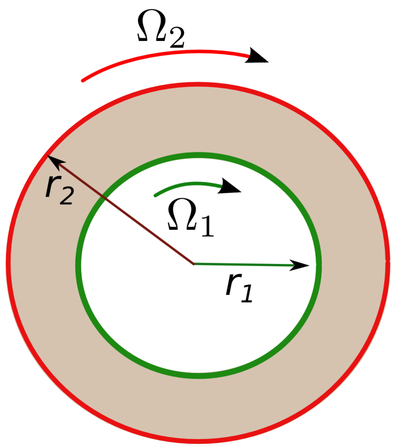

FIGURE 1. Two concentric co-axial cylinders each rotating at a different angular rotation rate. A viscous fluid lies between the two cylinders.

1. Couette Flow

A viscous fluid lies in the space between two infinitely long, concentric co-axial cylinders of radius r_1 and $r_2 > r_1$. The cylinders rotate with angular speeds of Ω_1 and Ω_2, respectively. We assume a high fluid viscosity. Such flow becomes unstable to formation of Taylor vortices if $\Omega_1 \gg \Omega_2$.

(a) Using the ϕ component of the Navier Stokes equation in cylindrical coordinates, show that in steady state the fluid rotates with tangential velocity

$$u_\phi = Ar + \frac{B}{r} \tag{1}$$

where constants

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \tag{2}$$

$$B = \frac{(\Omega_1 - \Omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2} \tag{3}$$

Assume that $u_r = u_z = 0$ and that the flow is steady state and axisymmetric.

(b) Assuming an incompressible flow find the difference in pressure between the inner and outer cylinder for the limit $r_2 - r_1$ much smaller than r_1 or r_2. Use the radial component of the Navier Stokes equation. To do this expand A and B in terms of a where $a = r_2 - r_1$ and $a \ll r_1$. Then ΔP (the pressure difference) is $\Delta P \sim a\frac{\partial P}{\partial r}$.

In cylindrical coordinates the components of the Navier Stokes equations are

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right]u_r \\
&\quad -\nu\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi} \\
\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_\phi u_r}{r} &= -\frac{1}{r\rho}\frac{\partial p}{\partial \phi} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right]u_\phi \\
&\quad +\nu\frac{2}{r^2}\frac{\partial u_r}{\partial \phi} \\
\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right]u_z
\end{aligned}$$

Note the non-trivial extra terms arising from the coordinates.

2. Dynamic and kinematic viscosity

The dynamic and kinematic shear viscosities, μ and ν are related by $\mu = \rho\nu$ where ρ is the density. Kinematic theory allows us to estimate that $\nu \sim v\lambda$ where v is a thermal velocity and λ is the mean free path. It may be useful to define $\rho = mn$ where m is the mass of the particles and n is the number of particles per unit volume. Show that the dynamic shear viscosity is not dependent on the particle number density, n, if the temperature and collision cross section don't vary.

3. Accretion and Excretion Disks

By combining the ϕ component of the Navier Stoke's equation and conservation of mass in cylindrical coordinates it is possible to show that the basic equation that describes viscous evolution of an accretion disk around a point mass is

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right] \tag{4}$$

where $\Sigma(R, t)$ is the disk surface density (mass per unit area) as a function of radius and time. Here ν is the kinematic viscosity (units cm^2/s).

(a) Consider a steady state disk with $\frac{\partial \Sigma}{\partial t} = 0$. In class we found $\dot{M} \sim 3\pi\Sigma\nu$ for a steady state Keplerian disk. Show that when $\Sigma\nu$ is constant (and independent of radius) the equation above gives a steady state solution. Are there other situations that would give a steady state solution?

The more general equation for angular momentum transport can be written

(5) $$\frac{\partial j}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(j u_R) = \frac{1}{R}\frac{\partial}{\partial R}(R^3 \nu \Sigma \frac{d\Omega}{dR})$$

where the angular momentum per unit area $j = R\Sigma u_\phi$. The second term on the left is the angular momentum flux caused by radial transport. Here we have not assumed Keplerian rotation (and consequently the equation contains a factor of $d\Omega/dr$). The term on the right hand side can be thought of as a torque per unit area $\frac{\partial j}{\partial t}$. With vanishing viscosity, $\nu = 0$, we expect that there is no angular momentum transport or $u_R = 0$.

(b) Find a condition on the disk that would lead to excretion rather than accretion. This means the radial velocity u_R is positive instead of negative.

Recently excretion disks have been considered for circumstellar disks externally truncated by photo-evaporation (proplyds) and for our solar system (work by Steve Desch). Bill Ward and Robin Canup have considered them in the context of a primordial circum-Jovian accretion disk.

(c) Consider the radial component of the Navier Stokes equation in steady state.

(6) $$u_R \frac{\partial u_R}{\partial R} + \frac{u_\phi}{R}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi^2}{R} = -\frac{1}{\Sigma}\frac{\partial p}{\partial R} - \frac{\partial \Phi}{\partial R}$$

We can neglect the viscous term because we expect the velocity shear only gives a strong viscous force in the ϕ direction, not in the radial direction. We do not expect strong dependence on ϕ so derivatives with respect to ϕ can be dropped. Show that in the limit of $u_\phi \gg u_R$

(7) $$u_\phi^2 = v_c^2 + c_s^2 \frac{R}{\Sigma}\frac{\partial \Sigma}{\partial R}$$

where c_s is the sound speed and v_c is the velocity of a particle in a circular orbit not affected by the gas (Keplerian velocity);

$$v_c \equiv \sqrt{R\frac{\partial \Phi}{\partial R}}.$$

The mean tangential velocity is slightly below that of the velocity of a particle in a circular orbit if the density drops with increasing radius.

4. Head winds in the minimum mass solar nebula

Many papers refer to a surface density that is called the "Minimum mass solar nebula." This is estimated from the masses and spacing of the 4 giant planets in our Solar system. The gas density

(8) $$\Sigma_{gas} = 2400 \left(\frac{R}{1 \text{ AU}}\right)^{-3/2} \text{g cm}^{-2}$$

The dust density

(9) $$\Sigma_{dust} = 10 \left(\frac{R}{1 \text{ AU}}\right)^{-3/2} \text{g cm}^{-2}$$

The density of solids (ices) is 3-4 times that of the dust. The above are by Hyashi, C. 1981, Prog. Theor. Physics Supp. 70, 35.

(a) Use equation 7 to estimate the difference between the circular velocity of a particle in a circular orbit about the Sun and the gas in the minimum mass solar nebula. Use a disk aspect ratio of $h/R = 0.1$ and give your velocity difference in units of the circular velocity.

(b) Using an exponential scale height and $h/R = 0.1$, estimate the gas density, ρ_{gas}, in the midplane at $R = 1$ AU.

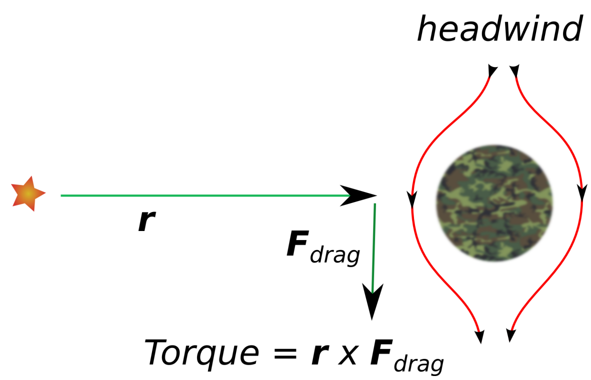

FIGURE 2. A planetesimal moving in a disk is moving faster than the ambient gas so it feels a headwind. The drag force caused by this headwind exerts a torque on the planetesimal causing it to spiral inwards.

The drag force on a planetesimal that is embedded in a gas disk

(10) $$F_D = \frac{1}{2} C_D \rho_{gas} \pi s^2 v^2$$

where C_D is a drag coefficient, ρ_{gas} is the gas density, v is the difference in velocity between the gas and the planetesimal and s is the radius of the planetesimal. (Note the drag force depends on the area of the object; here $A = \pi s^2$). A planetesimal may be orbiting at the Keplerian speed, however equation 7 implies that when the surface density of the gas drops with increasing radius then the gas is moving slower than the Keplerian speed. Consequently a planetesimal in a circular orbit would feel a headwind. This headwind would remove angular momentum from the planetesimal causing it to spiral inwards. The torque can be estimated from the drag force. The angular momentum of the planetesimal is mRv_c where m is the mass of the planetesimal, R is the radius from the star and v_c the Keplerian velocity.

(c) Show that a slowly in-spiraling planetesimal in a nearly circular orbit at radius r that is drifting inwards at a speed $\dot{R}$ looses angular momentum at a rate

$$\dot{L} \sim m\frac{v_c}{2}\dot{R} \tag{11}$$

where v_c is the Keplerian velocity of a particle in a circular orbit. Using the torque caused by the drag force show that

$$t_{inspiral} = \frac{R}{\dot{R}} \sim \left(\frac{\rho_d}{\rho_{gas}}\right)\frac{sv_c}{C_D v^2} \tag{12}$$

where ρ_d is the density of the planetesimal.

(d) Assuming drag coefficient $C_D \sim 1$ and planetesimal density $\rho_d \sim 1$ g cm^{-3} estimate a timescale in years for the inspiral $R/\dot{R}$ of a meter sized planetesimal in the minimum mass solar nebula at 1AU. Your timescale should be shorter than a thousand years, presenting a challenging problem for current planetesimal formation models.

5. Radial temperature profiles for accretion disks

Consider an accretion disk with an accretion rate sufficiently high that its thermal structure is due to energy viscously dissipated in the disk.

$$\nu\Sigma\Omega^2 \sim \sigma_{SB}T^4 \tag{13}$$

where ν is the viscosity,

$$\Omega = \sqrt{GM/R^3} \tag{14}$$

the angular rotation rate, M the mass the central object, Σ the mass surface density, σ_{SB} the Stefan-Boltzmann constant. The above temperature is that of the surface if the disk is optical thick, otherwise it is approximately the average temperature. The quantities Σ, ν, T, Ω can vary with radius R.

(a) If the disk is optically thin how would its temperature scale with radius? In other words T is proportional to R to what power? Assume a steady accretion rate independent of radius $\dot{M} \sim 3\pi\Sigma\nu$ and use equations 13, 14. (Do not assume a minimum mass Solar nebula.)

(b) If the disk is optically thick but the disk opacity κ is independent of temperature and density how would its mid-plane temperature scale with radius? Assume viscous forces dissipate energy in the disk mid-plane but the surface temperature is set by equation 13. Assume an α disk with α independent of radius and viscosity set by the properties of the gas in the midplane. Assume $\dot{M} \sim 3\pi\Sigma\nu$. The opacity of the disk relates the mid-plane and surface temperatures. It may be useful to remember that the sound speed $c_s \propto T^{1/2}$, how hydrostatic equilibrium relates c_s, h, Ω where h is a scale height and Ω is the angular rotation rate (equation 14) and how viscosity ν is defined for an α disk.