PROBLEM SET #3B **AST242**

1. Are galaxy gas disks accretion disks?

Consider a gas rich spiral disk galaxy. A typical velocity dispersion for turbulent motions in the HI gas is of order 10 km/s and a typical radial sizescale for the disk is of order 10 kpc. A typical circular rotation velocity would be 200 km/s.

(a) Using the velocity dispersion in the HI disk and hydrostatic equilibrium, estimate the scale height of the atomic gas component in parsec. Here the velocity dispersion plays the role of the sound speed.

(b) Using the scale height and velocity dispersion in the HI disk, estimate a turbulent viscosity.

(c) Using the radius of the galaxy, estimate an accretion timescale in years, or a timescale for accretion to take place over a large radius.

(d) Compare an accretion timescale to the Hubble time. Is viscous accretion likely to be important in galactic gas disks?

It may be helpful to check that 1 km/s is approximately 1 pc/Myr is approximately 1 kpc/Gyr.

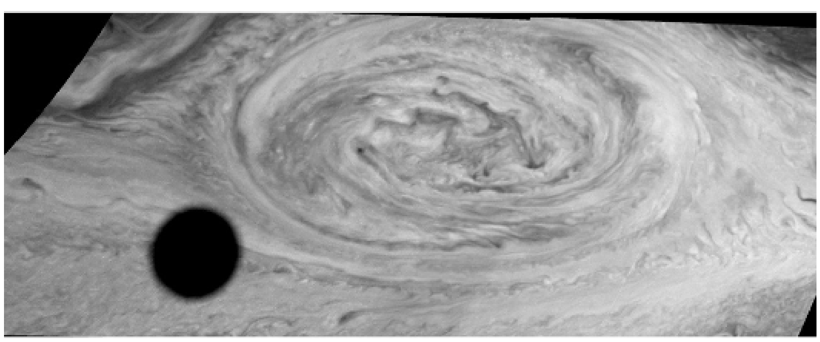

Fig. 1.— One of the mosaics analyzed by our feature tracker. The shadow in the mosaic belongs to Europa.

FIGURE 1. Galileo image of Jupiter's red spot used by Choi et al. (2007).

2. Estimating the vorticity of Jupiter's great red spot

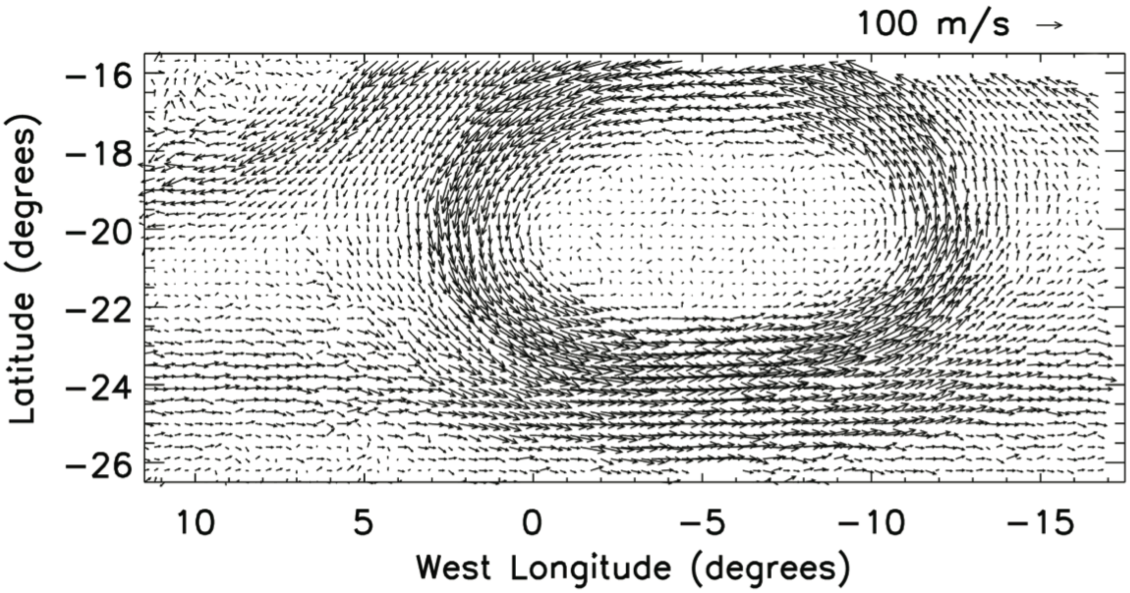

FIGURE 2. Velocity vectors measured by Choi et al. 2007.

Using Figure 2, and taking into account the rotation of Jupiter and latitude of the red spot, estimate the vorticity in Jupiter's red spot, in units of s^{-1}.

Figures 1 and 2 are from the following paper: *Velocity and Vorticity Measurements of Jupiter's Great Red Spot Using Automated Cloud Feature Tracking*, Choi, D. S., Bandfield, D., Gierasch, P. J. & Showman, A. P. Icarus, 188, 35-46 (2007).

The radius of Jupiter is approximately 70,000 km. The sidereal rotation period of Jupiter is $P = 9.925$ hours $= 3.573 \times 10^4$ s giving a spin of $\Omega = 2\pi/P = 1.758 \times 10^{-4}$ s^{-1}.

The cross product in cylindrical coordinates

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}A_{z,\phi} - A_{\phi,z}\right)\hat{\mathbf{r}} + \left(A_{r,z} - A_{z,r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - A_{r,\phi}\right)\hat{\mathbf{z}}$$

3. Potential flow

Consider a flow where the vorticity is initially zero and the flow is everywhere inviscid, barotropic and incompressible.

(a) Explain using the Helmholtz equation or Kelvin's circulation theorem why is it possible to construct a velocity potential function Ψ with velocity

$$\mathbf{u} = \boldsymbol{\nabla}\Psi$$

(b) Show that

$$\nabla^2\Psi = 0$$

(c) Show that at each point streamlines are perpendicular to equipotential surfaces.

(d) If the flow is also steady state, show that Bernoulli's constant is constant everywhere in the flow, not just conserved along each streamline.

4. Burger's vortex

Burger's vortex is one of a few known simple steady state analytical solutions to the Navier-Stokes equation that exhibit vorticity. It can be used as an analogy for how water rotates as it goes down a drain, or perhaps for a tornado.

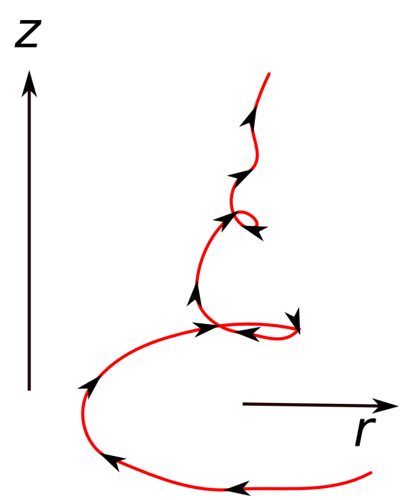

FIGURE 3. Streamlines for Burger's vortex. If z is flipped then the flow is like water going down a drain.

Consider a steady flow in cylindrical coordinates with velocity vector

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_z\hat{\mathbf{z}} + v_\phi\hat{\boldsymbol{\phi}} \tag{1}$$

with

$$\begin{aligned} v_r &= -\frac{1}{2}\alpha r \\ v_z &= \alpha z \\ v_\phi &= v_\phi(r) \end{aligned} \tag{2}$$

and $\alpha > 0$ a constant that describes the strain or rate of shear in the flow.

(a) Show that this flow is incompressible.

Let the vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} = \omega_r \hat{\mathbf{r}} + \omega_\phi \hat{\boldsymbol{\phi}} + \omega_z \hat{\mathbf{z}} \tag{3}$$

(b) Show that the vorticity only contains a $\hat{\mathbf{z}}$ component and

$$\omega_z = \frac{1}{r}\frac{d}{dr}(r v_\phi(r)) \tag{4}$$

For incompressible flow the Navier-Stokes equation can be manipulated to give an equation for the evolution of the vorticity,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega} \tag{5}$$

where ν is the kinematic viscosity.

(c) Show that equation 5 for this flow can be written as

$$\begin{aligned} \frac{D\omega_z}{Dt} &= \omega_z \frac{\partial v_z}{\partial z} + \nu \boldsymbol{\nabla}^2 \omega_z \\ \frac{D\omega_z}{Dt} &= \omega_z \alpha + \nu \boldsymbol{\nabla}^2 \omega_z \end{aligned} \tag{6}$$

(d) Show that equation 6 can be written

$$\left(\partial_r + \frac{1}{r}\right)\left(\partial_r + \frac{\alpha r}{2\nu}\right)\omega_z = 0 \tag{7}$$

with $\partial_r = \frac{\partial}{\partial r}$.

(e) Show that a steady state solution to equation 6 is

$$\omega_z = \omega_0 \exp\left(-c r^2\right) \tag{8}$$

with constant c, and find the constant c. This constant depends on the strain α and the kinematic viscosity ν.

Here the viscosity causes the vorticity to diffuse outward, whereas the strain, with strength α, causes a stretching of vortex lines and so an increase of vorticity. In steady state there is a balance between the two affects.

(f) The solution is a Gaussian distribution of vorticity. If the strain increases how (with what power index) does the width of the Gaussian vary? If the viscosity increases how does the width of the Gaussian change?

The following may be handy:

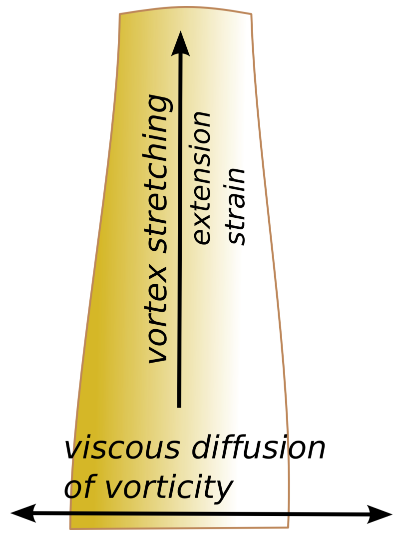

FIGURE 4. The strain of the flow stretches the vortex tube causing vorticity to increase. Viscosity causes vorticity to diffuse outward. In steady state there is a balance between the processes.

In cylindrical coordinates[1] the divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial (r A_r)}{\partial r} + \frac{\partial A_z}{\partial z} \tag{9}$$

and the cross product

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}A_{z,\phi} - A_{\phi,z}\right)\hat{\mathbf{r}} + \left(A_{r,z} - A_{z,r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial (r A_\phi)}{\partial r} - A_{r,\phi}\right)\hat{\mathbf{z}} \tag{10}$$

The z component of the Lagrangian derivative

$$[(\mathbf{u}\cdot\nabla)\mathbf{A}]\cdot\hat{\mathbf{z}} = u_r A_{z,r} + \frac{u_\phi}{r}A_{z,\phi} + u_z A_{z,z} \tag{11}$$

The Laplacian operator

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{12}$$

The vector identity

$$\nabla \times (\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{v} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{13}$$

5. Spiral Density Waves in a Self Gravitating Gas Disk

[1] http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

Consider a thin rotating gas disk with surface density Σ. We denote the radial velocity u and the tangential velocity v. In cylindrical coordinates (R, ϕ) the radial and tangential components of Euler's equation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial R} + \frac{v}{R}\frac{\partial u}{\partial \phi} - \frac{v^2}{R} = -\frac{\partial h}{\partial R} - \frac{\partial \Phi}{\partial R} \tag{14}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial R} + \frac{v}{R}\frac{\partial v}{\partial \phi} + \frac{uv}{R} = -\frac{1}{R}\frac{\partial h}{\partial \phi} - \frac{1}{R}\frac{\partial \Phi}{\partial \phi} \tag{15}$$

where h is the enthalpy. We consider an unperturbed system with with radial velocity $u_0 = 0$, and tangential velocity $v_0 = R\Omega$ where $\Omega = \frac{d\Phi_0}{dR}$ and the unperturbed gravitational potential axisymmetric, $\Phi_0(R)$. The unperturbed system has enthalpy, h_0, surface density, Σ_0, gravitational potential, Φ_0, and tangential velocity, v_0, that are independent of time and ϕ.

Define the epicyclic frequency as

$$\kappa^2 \equiv 3\Omega^2 + \frac{d^2\Phi_0}{dR^2} \tag{16}$$

(a) Show that to first order Euler's equations are

$$\frac{\partial u_1}{\partial t} + \Omega\frac{\partial u_1}{\partial \phi} - 2\Omega v_1 = -\frac{\partial}{\partial R}(h_1 + \Phi_1) \tag{17}$$

$$\frac{\partial v_1}{\partial t} + u_1\frac{\kappa^2}{2\Omega} + \Omega\frac{\partial v_1}{\partial \phi} = -\frac{1}{R}\frac{\partial}{\partial \phi}(h_1 + \Phi_1) \tag{18}$$

and the continuity equation is

$$\frac{\partial \Sigma_1}{\partial t} + u_1\left(\frac{\Sigma_0}{R} + \frac{\partial \Sigma_0}{\partial R}\right) + \Sigma_0\frac{\partial u_1}{\partial R} + \Omega\frac{\partial \Sigma_1}{\partial \phi} + \frac{\Sigma_0}{R}\frac{\partial v_1}{\partial \phi} = 0 \tag{19}$$

Assume first order perturbations in the form

$$\propto e^{i(kR + m\phi - \omega t)} \tag{20}$$

These perturbations have m spiral arms.

(b) Show that Euler's equations become

$$i(m\Omega - \omega)u_1 - 2\Omega v_1 = -ik(h_1 + \Phi_1) \tag{21}$$

$$i(m\Omega - \omega)v_1 + u_1\frac{\kappa^2}{2\Omega} = \frac{im}{R}(h_1 + \Phi_1) \tag{22}$$

and the continuity equation becomes

$$i(m\Omega - \omega)\Sigma_1 + u_1\left(\frac{\Sigma_0}{R} + \frac{\partial \Sigma_0}{\partial R}\right) + ik\Sigma_0 u_1 + \frac{im\Sigma_0 v_1}{R} = 0 \tag{23}$$

Assume $kR \gg 1$ so that radial derivatives are dominated by k. Taking the limit of large k is equivalent to a tight winding approximation or a WKB approximation.

In this limit terms proportional to $\frac{1}{R}$ can be dropped as they are smaller than terms proportional to k.

(c) Show that in the WKB limit

$$u_1 = -k(h_1 + \Phi_1)(m\Omega - \omega)\Delta^{-1} \tag{24}$$

and

$$(m\Omega - \omega)\frac{\Sigma_1}{\Sigma_0} + ku_1 = 0 \tag{25}$$

where

$$\Delta \equiv (m\Omega - \omega)^2 - \kappa^2 \tag{26}$$

Combine these equations together to show that

$$\Sigma_1 = k^2\Sigma_0(h_1 + \Phi_1)\Delta^{-1} \tag{27}$$

(d) Assume that the disk is very thin. Using LaPlace's equation $\nabla^2\Phi = 0$ above the disk and Gauss' law in a pillbox containing the disk and Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, show that

$$\Phi_1 \approx -\frac{2\pi G\Sigma_1}{|k|} \tag{28}$$

(e) Using $h_1 \sim c_s^2\frac{\Sigma_1}{\Sigma_0}$ and the above expression for Φ_1 derive the following dispersion relation for spiral density waves valid for a thin disk in the WKB approximation

$$(m\Omega - \omega)^2 - \kappa^2 = k^2c_s^2 - 2\pi G\Sigma_0|k| \tag{29}$$

(f) For axisymmetric perturbations $m = 0$. Show that all wavevectors gives wavelike solutions and all perturbations are stable if

$$Q \equiv \frac{c_s\kappa}{\pi G\Sigma_0} > 1 \tag{30}$$

The above parameter is called the *Toomre* Q parameter.

Notes: equation (24) implies that velocity perturbations become infinite as $\Delta \to 0$. These are known as *Lindblad* resonances. Equation (25) implies that density perturbations can be large where $m\Omega \sim \omega$. This is known as a *corotation* resonance. Lindblad resonances are locations where spiral density waves can be excited or driven, for example by planets or satellites embedded in the disk. Numerical N-body simulations if they are begun with Q near 1 will exhibit spiral density waves or bar instability. Here we have considered a gas disk however a similar stability criterion exists for a particle or stellar disk, but with the velocity dispersion of the stars or particles playing the role of the sound speed.