# Solar System Dynamics

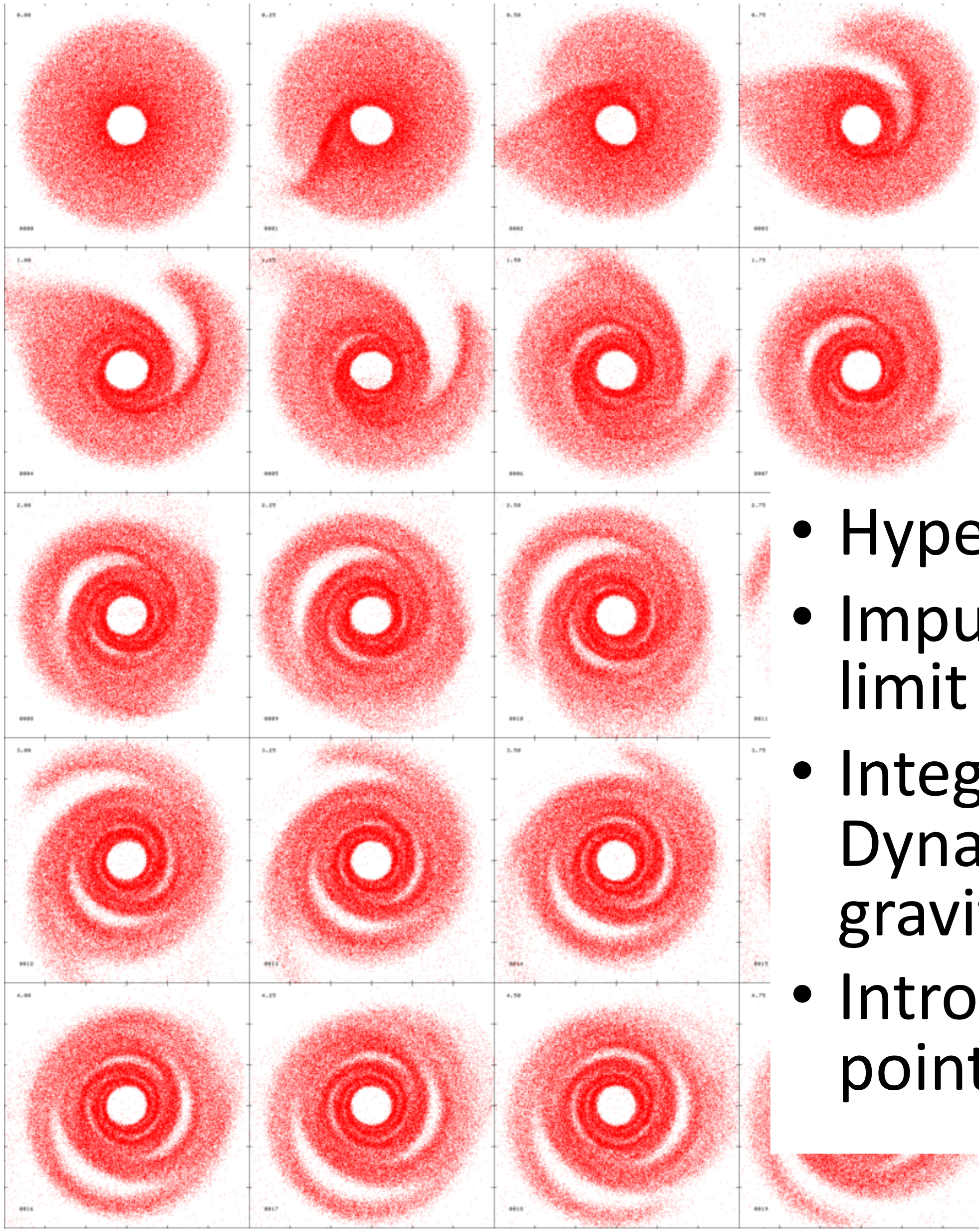
- Instructor *Alice Quillen*
  - Seminar class *AST570 TR12:30-1:45pm B+L315*
    http://astro.pas.rochester.edu/~aquillen/ast570/
- Dynamics of circumstellar disks, planetary systems, solar system formation and evolution, exoplanets
- Overlap with Galactic Dynamics
- Recommended Texts:
  - *Solar System Dynamics* by Murray and Dermott
  - *Galactic Dynamics* by Binney and Tremaine
  - *Modern Celestial Mechanics* by Morbidelli
    http://www.oca.eu/morby/celmech.ps

# Some Topics for Lecture

1. Keplerian orbits, impulse approximation, hyperbolic orbits, dynamical friction
2. Multiple planet interactions, secular perturbations, pericenter glow model, apsidal resonance
3. Mean motion resonances, chaotic zone boundary near corotation, timescales and size scales in resonances, resonance capture
4. Symplectic integrators
5. Collisional cascades, dust production, debris disk evolution
6. Planet migration in both gaseous and planetesimal disks
7. Planetesimal growth models
8. Chaotic motion and toy models for chaotic dynamics, lifetimes and diffusion
9. Tidal evolution
10. Recommended additional topics

# Class format

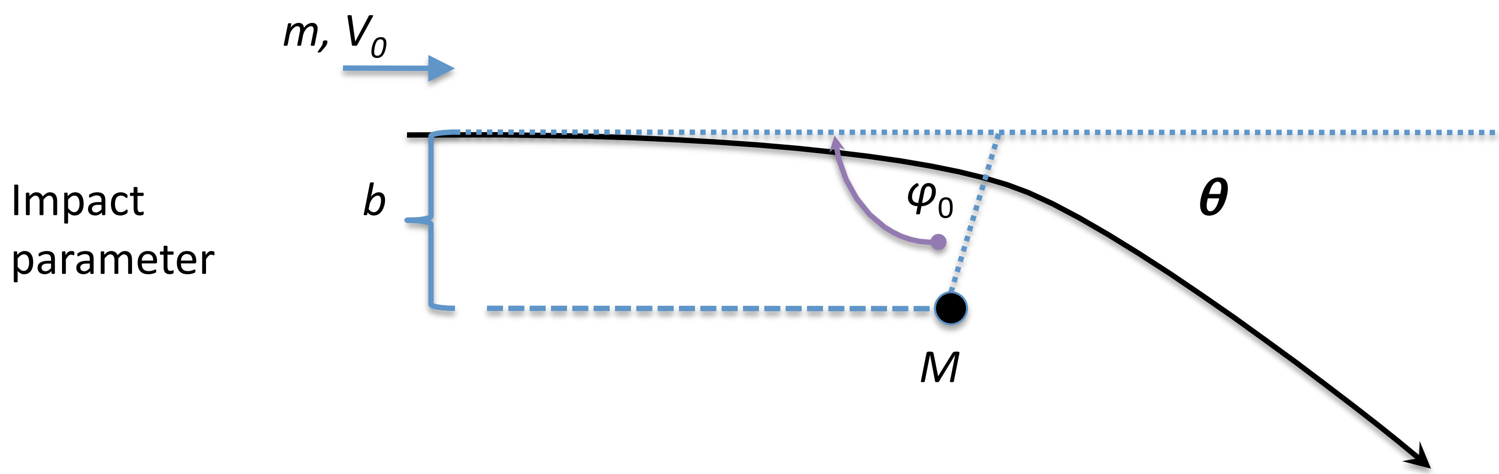
- One lecture per week (Tues)
- One group discussion/problem session per week (Thursday)
- No exams
- A final research project
- Problems to be worked at home or in class



# Gravitational Interactions

- Hyperbolic orbit
- Impulse approximation --- limit of fast close approach
- Integrating the effect: Dynamical friction and gravitational stirring
- Introduction to Lagrange points and tidal force

# Hyperbolic orbit




$$\theta = 2\phi_0 - \pi$$

Deflection angle

$$L = bV_0$$

Angular momentum, conserved

# The center of mass

The location of the center of mass is given by

$$\mathbf{X} = \frac{\sum_i M_i \mathbf{x}_i}{\sum_i M_i}$$

The velocity of the center of mass is given by taking the time derivative of the previous expression.

$$\mathbf{V} = \frac{\sum_i M_i \mathbf{v}_i}{\sum_i M_i}$$

# Relation between velocity changes

Two bodies, *M*, *m* with velocities $v_m$ , $v_M$

$\mathbf{V} = v_M - v_m$
is the velocity difference between bodies

$$mv_m + Mv_M = \text{constant}$$

$$m\mathbf{v}_m + M\mathbf{v}_M - M\mathbf{v}_m + M\mathbf{v}_m = \text{constant}$$

$$(m+M)\mathbf{v}_m + M\mathbf{V} = \text{constant}$$

$$\begin{aligned} \mathbf{v}_m &= -\frac{M}{m+M}\mathbf{V} + \text{constant} \\ \mathbf{v}_M &= \frac{m}{m+M}\mathbf{V} + \text{constant} \end{aligned}$$

with a sign flip depending on sign convention for **V**

$$\begin{aligned} \Delta\mathbf{v}_m &= -\frac{M}{m+M}\Delta\mathbf{V} \\ \Delta\mathbf{v}_M &= \frac{m}{m+M}\Delta\mathbf{V} \end{aligned}$$

$\Delta\mathbf{V}$ Velocity change between the two bodies before and after encounter

# The two body problem

For two massive bodies, the total energy is

$$E = \frac{M_1 v_1^2}{2} + \frac{M_2 v_2^2}{2} - \frac{G M_1 M_2}{|\mathbf{r_1} - \mathbf{r}_2|}, \text{ Kinetic Energy + Potential Energy}$$

We can rewrite this as: $\boxed{E = \frac{M\mathbf{V}^2}{2} + \frac{\mu \mathbf{v}^2}{2} - \frac{GM\mu}{|\mathbf{r}|}}$

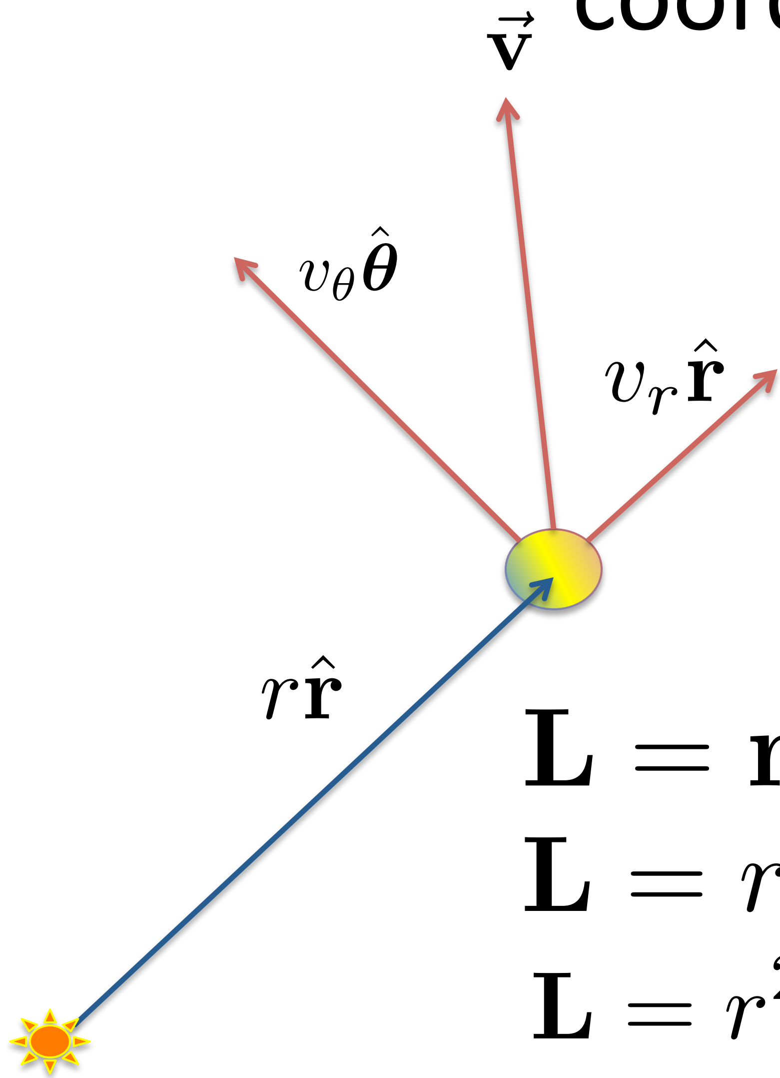
where $M = M_1 + M_2$ is the total mass

$\mu = \left( \frac{M_1 M_2}{M_1 + M_2} \right)$ is the reduced mass

$\mathbf{V}$ = velocity of center of mass

$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, is the velocity difference, and $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$

# Angular momentum in polar coordinates




$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$
$$\mathbf{L} = r v_\theta \hat{\mathbf{z}}$$
$$\mathbf{L} = r^2 \dot{\theta} \hat{\mathbf{z}}$$

Only depends on tangential velocity component

# Keplerian orbit

Radial force

$$\frac{d^2\mathbf{r}}{dt} = -\frac{G(M+m)}{r^2}\mathbf{r}$$

vector $\mathbf{r}$ between the two masses

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G(M+m)}{r^2}$$

$$h = r^2\dot{\theta} \equiv L$$

angular momentum

$$u = \frac{1}{r}$$

$$\dot{u} = \frac{du}{d\theta}\dot{\theta} = \frac{du}{d\theta}\frac{h}{r^2} = -\frac{\dot{r}}{r^2}$$

$$\ddot{r} = -\frac{d^2u}{d\theta^2}h^2u^2$$

$$\frac{d}{dt}$$

Get rid of time derivative using conservation of angular momentum

$$\frac{d^2u}{d\theta^2} + u = \frac{G(M+m)}{h^2}$$

Solved by $\quad r = \dfrac{p}{1 + e\cos(\theta - \varpi)}$

with $\quad p = \dfrac{h^2}{G(M+m)}$

# Conic sections
# Keplerian orbit

orbit

$$r = \frac{p}{1 + e\cos(\theta - \varpi)}$$

*p* = semilatus rectum
*e* = eccentricity
*a* = semi-major axis
*q*= pericenter distance
ϖ = longitude of pericenter
θ = true longitude

$$\begin{aligned} 0 < e \le 1 \quad & p = a(1-e^2) \\ e = 1 \quad & p = 2q \\ e > 1 \quad & p = a(e^2 - 1) \end{aligned}$$

ellipse,
parabola,
hyperbola

note sometimes a negative *a* is used for hyperbolic orbits so that formula for Energy is the same

pericenter for ellipse and hyperbola q=a|e-1|

# True anomaly

$$r = \frac{p}{1 + e\cos(\theta - \varpi)}$$

*f* = true anomaly

$$r = \frac{p}{1 + e\cos f}$$

$$x = r\cos f$$
$$y = r\sin f$$

heliocentric coordinates for Keplerian problem

Angles with respect to pericenter tend to be called ***anomali***, those respect to the equinox direction tend to be called ***longitudes***

# Energy and semi-major axis

For the two body system the energy is the sum of the center of mass motion plus the Keplerian energy

$$E = \frac{mV_0^2}{2} = \frac{GMm}{2a} + \frac{(m+M)V_{com}^2}{2}$$

$$\mathbf{V}_{com} = \frac{m}{m+M}\mathbf{V}_0$$

$$\longrightarrow \quad \boxed{a = \frac{G(M+m)}{V_0^2}}$$

a>0 as orbit is unbound

# Angular momentum, p and e

Two ways to write *p*

$$p = \frac{h^2}{G(M+m)} = a(e^2-1)$$

$$e^2 - 1 = \frac{h^2}{G(M+m)a} = \frac{b^2V_0^4}{G^2(M+m)^2}$$

# Gravitational focusing

Another use of hyperbolic orbit. The difference between the impact parameter *b* and the pericenter distance *q*

at closest approach $q=a(e-1)$

$$a = \frac{G(M+m)}{V_0^2}$$

$$e^2 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2}$$

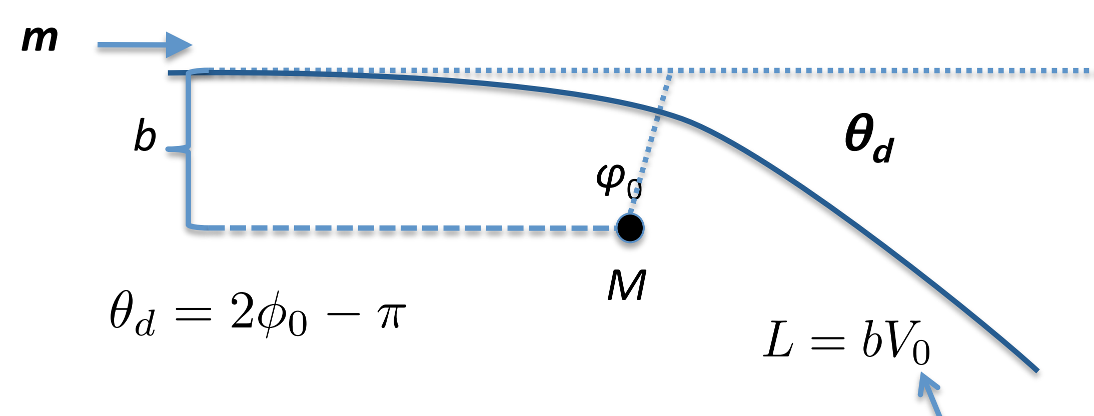
$$\begin{aligned} q &= b\left[\sqrt{1+A^2} - A\right] \\ A &= \frac{G(M+m)}{bV_0^2} \end{aligned}$$

Limit of high velocity compared to
Gives q~b,
For velocities slow compared to this velocity there is focusing and $q<b$

$$v_b = \sqrt{\frac{G(M+m)}{b}}$$

$$A = \left(\frac{v_b}{V_0}\right)^2$$

Impact parameter *b*
Deflection angle $\theta_d$




- For a hyperbolic orbit *e*>1

$$r = \frac{p}{1 + e\cos f}$$

- Large *r* when denominator is small or when $\sec\phi_0 = -e$

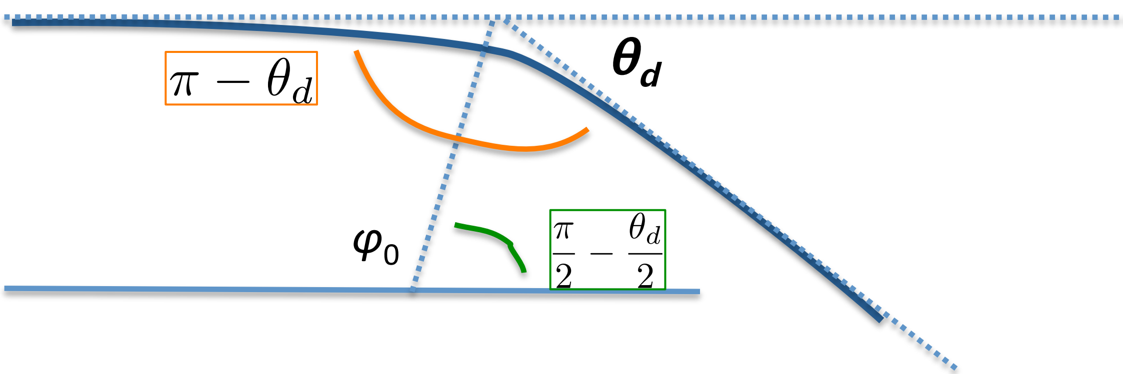
or when $\tan^2\phi_0 = e^2 - 1$

$$1 + \tan^2\phi_0 = e^2$$

by symmetry this happens at two different values of angle ϕ

# Deflection angle

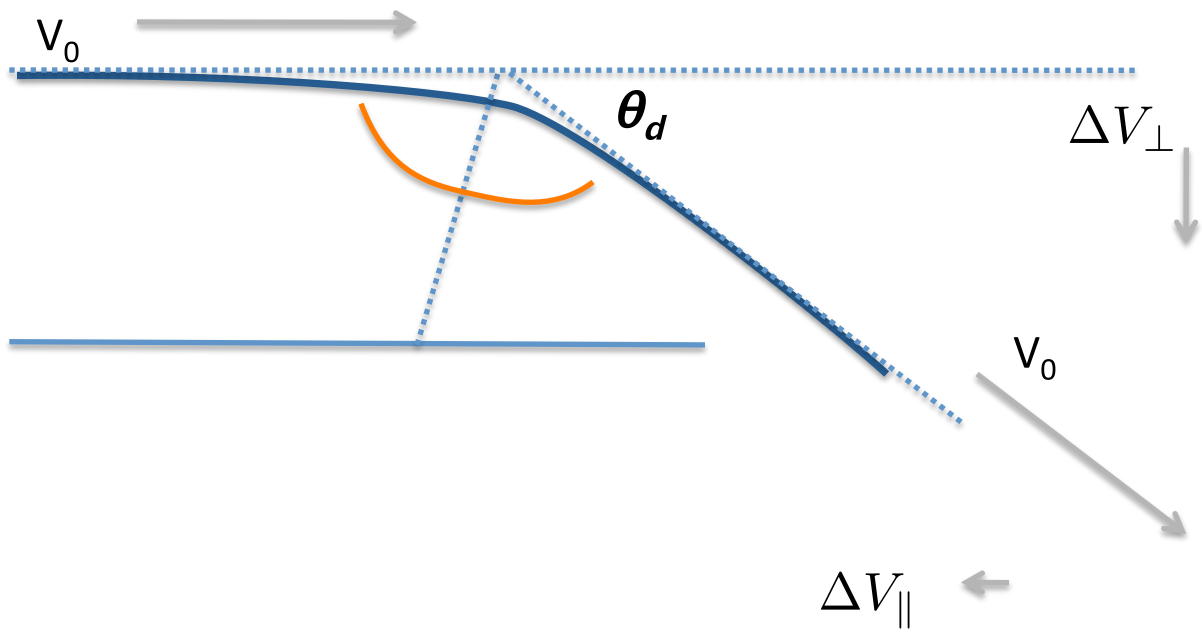
Φ=0 at closest approach




$$\theta_d = 2\phi_0 - \pi$$

# Velocity Changes

$$\Delta V_\perp = V_0 \sin\theta_d$$
$$\Delta V_\| = V_0(1 - \cos\theta_d)$$



# Writing Velocity change in terms of tan $\Phi_0$

$$\theta_d = 2\phi_0 - \pi$$

$$\begin{aligned} \Delta V_\perp &= V_0 \sin\theta_d \\ \Delta V_\parallel &= V_0(1-\cos\theta_d) \end{aligned}$$

First relate these to functions involving $\varphi_0$
Then convert out of center of mass frame

$$\begin{aligned} V_0 \sin\theta_d &= -V_0 \sin 2\phi_0 \\ \sin 2\phi_0 &= 2\sin\phi_0\cos\phi_0 \\ &= 2\tan\phi_0 \sec^2\phi_0 \\ &= \frac{2\tan\phi_0}{1+\tan^2\phi_0} \end{aligned}$$

$$\begin{aligned} 1-\cos\theta_d &= 1+\cos 2\phi_0 \\ &= 2\cos^2\phi_0 \\ &= \frac{2}{1+\tan^2\phi_0} \end{aligned}$$

Can be derived using energy to get *a* and the relation between *e* and tan $\phi_0$

$$C = 1+\tan^2\phi_0 = 1 + \frac{b^2V_0^4}{G^2(M+m)^2}$$

$$p = \frac{h^2}{G(M+m)} = a(e^2-1)$$

# Velocity changes

$$\begin{aligned}\Delta V_\perp &= V_0 \sin\theta_d \\ \Delta V_\parallel &= V_0(1-\cos\theta_d)\end{aligned}$$

First relate these to functions involving $\varphi_0$
Then convert out of center of mass frame using

$$\begin{aligned}\Delta \mathbf{v}_m &= -\frac{M}{m+M}\Delta \mathbf{V} \\ \Delta \mathbf{v}_M &= \frac{m}{m+M}\Delta \mathbf{V}\end{aligned}$$

$$\begin{aligned}\Delta v_{M,\perp} &= \frac{2mbV_0^3}{G(M+m)^2}\frac{1}{C} \\ \Delta v_{M,\parallel} &= \frac{2mV_0}{(M+m)}\frac{1}{C}\end{aligned}$$

Perpendicular component relevant for impulse approximation

Parallel component Important for dynamical friction

$$C = 1+\tan^2\phi_0 = 1+\frac{b^2V_0^4}{G^2(M+m)^2}$$

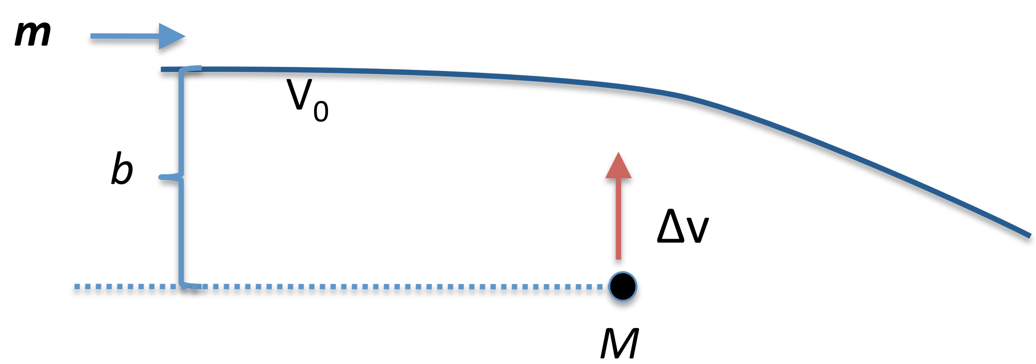
# Impulse Approximation

$$\begin{aligned}\Delta v_{M,\perp} &= \frac{2mbV_0^3}{G(M+m)^2}\frac{1}{C} \\ \Delta v_{M,\parallel} &= \frac{2mV_0}{(M+m)}\frac{1}{C}\end{aligned} \qquad C = 1 + \tan^2\phi_0 = 1 + \frac{b^2V_0^4}{G^2(M+m)^2}$$

- Ignore mass of *M*, only consider *m*
- Take the limit of large $V_0$ →

$$\begin{aligned}\Delta V_\perp &= \frac{2Gm}{V_0 b} \\ \Delta V_\parallel &= 0\end{aligned}$$

# Impulse approx to order of mag




Assume velocity is high

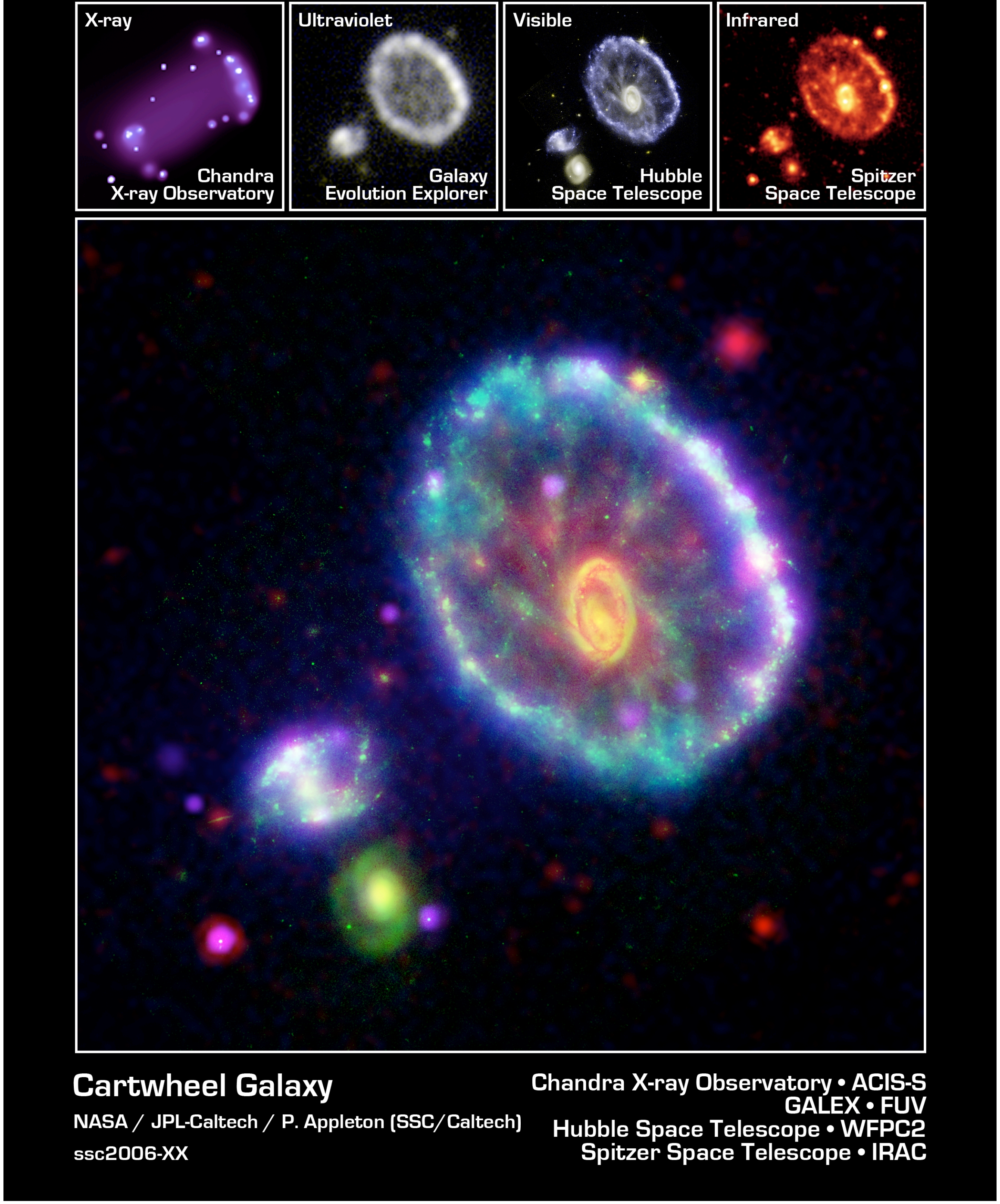
The encounter is only important for a timescale $dt=2b/V_0$

The force on M during the encounter is $F \sim Gm/b^2$

$F=dv/dt$ so $dv \sim F\,dt$

$$\Delta v \sim \frac{Gm}{b^2} \times \frac{2b}{V_0} \sim \frac{2Gm}{bV_0}$$

Velocity change direction is approximately perpendicular to direction to periapse



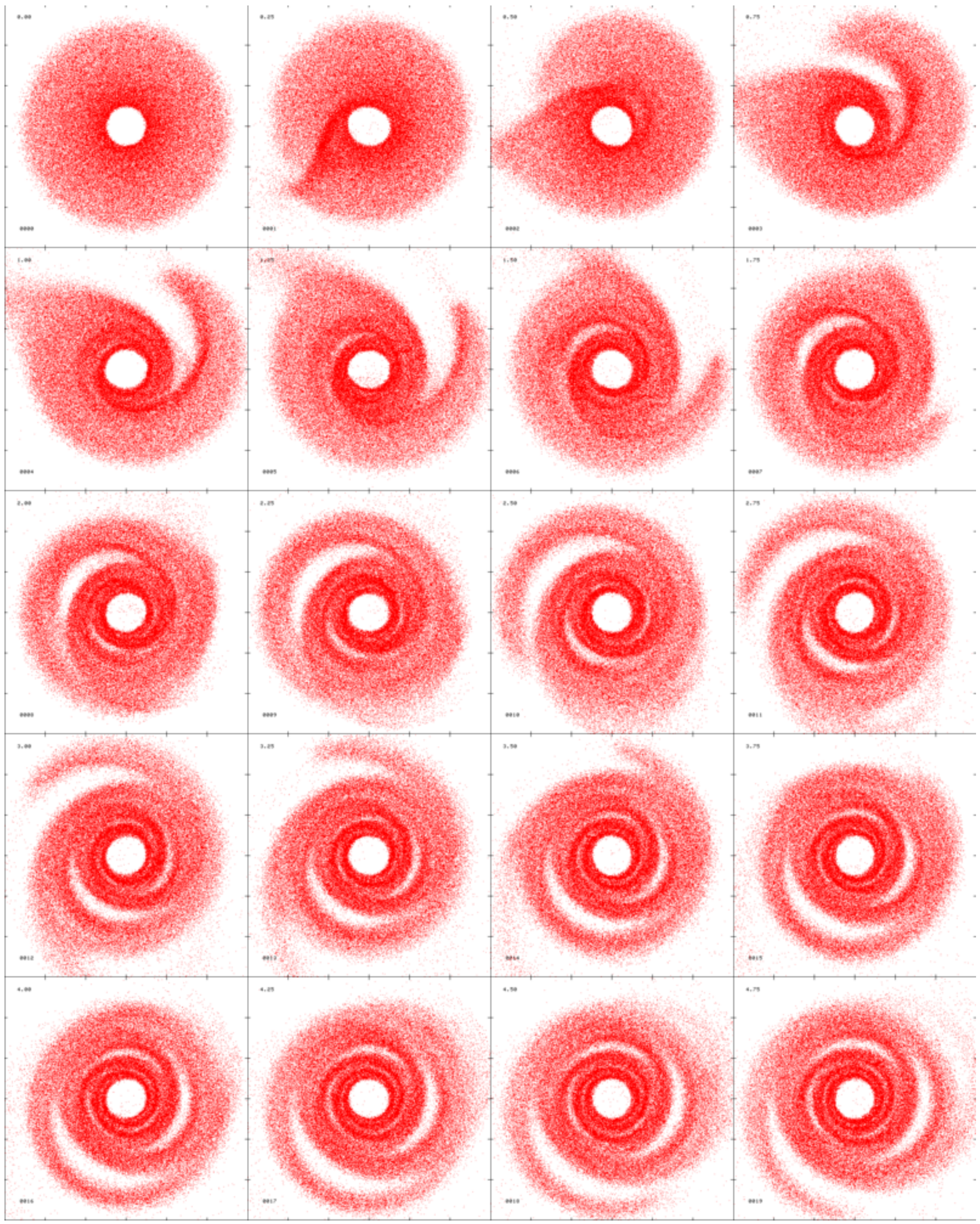

- Impulse is toward position of closest approx
- Impulse approximation is good in the limit that position of object during collision is fixed

# Applications of the Impulse Approximation

- Effect of stellar flybys on the production of long period comets from Oort cloud bodies.
  - Impulse approximation is good as these bodies are moving slowly compared to passing stars. For a comet, the position during the collision is unchanged, however the velocity changes. If the velocity perturbation is against the direction of rotation, the angular momentum drops. That means the eccentricity increases and the pericenter can be small, so the object can be sent into the inner solar system.
  - Note that Galactic tidal field is also thought to be important, causing periodic eccentricity evolution.

# Applications Impulse Approx

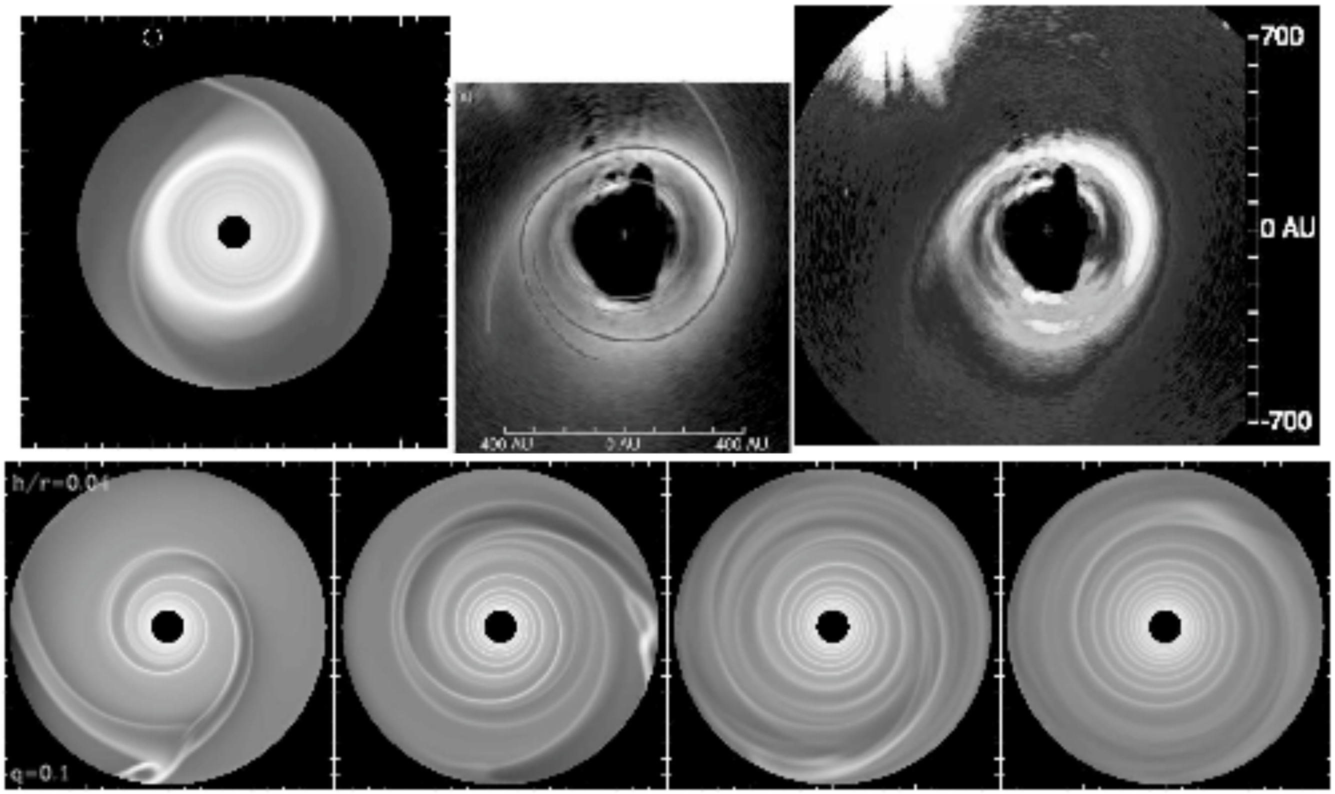
- Tidal heating and evaporation of clusters as they pass through the Galactic plane
  - As positions are not changed during collisions potential energy is constant
  - Kinetic energy is changed leading to heating and so evaporation of some stars from star clusters
- Disruption and evolution of wide stellar binaries
  - Semi-major axis distribution for resolved binaries drops at about $10^4$ AU similar to the boundary of the Oort cloud
  - flyby perturbations and Galactic tide would predict a change in the semi-major axis and eccentricity distribution of very wide binaries.



Response of a Galactic disk to a perturber on the orbit of the Sgr Dwarf galaxy using the impulse approximation for the Dwarf at its last pericenter less than 1Gyr ago.

With the goal of seeing whether we could create the Galactic warp and the Monocerous stellar stream with a close passage

# Stellar flybys and binary interactions on a gaseous circumstellar disk




Prograde encounters are more damaging though this is not predicted by the Impulse approximation

# Dynamical Friction

- Number density of stars a function of *v*: *f(v)*
- Number of stars interacting for each impact parameter b gives a rate of encounters

$$2\pi b\ db\ V_0 f(v) d^3 v$$

- Integrate $\Delta v_{||}$ as a function of *b* and *v*

$$\Delta v_{M,\|} \quad = \quad \frac{2mV_0}{(M+m)}\frac{1}{C} \qquad\qquad C = 1 + \tan^2\phi_0 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2}$$

In the limit of large b $\quad \Delta v_{\|} \propto b^{-2}$

$$\int 2\pi b dv \Delta v_{\|} \propto \ln b$$

particles at large distances are important

# Dynamical Friction

$$\frac{d\mathbf{v}_M}{dt} = 2\pi \ln(1+\Lambda^2) G^2 m(M+m) \int f(v) d^3v \frac{(\mathbf{v}-\mathbf{v}_M)}{(v-v_M)^3}$$

$$\Lambda = \frac{b_{max} V_0^3}{G(M+m)}$$

Coulomb log depends on size of system
Large but weak interactions are important

# Chandrasekhar's formula

$$\frac{d\mathbf{v}_M}{dt} \approx -16\pi \ln\Lambda \ G^2 m(M+m) \ \frac{\mathbf{v}_M}{v_M^3} \int_0^{v_M} f(v) v^2 dv$$

- Chandrasekhar's dynamical friction formula
  - Depends on *v* so is a frictional type of force

$$\frac{d\mathbf{v}_M}{dt} \sim -\frac{4\pi \ln\Lambda \ G^2 M\rho}{v_M^3}\mathbf{v}_M$$

Times a unitless factor that depends on velocity dispersion
*ρ=nM* is stellar mass density

- Is stronger for more massive bodies --- large objects are damped in planetesimal disks and large satellite galaxies merger quickly whereas Globular clusters can orbit in the halo for a Hubble time

# Dynamical Friction

$$\frac{d\mathbf{v}_M}{dt} \sim -\frac{4\pi \ln \Lambda \ G^2 M \rho}{v_M^3} \mathbf{v}_M$$

- The formula itself is not accurate to a factor of 3 or so because of uncertainty in the Coulomb log, neglecting self gravity of wake and actual orbit shapes for the important long range interactions
- Friction is stronger for lower velocities
- An acceleration in direction opposite motion --- loss of angular momentum, spiral inwards for massive objects orbiting in a Galaxy

# Dynamical friction

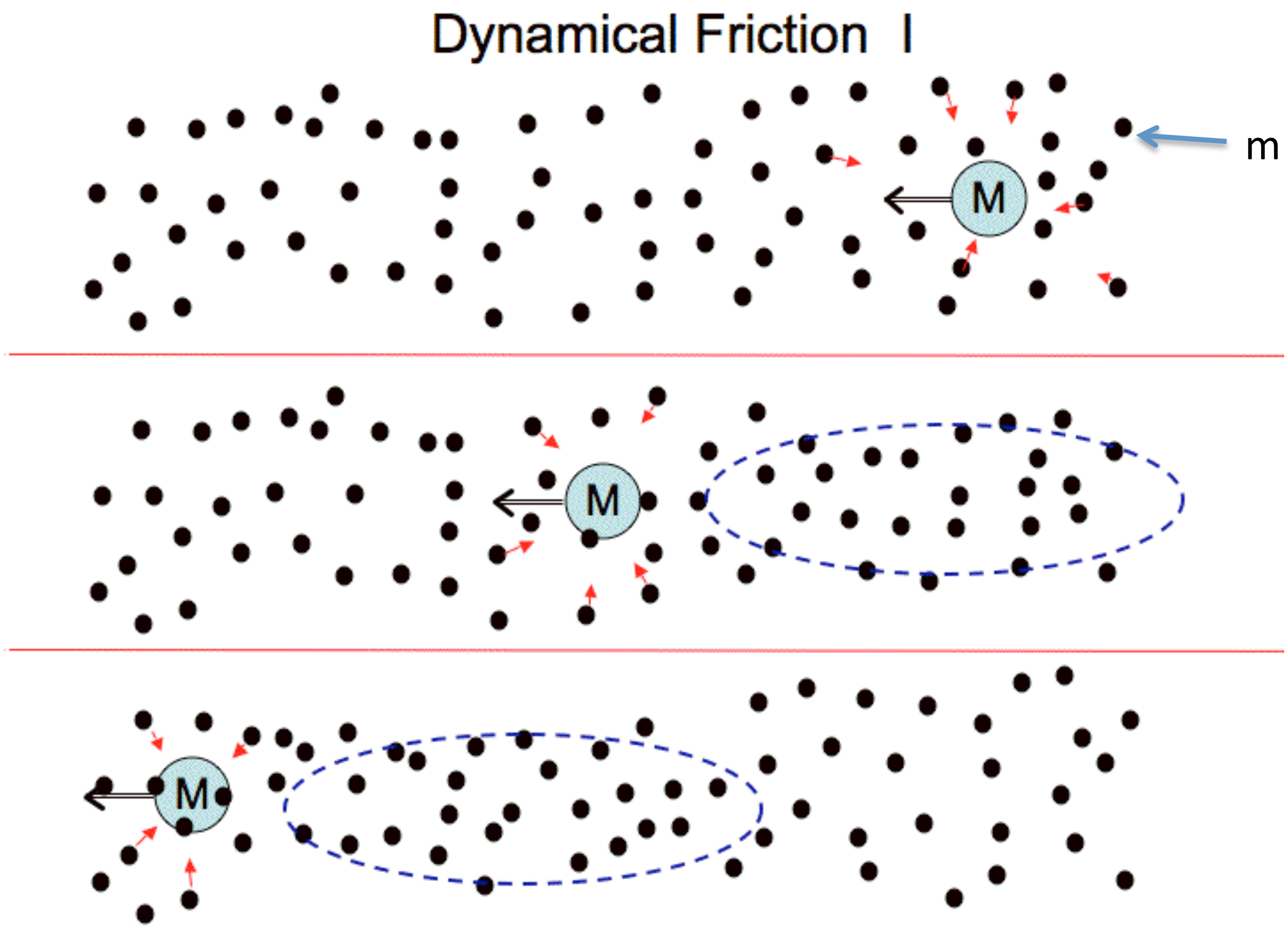
- More accurate integration of velocity distribution can be done if the velocity distribution is Maxwellian. In this case the friction force depends on the ratio $X \equiv \frac{v_M}{2\sigma}$

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi \ln\Lambda \; G^2 M\rho}{v_M^3}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}e^{-X^2}\right]\mathbf{v}_M$$

  where σ is the velocity dispersion

- The friction acceleration is proportional to *M* so the force is proportional to $M^2$

# Dynamical Friction --- Wake



# Diffusion coefficients

- Consider phase space distribution $f(v, x, t)$
- Rate of change of velocity (units v/t) $D(\Delta v_{||})$
- Rate of change of dispersion $D(\Delta v_{\perp}^2), \quad D(\Delta v_{||}^2),$
  - Leading to gravitational stirring or heating by scattering
- How to compute Diffusion coefficients: Compute Δv or Δv$^2$ using perturbations from a hyperbolic orbit. Integrate all possible impact parameters $\times 2\pi b db$. Integrate over velocity distribution.

# Diffusion coefficients for a Maxwellian distribution

$$
\begin{aligned}
D(\Delta v_{\|}) &= -\frac{4\pi G^2 \rho (m + m_a) \ln\Lambda}{\sigma^2} G(X) \\
D(\Delta v_{\|}^2) &= \frac{4\sqrt{2}\pi G^2 \rho m_a \ln\Lambda}{\sigma} \frac{G(X)}{X} \\
D(\Delta v_{\perp}^2) &= \frac{4\sqrt{2}\pi G^2 \rho m_a \ln\Lambda}{\sigma} \left[\frac{\mathrm{erf}(X) - G(X)}{X}\right] \\
G(X) &= \frac{1}{2X^2}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2}\right]
\end{aligned}
$$

See Binney & Tremaine on the Fokker Planck equation (in my edition chap 8, equations 8.68)

# Equipartition of Energy

- Kinetic energy

$$
\begin{aligned}
E &= \sum_i m \frac{v_i^2}{2} \\
D(\Delta E) &= m \sum_i \left[ v_i D(\Delta v_i) + \frac{1}{2} D(\Delta v_i^2) \right] \\
&= m v D(\Delta v_\parallel) + \frac{m}{2} \left( D(v_\parallel^2) + D(v_\perp^2) \right)
\end{aligned}
$$

- where convention is the perpendicular part takes into account both perpendicular directions
- Dynamical friction term is negative so is a cooling term. Other terms are heating terms.
- When different mass bodies are present, the two terms can balance leading to equipartition

  $m\ v = m_a <v_a>$

# Heating and dynamical friction

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r_0^2 \sigma^* M_\odot^{-2}}{\sqrt{\pi}(\langle e^2\rangle + \langle e^{*2}\rangle)^{1/2}(\langle i^2\rangle + \langle i^{*2}\rangle)^{1/2}} \times \left[ B J_e m^* + 1.4 A H_e \left( \frac{m^*\langle e^{*2}\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e^{*2}\rangle} \right) \right],$$

From Stewart and Ida 2000 (Icarus, 143, 28) discussing a population of planetesimals affected by another population of planetesimals.

$\sigma^*$ is mass density of planetesimals of mass $m^*$.

$\Omega$ is angular rotation rate for an object in circular orbit at $r_0$

This equation gives eccentricity dispersion growth on mass $m$ objects in the disk.

Two heating terms, one damping term.

# Eccentricity and inclination evolution

m is self
m* is other

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r_0^2 \sigma^* M_\odot^{-2}}{\sqrt{\pi}(\langle e^2\rangle + \langle e^{*2}\rangle)^{1/2}(\langle i^2\rangle + \langle i^{*2}\rangle)^{1/2}} \times \left[ B J_e m^* + 1.4 A H_e \left( \frac{m^*\langle e^{*2}\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e^{*2}\rangle} \right) \right],$$

Heating depends on mass of other particles

Dynamical friction depends on mass of self and cools disk

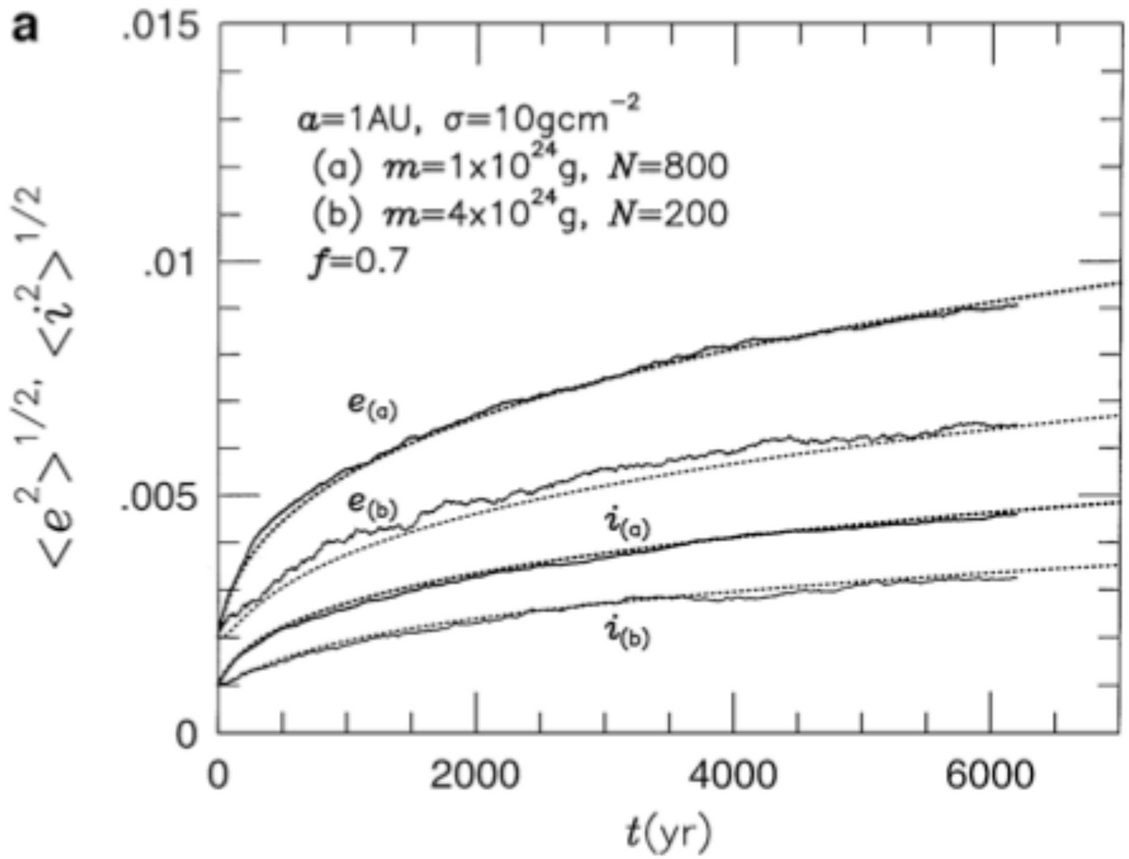
When these two cancel we have what is known as equipartition

# Dependence on mass

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r_0^2 \sigma^* M_\odot^{-2}}{\sqrt{\pi}(\langle e^2\rangle + \langle e^{*2}\rangle)^{1/2}(\langle i^2\rangle + \langle i^{*2}\rangle)^{1/2}} \times \left[ B J_e m^* + 1.4 A H_e \left( \frac{m^*\langle e^{*2}\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e^{*2}\rangle} \right) \right],$$

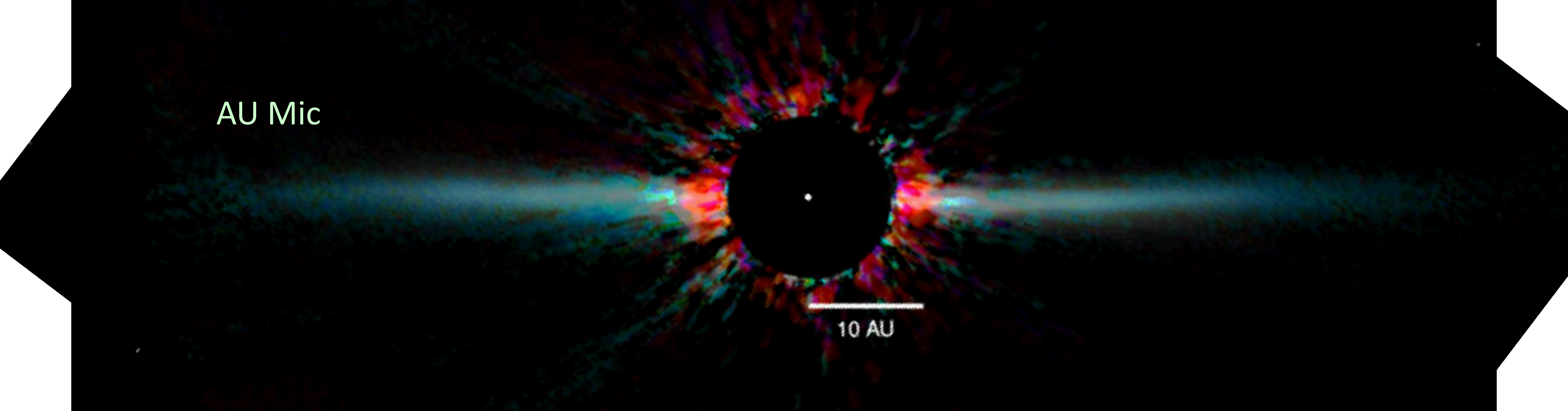
Heating rates depend on produce of surface density times mass. Strongly dependent on the most massive objects in the disk for most size distributions you could envision.

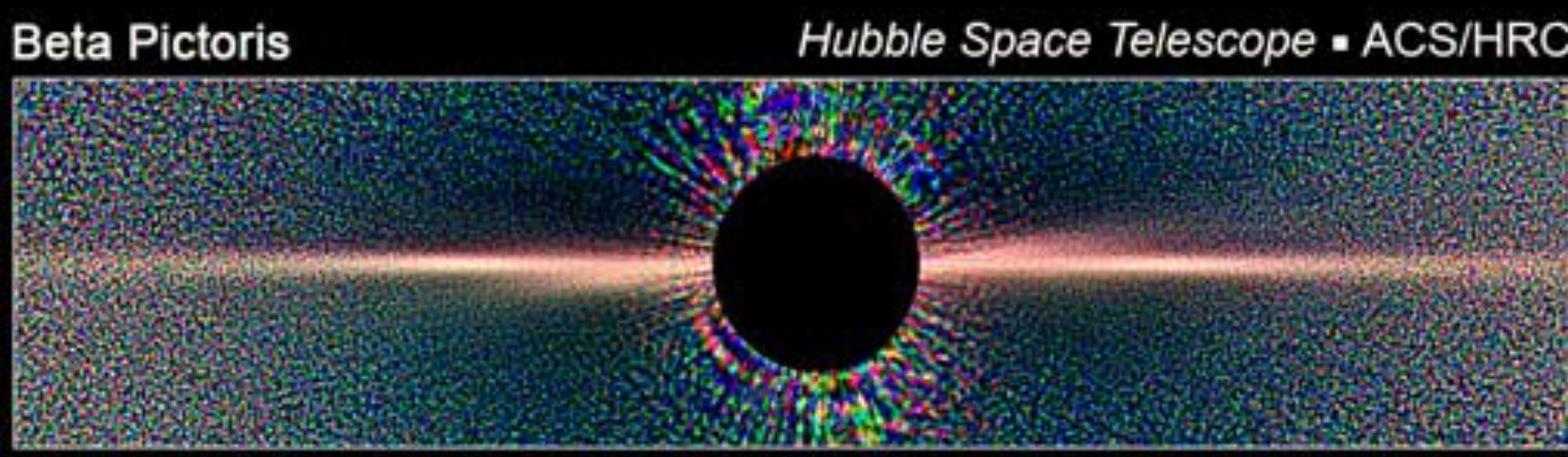
# Evolution of 2 populations




Ability of a low mass swarm to cool larger planetesimals invoked in oligarchic planet formation scenario for outer planets by Lithwick and Goldreich

# Resolved Edge on Debris disks




# Inclination evolution

- Similar evolution equation for inclination. For dispersion dominated regime taking into account heating only

$$\frac{1}{\Omega}\frac{d\langle i^2\rangle}{dt} \sim \frac{\sigma\mu}{\langle i^2\rangle}$$

- With solution

$$\bar{i}(t) \sim (\sigma\mu\Omega t)^{1/4}$$

μ is ratio of planetesimal to stellar mass
Ω angular rotation rate
σ mass density times $r^2/M_*$

- (note coefficients dependent on Coulomb log and of order 1 are not given)
- Disk thicknesses for a few debris disks are observed (AU Mic, Beta Pic, maybe Fomalhaut) leading to the speculation that Pluto sized planetary embryos reside in these disks

# Sheer dominated and dispersion dominated regimes

- Relative velocity set by differential rotation --- in which case *sheer dominated*
- Relative velocity set by velocity dispersion of particles – which case *dispersion dominated*

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r_0^2 \sigma^* M_\odot^{-2}}{\sqrt{\pi}(\langle e^2\rangle + \langle e^{*2}\rangle)^{1/2}(\langle i^2\rangle + \langle i^{*2}\rangle)^{1/2}} \times \left[ B J_e m^* + 1.4 A H_e \left( \frac{m^*\langle e^{*2}\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e^{*2}\rangle} \right) \right],$$

- Two heating terms, first dominates when eccentricity of perturbers are low, (sheer dominated)
- Regime depends on Hill radius ….

# Tidal forces

- Expand the gravitational potential from a perturber about another body

$$\Phi(\mathbf{r}-\mathbf{R}_0) = \Phi(\mathbf{R}_0) + \frac{d\Phi(\mathbf{R}_0)}{dx_i} r_i + \frac{d^2\Phi(\mathbf{R}_0)}{dx_i dx_j}\frac{r_i r_j}{2}$$

$$\mathbf{F}_T = \frac{GM}{D^3}(2x, -y, -z) \qquad \mathbf{R}_0 = (D,0,0) \qquad \mathbf{r} = (x,y,z)$$

- Set tidal force equal to self gravitating force $F_{SG}=Gm/r^2$

  Tidal force exceeds self-gravity when

$$\left(\frac{r}{D}\right)^3\left(\frac{M}{m}\right) > 1$$

Note that this can always be written as a density ratio

M external (like Sun)
m local object (like Earth)

# Expanding Gravitational force in a Taylor series

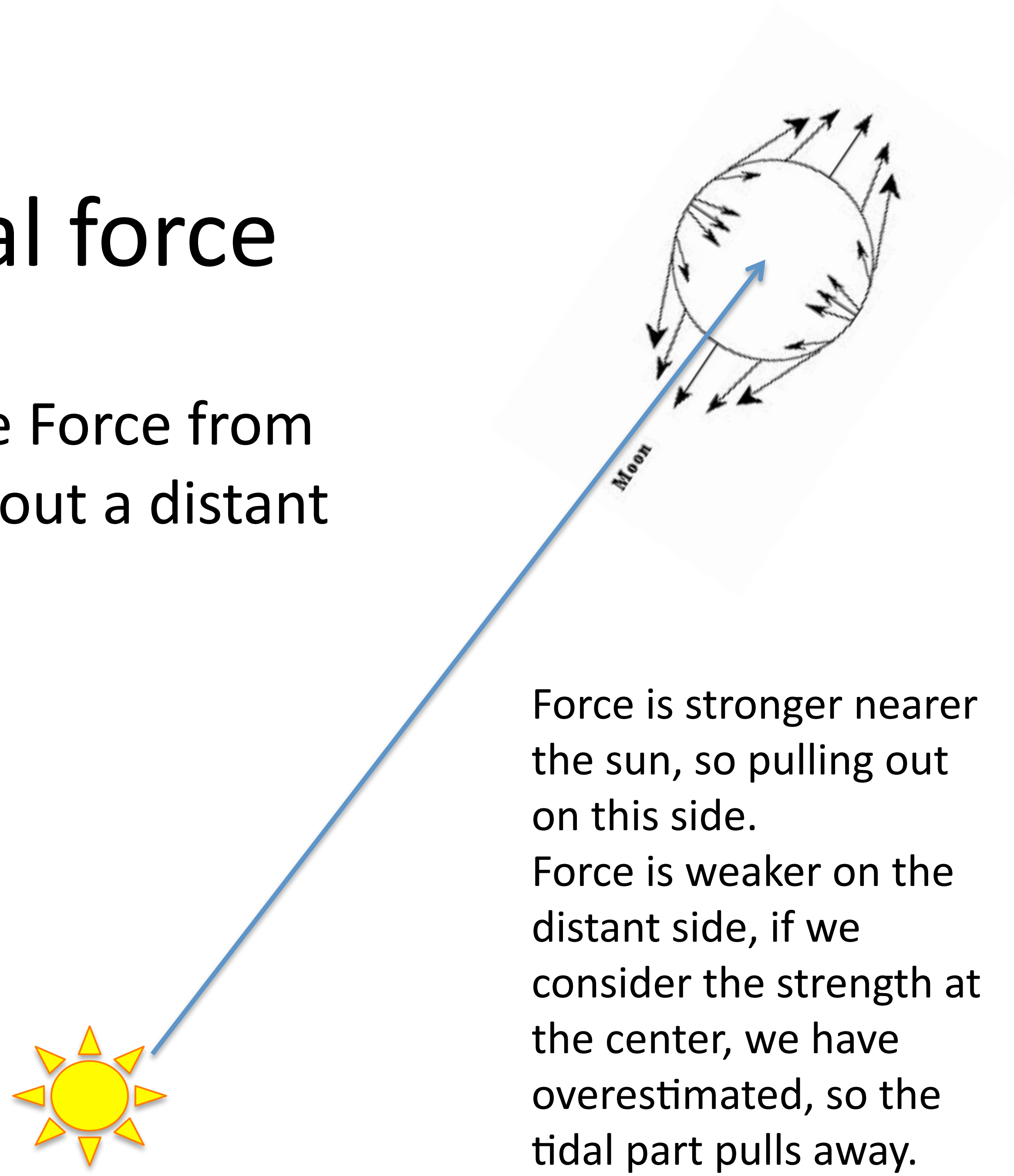
$$f(x+\delta) = f(x) + f'(x)\delta + f''(x)\frac{\delta^2}{2}$$

- Force is a vector and we need to know how the force depends for three directions of varying the position. Expand each component separately. (Or expand the potential and then take the gradient of it.)

$$\begin{aligned} f(x+\delta_x, y+\delta_y, z+\delta_z) = f(x,y,z)+ \\ + \frac{\partial f(x,y,z)}{\partial x}\delta_x + \frac{\partial f(x,y,z)}{\partial y}\delta_y + \frac{\partial f(x,y,z)}{\partial z}\delta_z \\ + \frac{\partial^2 f(x,y,z)}{\partial x^2}\frac{\delta_x^2}{2} + \frac{\partial^2 f}{\partial x \partial y}\delta_x\delta_y + \ldots . \end{aligned}$$

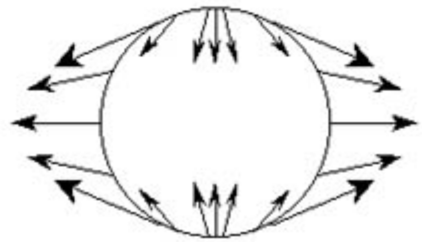
# Tidal force

- Expand the Force from the Sun about a distant point.




Force is stronger nearer the sun, so pulling out on this side.
Force is weaker on the distant side, if we consider the strength at the center, we have overestimated, so the tidal part pulls away.

# Tidal Force (continued)




$$F = -\frac{GM_\odot}{d^2}\hat{\mathbf{d}} + \frac{GM_\odot}{d^3}(3(\hat{\mathbf{d}}\cdot\vec{\mathbf{r}})\hat{\mathbf{d}} - \vec{\mathbf{r}}).....$$

Here $d$ is the distance between the Sun and Earth,

and $r$ is the distance from the center of the Earth.

$\hat{\mathbf{d}}$ represents the unit vector between Earth and Sun

$\vec{\mathbf{r}}$ represents a vector from the Center of the Earth

The direction of the tidal force depends on the direction of $\mathbf{r}$

$$\boxed{F_t = \frac{2GM_\odot}{d^3}r}$$

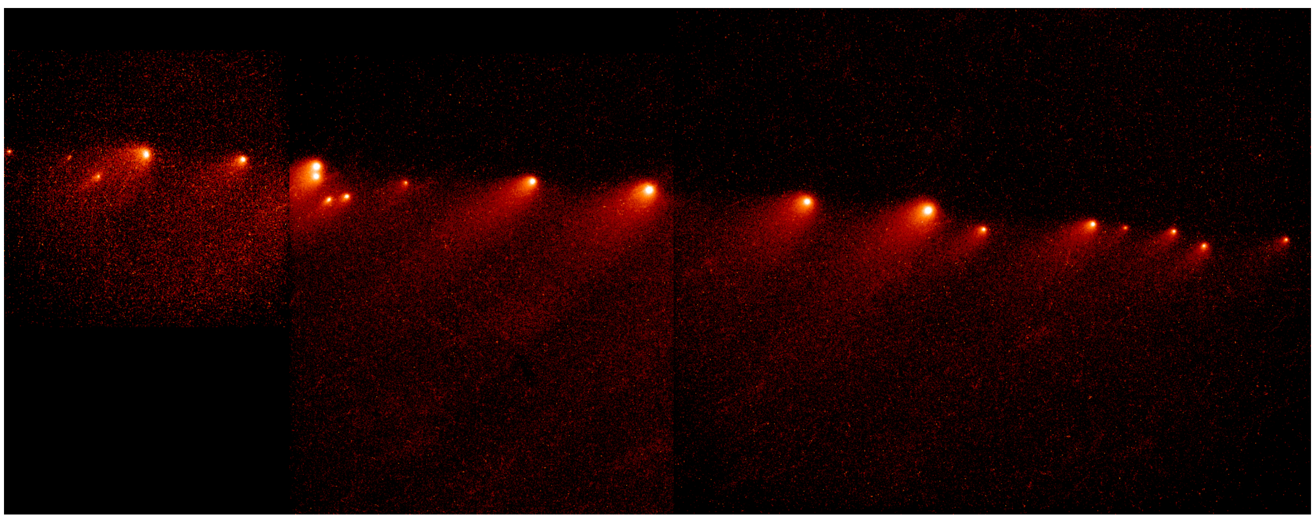
gravitational force was $d^{-2}$
tidal force here $d^{-3}$

Tidal force is $2F_t$ outward for $\mathbf{r}$ toward or away from the Sun,

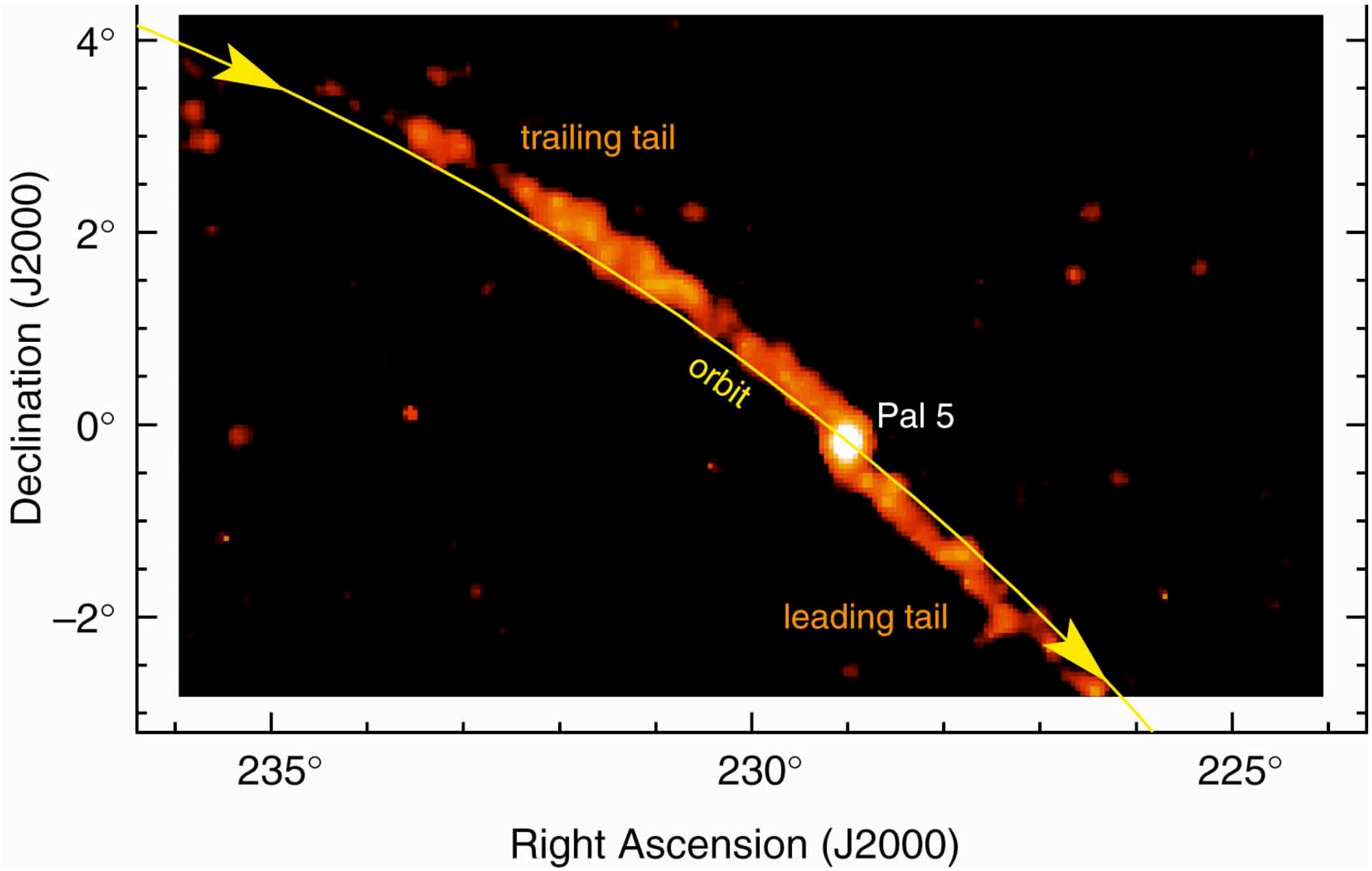
and is $-F_t$ in the plane perpendicular to this line.

# Tidal disruption

- If Jupiter has a mean density of about 1g $cm^{-3}$ what can you say about the progenitor comet for Shoemaker Levy 9 that disrupted upon close passage to Jupiter?



# Tidal stripping




Leading tail is has higher angular momentum so moves faster than cluster center. For elegant semi- analytical formulation see Johnston, K. et al. 1999

# Tidal stripping or disruption – many settings

- Disruption of comets near planets (Shoemaker-Levy 9)
- Disruption of stellar binaries near the Milky Way's central black hole
- Tidal stripping satellite galaxies, globular clusters
  - Formation of eccentric disks in centers of galaxies (M31)
- Useful to remember mean density of objects
  - For Sun and Jupiter ~1.0g/cm$^3$,
  - larger stars 0.1
  - Rocky planets ~3 (excepting Mercury which is 6 or 7)
  - Comets ~0.5 (rubble pile and ice)
  - Stony asteroids maybe 1-3
  - Galaxies – depends on rotational velocity

$$M(r) \sim \frac{v_c^2 r}{G}$$

$$\rho \sim \frac{v_c^2}{G r^2}$$

# Roche or Hill Radius

- Near a planet its gravitational field is more important than that of the Sun inside the Hill radius

$$r_H = a_p \left( \frac{M_p}{3M_\odot} \right)^{1/3}$$

- Associated Hill velocity

$$v_H = r_H n \qquad\qquad n = \sqrt{\frac{GM}{a_p^3}}$$

- Where *n* is the "mean motion" (angular rotation rate for a circular orbit), or for any orbit $P = 2\pi/n$

# Sheer vs Dispersion dominated

- Difference in velocity from a circular orbit in units of the Hill velocity greater than 1 then dispersion dominated
- Or

$$e \gtrsim \left(\frac{M}{3M_0}\right)^{1/3}$$

- Steward and Ida tend to work in units of Hill velocities and related eccentricity and inclination.

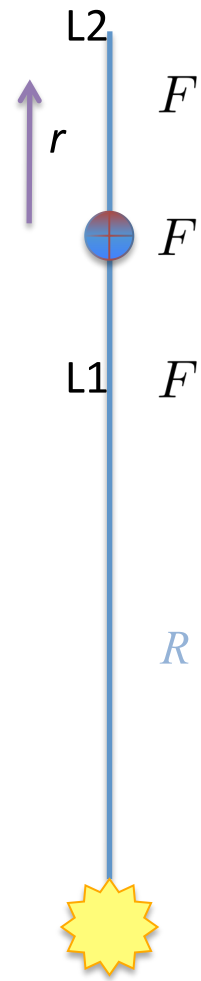
# Lagrange points

- Balance the force from the planet with that of the Sun
- At L1 the Earth's force exactly cancels the larger force from the Sun so that an object feels slightly less force, allowing it to remain in a orbit with the same orbital period as the Earth which is slightly further out
- At L2 the Earth's force adds to the Sun's allowing an object to orbit with a orbital period equivalent to that of the Earth even though the object is further away from the Sun

**Force from Earth**

$$F = \frac{GM_{\oplus}}{r^2}$$

$$F = -\frac{GM_{\oplus}}{r^2}$$




**Force from Sun**

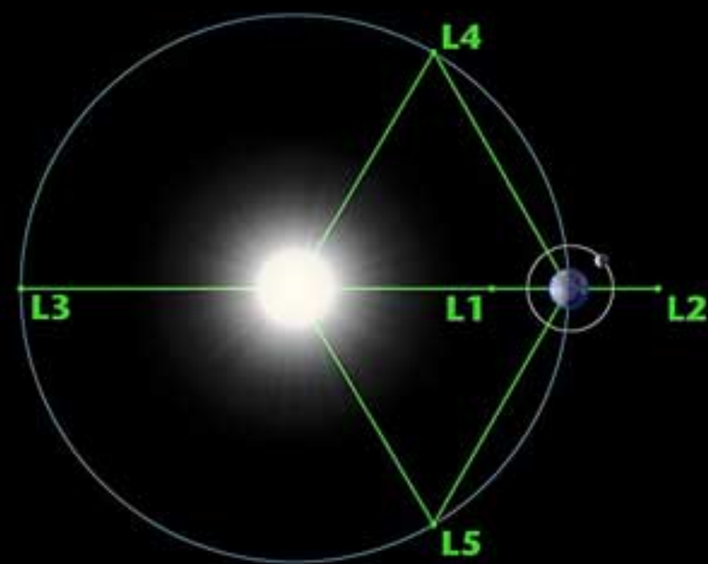
$$F = \frac{GM_{\odot}}{R^2}\left(1 - \frac{2r}{R}\right)$$

$$F = \frac{GM_{\odot}}{R^2}$$

$$F = \frac{GM_{\odot}}{R^2}\left(1 + \frac{2r}{R}\right)$$

# Lagrange Points

- There are special points where a particle in a frame rotating with a planet feels no net force.
- These are known as Lagrange points.
- There are 5 of them.
- We can think of L1 and L2 as places where the tidal force from the Sun is balanced against gravity from the planet.



# Restricted Three-body Problem

- Two massive bodies, in a circular orbit. Like Jupiter+Sun. Orbit is Keplerian.
- Consider the dynamics of a third massless particle.

$$\frac{d\mathbf{v}}{dt} = -\frac{GM_*(\mathbf{r}-\mathbf{r}_*)}{|\mathbf{r}-\mathbf{r}_*|^2} - \frac{GM_p(\mathbf{r}-\mathbf{r_p})}{|\mathbf{r}-\mathbf{r}_p|^2}$$

$$\frac{d\mathbf{v}}{dt} - 2\mathbf{\Omega} \times \mathbf{v} - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r} = -\nabla\Phi$$

In the rotating frame with gravitational potential from each interaction

$$\frac{d\mathbf{v}}{dt} - 2\mathbf{\Omega} \times \mathbf{v} = -\nabla\Phi_{eff}$$

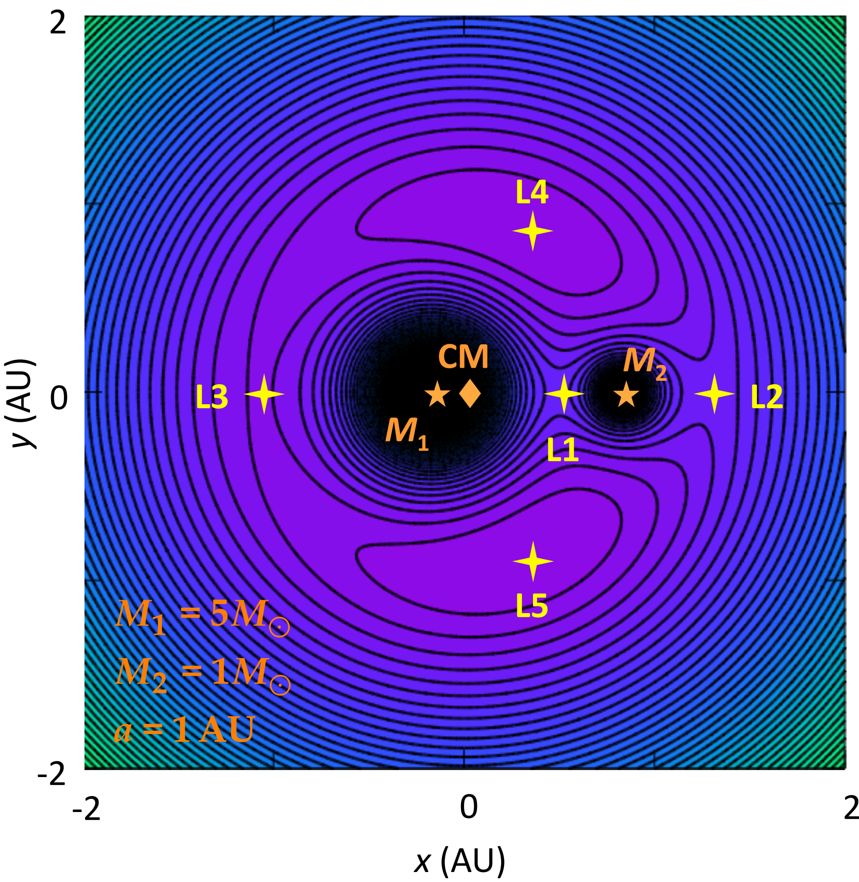
$$\Phi_{eff} = -\frac{GM_*}{|\mathbf{r}-\mathbf{r}_*|} - \frac{GM_p}{|\mathbf{r}-\mathbf{r}_p|} - \frac{\Omega^2}{2}r^2$$

pseudo or effective potential

# Effective potential contours

$$\Phi_{eff} = -\frac{GM_*}{|\mathbf{r}-\mathbf{r}_*|} - \frac{GM_p}{|\mathbf{r}-\mathbf{r}_p|} - \frac{\Omega^2}{2}r^2$$

$$= -\frac{GM_*}{|\mathbf{r}-\mathbf{r}_*|} - \frac{GM_p}{|\mathbf{r}-\mathbf{r}_p|} - \frac{G(M_*+M_p)}{2a^3}r^2$$

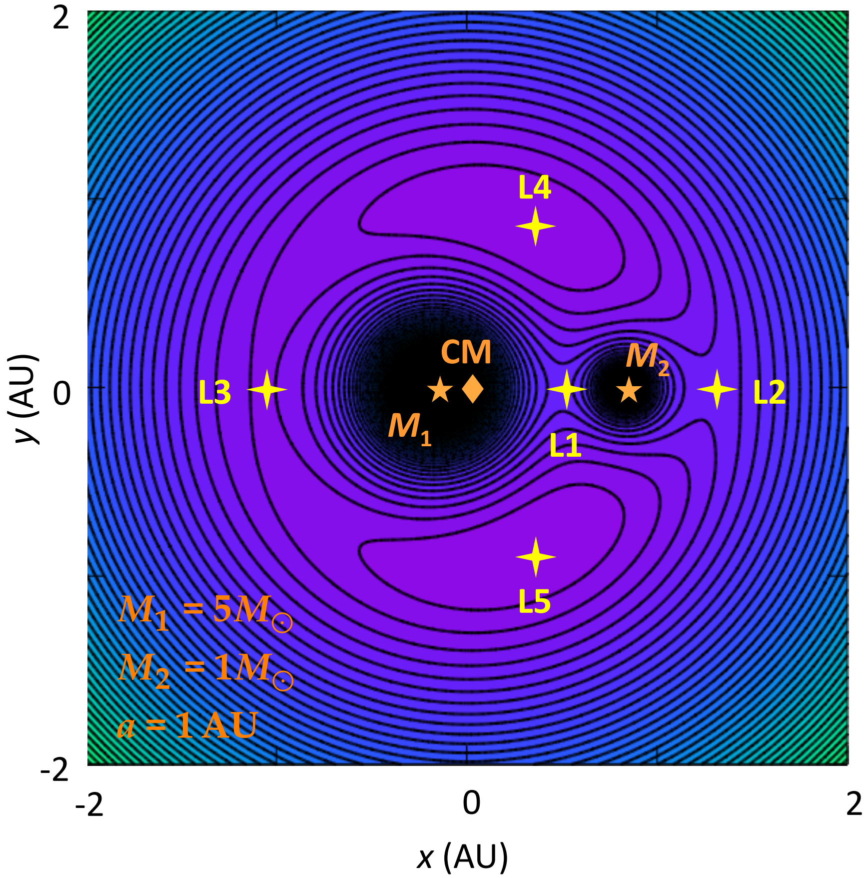



Consider an orbit with nearly zero velocity. The Coriolis force is zero.

$$\frac{d\mathbf{v}}{dt} - 2\mathbf{\Omega}\times\mathbf{v} = -\nabla\Phi_{eff}$$

Fixed points are extrema in the effective potential

# Effective potential contours

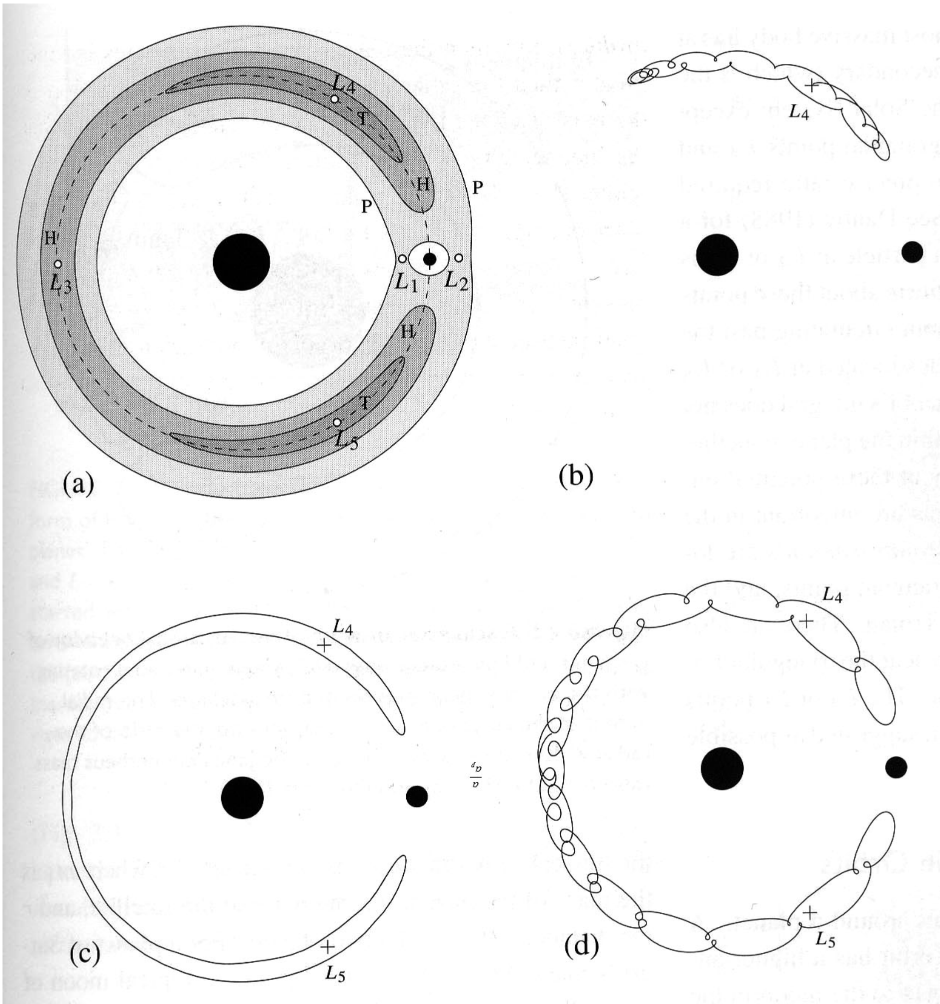



L4,L5 are potential minima.
Stable minima.

L1,L2 are saddle points.
Unstable minima

# Fixed points and oscillations about fixed points

- Lagrange points are fixed points in the rotating frame
- By expanding as a function of distance from a fixed point, it is possible to determine whether stable or not, and if stable, what period of oscillations about the fixed point is




From dePater and Lissauer

In the frame rotating with the Earth.

# Horseshoe and Tadpole Orbits

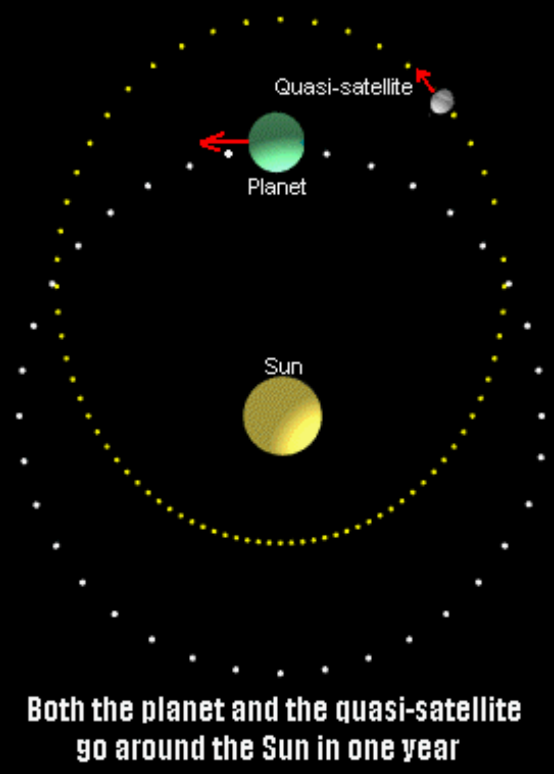
The L4 and L5 points are **stable**. Near these points there are small closed orbits.

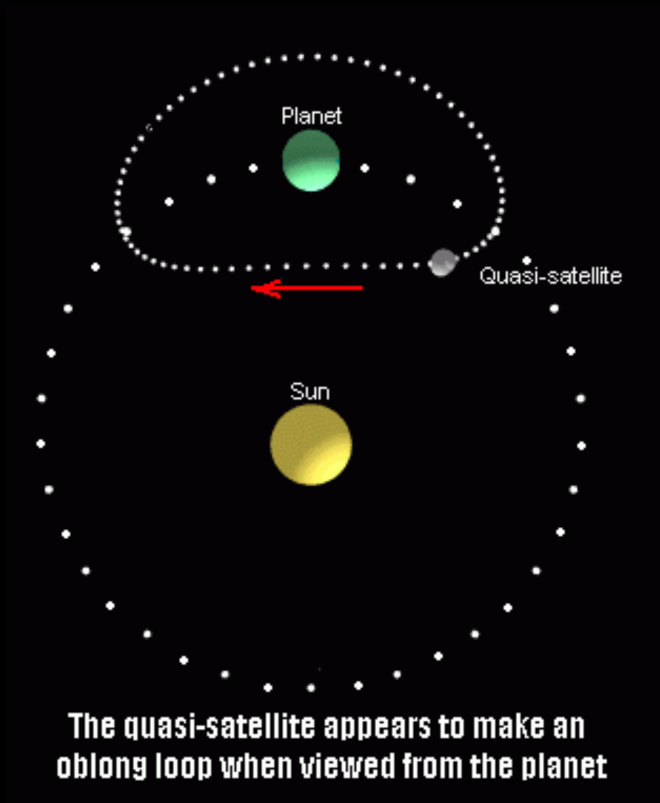
The other Lagrange points are not stable. This means that a small nudge away from the point will cause the particle to move far away in its orbit.

Space craft put in the L1 or L2 points must be maintained in these positions.

# Quasi satellites

Same semi-major axis as a planet.



# Quasi satellites

- Earth, Venus, Neptune and (recently discovered)  Pluto have quasi-satellites
- Goes outside the Hill sphere of the planet (this is different than regular satellites
- Stays in the vicinity of the planet (different than tadpole or horseshoe orbits
- Perturbations from the planet are important, lifetimes thousands of orbits but not necessarily the age of the solar system

# Gravitational potential in free space

- Outside a planet there is no mass density so the gravitational potential satisfies Laplace's equation $\nabla^2\Phi = 0$
- This can be re-written in the form of Legendre's equation, and this is satisfied by spherical harmonic functions multiplied by powers of r

$$\Phi(r,\mu,\phi) = \sum_{n=0}^{\infty}\left[A_n r^2 + B_n r^{-(n+1)}\right] S_n(\mu,\phi)$$

- By symmetry near a non-round body the solution should only depend on $\mu=\cos\theta$ and should be independent of $\phi$

# Gravitational potential

In the case of axi-symmetry and requiring the potential to be finite either work with

$$\Phi(r,\mu) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\mu) \qquad \text{or} \qquad \Phi(r,\mu) = \sum_{n=0}^{\infty} A_n r^n P_n(\mu)$$

-potential outside an oblate spinning planet

-potential due to tidal forces
-potential inside a non-round body

# Expansion and Legendre polynomials

$$\frac{1}{|\mathbf{r}-\mathbf{R}|} = \frac{1}{\sqrt{r^2+R^2-2\mathbf{r}\cdot\mathbf{R}}}$$

$$= \frac{1}{R}\frac{1}{\sqrt{1+\alpha^2-2\alpha\mu}} \qquad \mu = \frac{(\mathbf{r}\cdot\mathbf{R})}{rR}$$

expand in powers of α $\qquad \alpha = \frac{r}{R}$

$$\frac{1}{|\mathbf{R}-\mathbf{r}|} = \frac{1}{R}\sum_{n=0}^{\infty}\frac{d^n}{d\alpha^n}(1+\alpha^2-2\mu\alpha)^{-1/2}|_{\alpha=0}\left(\frac{r}{R}\right)^n\frac{1}{n!}$$

This function also satisfies Laplace's equation with axi-symmetry in spherical coordinates

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial R}\right)+\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right)=0$$

# Expansion and Legendre polynomials

Separating into powers of r and functions μ we can show that the functions of μ must satisfy Legendre's equation

$$(1-\mu^2)\frac{\partial^2 P_n(\mu)}{\partial \mu^2} - 2\mu\frac{\partial P_n(\mu)}{\partial \mu} + n(n+1)P_n(\mu) = 0$$

$$\frac{1}{|\mathbf{r}-\mathbf{R}|} = \frac{1}{R}\sum_0^\infty \left(\frac{r}{R}\right)^n P_n(\mu)$$

$P_n$ are the Legendre polynomials

# Gravitational potential due to tides

$$\Phi_T(r,\mu) \approx -\frac{GM}{R}\left(\frac{r}{R}\right)^2 P_2(\mu)$$

lowest order is the quadrupole

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$$

tidal potential is the quadrupolar term in this expansion

# Reading

- This lecture:
  - Binney and Tremaine Chap 7
  - Murray and Dermot Chap 2
  - Stewart & Ida 2000, Icarus, 143, 28