to capture

- Volume in resonance shrinking rather than growing (particle separating from planet)
- Non-adiabatic limit. Rapid drifting
- Initial particle eccentricity is high.
- Prevented by strong subresonances

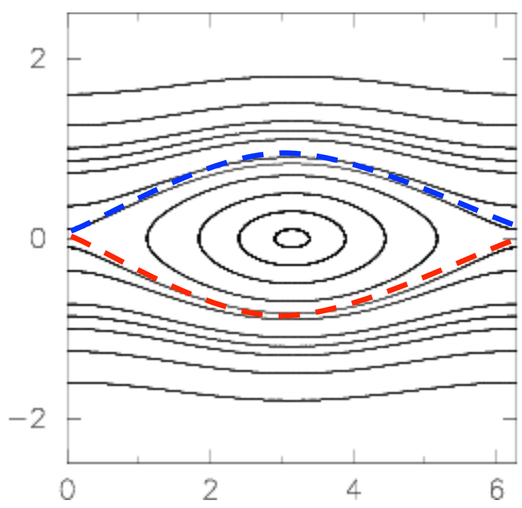
Adiabatic Capture Theory for Integrable Drifting Resonances

V_+ = Rate of Volume swept by upper separatrix

V_- = Rate of Volume swept by lower separatrix

$V_+ - V_-$ = Rate of Growth of Volume in resonance

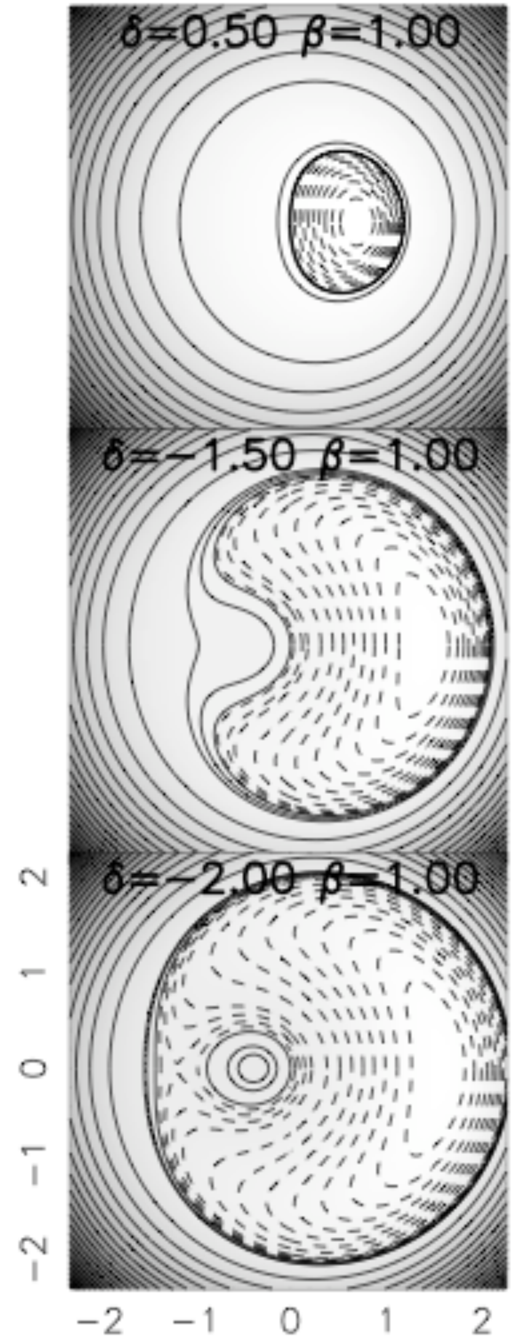
$P_c = \dfrac{V_+ - V_-}{V_+}$ = Probability of Capture



Theory introduced by Yoder and Henrard, applied toward mean motion resonances by Borderies and Goldreich

Critical eccentricity

- Below a critical eccentricity probability of capture is 100%
- This eccentricity can be estimated via dimensional analysis from the typical momentum size scale in the resonance
- It depends on order of resonance and a power of planet mass. Hence weak resonances require extremely low eccentricity for capture.
- Exact probability above e_{crit} in adiabatic limit can be computed via area integrals

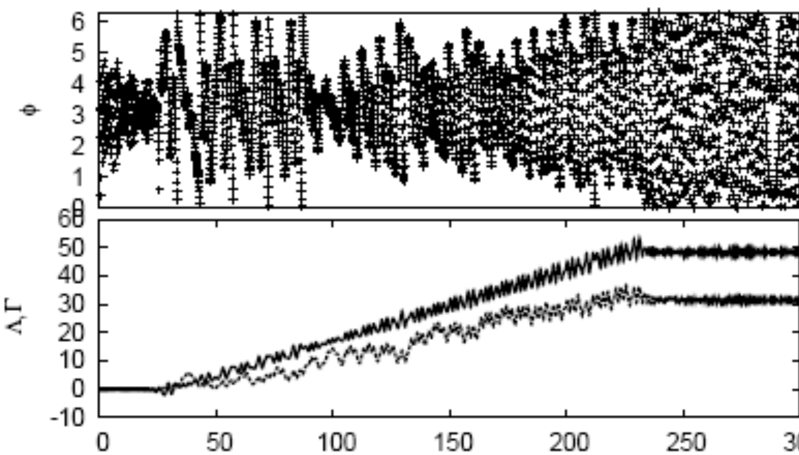


Limitations of Adiabatic theory

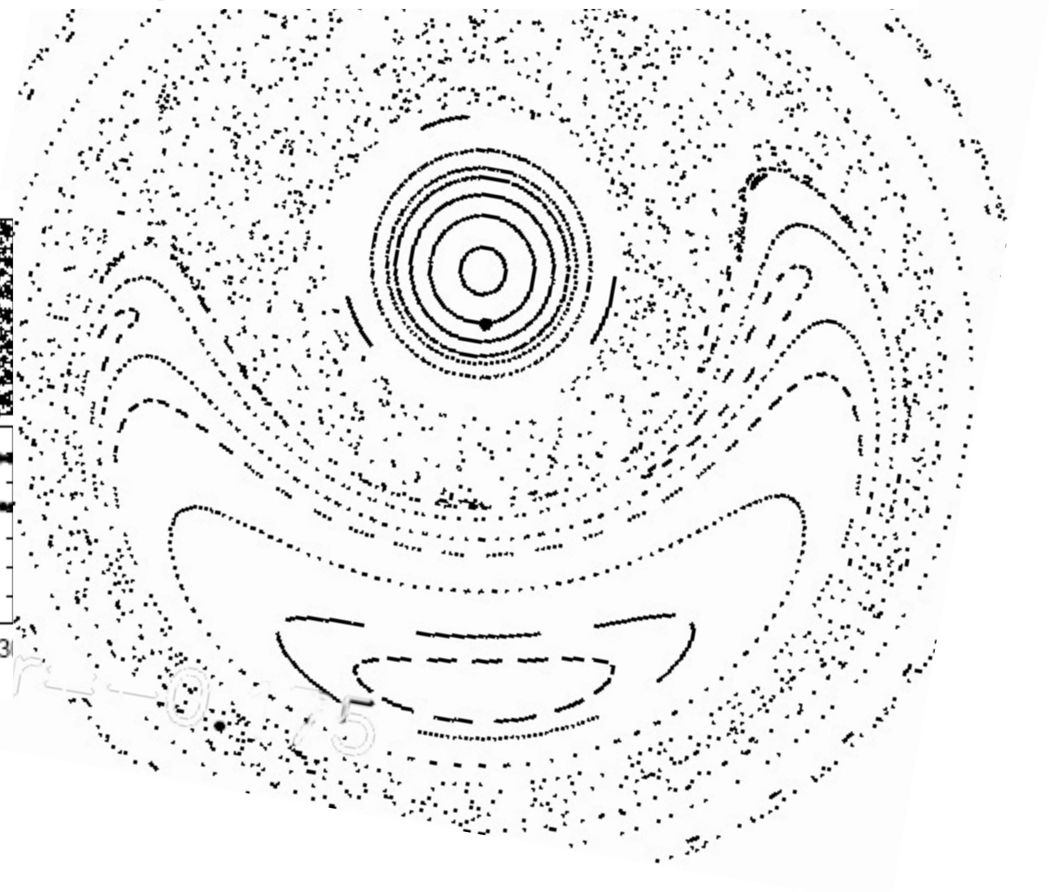
$$K(\Lambda,\psi;\Gamma,\gamma) = a\Lambda^2 + b\Lambda + c\Gamma - \sum_{p=0}^{k} \delta_{k,p}\Gamma^{(k-p)/2}\cos(\psi - (k-p)\varpi - p\varpi_p)$$

$\propto \mu e^p$

- At fast drift rates resonances can fail to capture -- the non-adiabatic regime.
- Subterms in resonances can cause chaotic motion.



temporary capture in a chaotic system



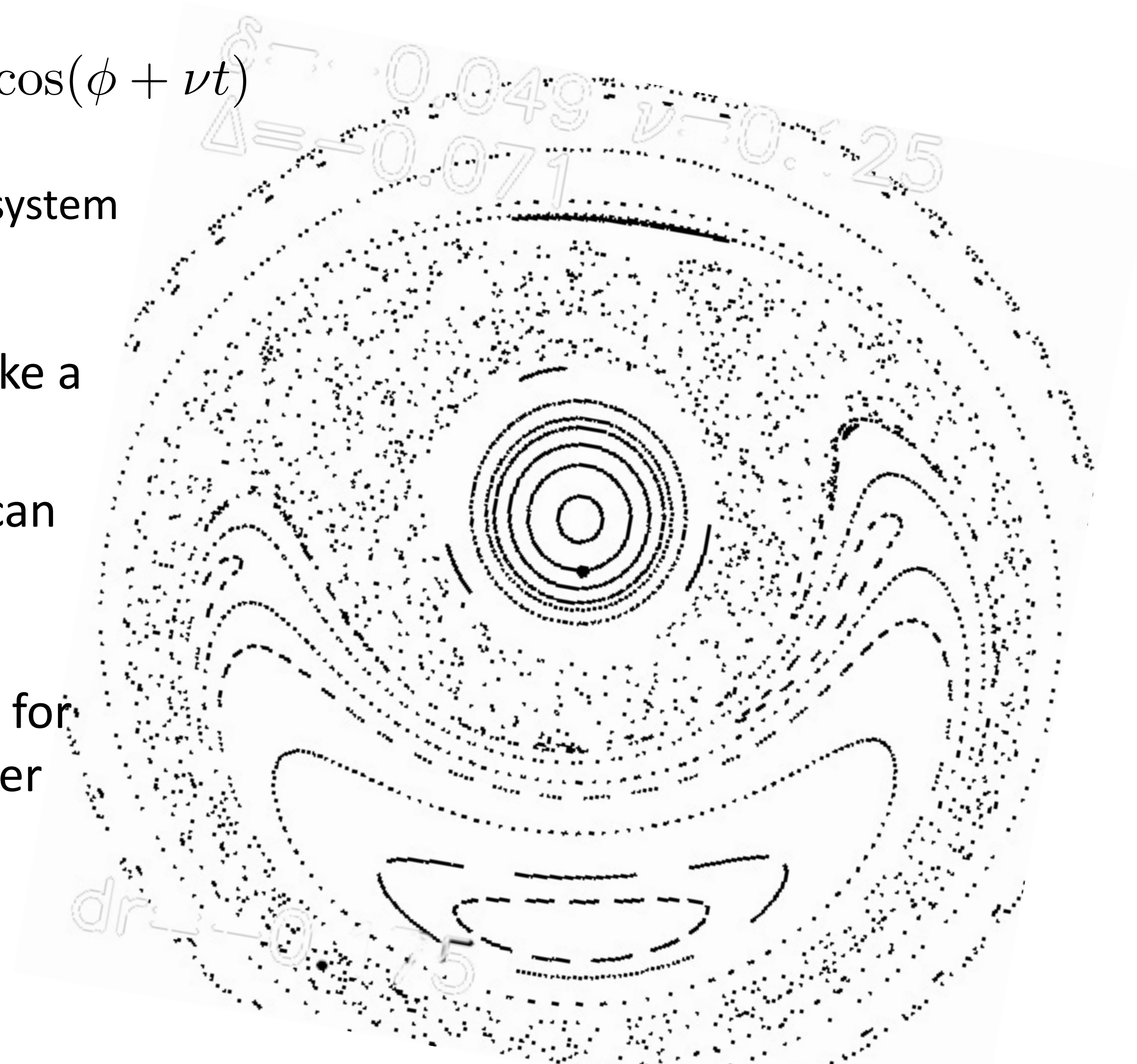
Coupled pendulum

$$H = p^2 + a\cos\phi + b\cos(\phi + \nu t)$$ forced pendulum

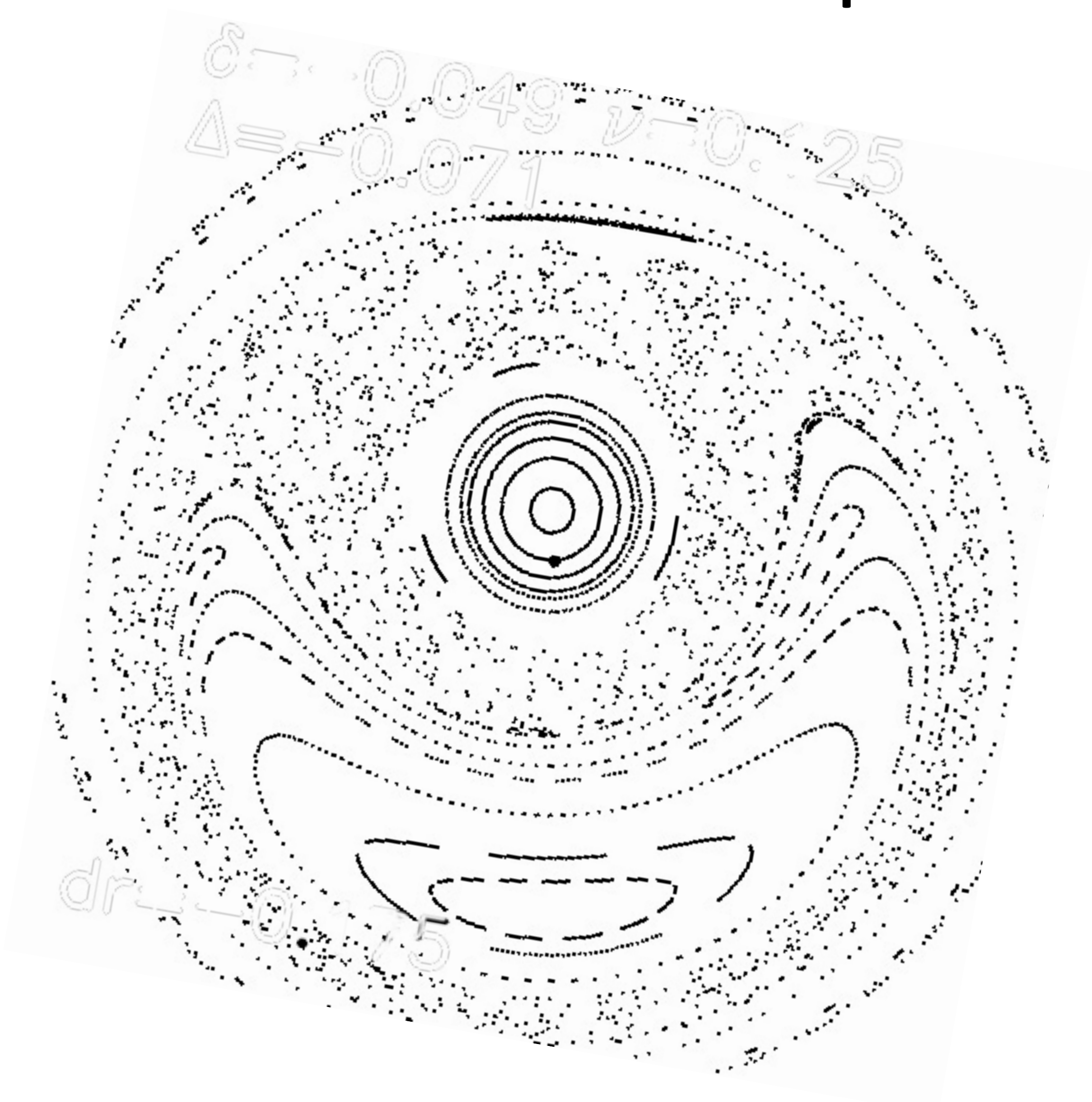
$$H + \Gamma^2 + \delta_0 \Gamma^{1/2}\cos\phi + \delta_1\cos(\phi + \nu t)$$

analogous first order resonance system

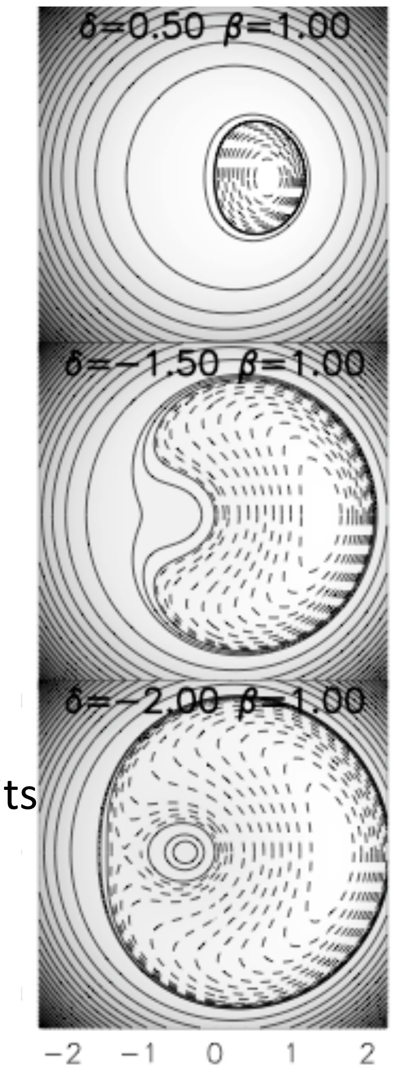
- plot points every t=1/ν to make a surface of section
- 2D system (4D phase space) can be plotted in 2D with area preserving map
- Nice work relating timescales for evolution to overlap parameter by Holman and Murray



Chaos first develops near separatrix



chaotic orbits
more likely
on this side



Non adiabatic limit

- If drift is too fast capture is unlikely. Drifting past resonance must be slow compared to libration timescale.
- Can be estimated via dimensional analysis
- Use the square of the timescale of the resonance.
- This follows as drift rates are in units of t^{-2}

Reading

- Morbidelli's book Chap 1
- Murray and Dermott (solar system dynamics) Chap 8
- Yoshida, H. 1993, Celestial Mechanics and Dynamical Astronomy, 56, 27, "Recent Progress in the Theory and Application of Symplectic Integrators"
- Duncan, Levison & Lee 1998, AJ, 116, 2067, "A multiple timestep symplectic algorithm for integrating close encounters"
- Holman, Matthew J., Murray, N. W. 1996, AJ, 112, 1278, "Chaos in High-Order Mean Resonances in the Outer Asteroid Belt"
- Quillen, A. C. 2006, MNRAS, 365, 1367, `Reducing the probability of capture into resonance'
- Mustill & Wyatt 2011, MNRAS, 413, 554, `A general model of resonance capture in planetary systems: first- and second-order resonances'