PROBLEM SET #2B **AST242**

1. Temperature Jumps at Shocks

The Rankine-Hugoniot conditions can be used to show that for an ideal gas with it adiabatic index γ

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \tag{1}$$

and

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \tag{2}$$

where the subscript 1 refers to upstream and subscript 2 refers to downstream. Here $M_1 = u_1/c_1$ is the Mach number of the pre-shock material and velocities are given in the shock's reference frame.

(a) Show that for strong ($M_1 \gg 1$) shock waves

$$\frac{T_2}{T_1} = \frac{(\gamma-1)p_2}{(\gamma+1)p_1} \tag{3}$$

While the density ratio approaches a limiting value with increasing Mach number, the pressure and temperature ratio can be arbitrarily large.

(b) Consider a shock passing into the interstellar medium of initial density $n_1 \sim 1$ cm^{-3} and temperature $T_1 = 10^4$K. What pre-shock speed (in the shock frame), u_1, is required to account for a post shock temperature of 10^7K? You can either do this to order of magnitude or assume $\gamma = 5/3$.

(c) Assume that the preshock gas has zero velocity with respect to the observer and that the shock normal is parallel to the line of sight. What is the post shock velocity in the observer frame?

2. Adiabatic shocks

For an adiabatic shock the Mach number downstream, M_2 can be written in terms of that upstream M_1

$$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)} \tag{4}$$

where we have assumed that the adiabatic index γ is the same on either side of the shock.

Show that if $M_1 > 1$ (and there is a shock) that $M_2 < 1$ and so that post-shock flow is subsonic in the shock frame. You may assume that $\gamma \geq 1$.

3. Blast wave estimates

A self-similar blast wave solution relates energy E, ambient density ρ, radius R, and time, t, with

$$R \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5} \tag{5}$$

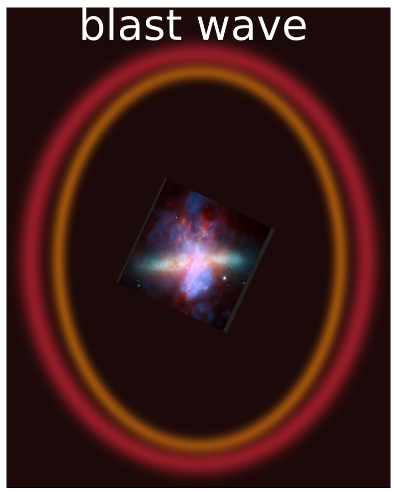

FIGURE 1. A hypothetical blast wave caused by a burst of star formation.

(a) Derive an approximate scaling relation for the blast wave's velocity as a function of time, E and ρ.

(b) Assume that the blast wave is made by a constant energy injection rate $\dot{E}$ rather than a single explosion of energy E. Derive a scaling relation (like the above one) for the blast wave's radius as a function of time in terms of $\dot{E}$ and ρ.

(c) For the constant $\dot{E}$ driven blast wave, derive a scaling relation for the blast wave's velocity as a function of time and $\dot{E}$.

In the center of a nearby elliptical galaxy you detect a spherical shell that has radius 500pc and is expanding at 200 km/s. You estimate the number density of the ambient ISM (outside the blast wave) in the galaxy from its X-ray emission as $n \sim 10^{-3}\text{cm}^{-3}$. The shell is probably too big to be caused by a single supernova. You consider the possibility that many supernovae have contributed to its energetics.

(d) Assuming a constant $\dot{E}$, how much energy per unit time ($\dot{E}$) in erg/s is needed to account for the shell?

(e) Assume that there is an on-going starburst at the center of the galaxy. Not all stars that are born go supernova. About 1 per 100 stars born goes supernovae. Each supernovae has 10^{51} ergs but of this energy only about 1% might be injected into the galaxy ISM because much is lost as radiation. Assume a constant star formation rate and that the average star has mass of 1 $M_\odot$. How much kinetic energy per unit time in ergs/s (this is $\dot{E}$) is injected with a star formation rate of 1 solar mass per year? You need to find a coefficient, X, such that

$$\dot{E} = X \text{erg s}^{-1} \times \left(\frac{SFR}{M_\odot \text{yr}^{-1}} \right) \tag{6}$$

where SFR is the star formation rate in solar masses per year. Approximately, what star formation rate in solar masses per year would be required to explain the shell in the context of a constant $\dot{E}$ blast wave model?

(f) Using its velocity and radius, estimate the age of the shell. This would give you an estimate for how long energy could have been injected in the medium.

4. Relativistic Jump Conditions

If a fluid is relativistic then energy and momentum density are dependent on the frame of the observer.

Consider a frame moving with the fluid. This frame we define with a four-vector $\mathbf{u}$. In a frame moving with the fluid $\mathbf{u} = (1, 0, 0, 0)$ with $u^0 = \frac{dt}{d\tau}$ giving how time advances in this frame. Here the zero-th coordinate is the time coordinate and the 1-3 coordinates are spatial coordinates x, y, z. In another (boosted) coordinate frame we could write $\mathbf{u} = (\gamma, \gamma\boldsymbol{\beta})$, where $\boldsymbol{\beta} = \frac{\mathbf{v}}{c}$ the boost velocity vector, c is the speed of light and the *Lorenz factor* $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. Here γ is a Lorenz factor not an adiabatic index.

A Lorenz boost transforms four-vectors

$$\boldsymbol{\Lambda}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & (\gamma-1)\frac{\beta_1^2}{\beta^2}+1 & (\gamma-1)\frac{\beta_1\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & (\gamma-1)\frac{\beta_2\beta_1}{\beta^2} & (\gamma-1)\frac{\beta_2^2}{\beta^2}+1 & (\gamma-1)\frac{\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & (\gamma-1)\frac{\beta_3\beta_1}{\beta^2} & (\gamma-1)\frac{\beta_3\beta_2}{\beta^2} & (\gamma-1)\frac{\beta_3^2}{\beta^2}+1 \end{pmatrix} \tag{7}$$

from one relativistic coordinate frame to another.

In the frame moving with the fluid the **energy momentum tensor** is

$$\mathbf{T} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{8}$$

with p the pressure and e the energy density. Ignoring internal degrees of freedom and photons $e = \rho c^2$. The T_{00} component gives the energy density, and the other diagonal components, T_{xx}, T_{yy}, T_{zz}, the momentum density or the pressure.

In another coordinate system, the stress tensor is more generally

$$\mathbf{T} = (e+p)\mathbf{u} \otimes \mathbf{u} + p\mathbf{g} \tag{9}$$

with $\mathbf{g}$ the metric tensor with $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$ and $g_{ij,i\neq j} = 0$. Recall that $\mathbf{u}$ is the flow four vector of the fluid. Using indices of the four vector $\mathbf{u}$

$$T^{ij} = (e+p)u^i u^j + pg^{ij}. \tag{10}$$

(a) Insert $\mathbf{u} = (1,0,0,0)$ into equation 9 to show that in the coordinate system moving with the fluid the stress energy tensor looks like that in equation 8.

(b) Taking into account only one spatial direction

$$\boldsymbol{\Lambda}(\beta) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \tag{11}$$

and in the frame of the fluid

$$\mathbf{T} = \begin{pmatrix} e & 0 \\ 0 & p \end{pmatrix} \tag{12}$$

Each index of a tensor transfers separately during a coordinate transformation. Here the coordinate transformation is the Lorenz boost.

$$\tilde{T}_{ij} = T_{lk}\Lambda_i^k \Lambda_j^l \tag{13}$$

Show using equation 12, 13, and the above boost (equation 11) that after a Lorenz transformation (but one for each index) **T** is equivalent to that given by equation 9 (but using one spatial dimension so the calculation is simpler).

It may be useful to show that $\gamma^2 - 1 = \gamma^2\beta^2$.

(c) Conservation of energy and momentum is equivalent to

$$T^{ij}_{\ ,j} = 0 \tag{14}$$

or

$$\frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{i1}}{\partial x_1} + \frac{\partial T^{i2}}{\partial x_2} + \frac{\partial T^{i3}}{\partial x_3} = 0$$

or

$$\frac{\partial T^{it}}{\partial t} + \frac{\partial T^{ix}}{\partial x} + \frac{\partial T^{iy}}{\partial y} + \frac{\partial T^{iz}}{\partial z} = 0$$

This is a conservation law (more like 4 conservation laws, one for each dimension).

In the frame of a shock, show that the jump conditions (for energy and momentum density) are

$$(e_1 + p_1)\gamma_1^2\beta_1 = (e_2 + p_2)\gamma_2^2\beta_2 \tag{15}$$

$$(e_1 + p_1)\gamma_1^2\beta_1^2 + p_1 = (e_2 + p_2)\gamma_2^2\beta_2^2 + p_2 \tag{16}$$

where subscripts refer to quantities on either side of the discontinuity.

(d) An **ultra relativistic shock** is one with large pre-shock Lorenz factor $\gamma_1 \gg 1$, corresponding to relativistic pre-shock velocity $\beta_1 \sim 1$ (in the shock frame). In this limit p_1 can be neglected on the left hand side of equation 16 giving

$$(e_1 + p_1)\gamma_1^2 = (e_2 + p_2)\gamma_2^2\beta_2^2 + p_2 \tag{17}$$

The post shock gas would be so hot it would act like a relativistic fluid with $p_2 \sim e_2/3$. This is like saying the particles are moving so fast that their pressure is set by the speed of light and $p \sim \frac{1}{3}\rho c^2$.

Show that in this ultra relativistic limit equations 15,17 imply that

$$\beta_2 \sim 1/3 \tag{18}$$

5. Computing the speed of a discontinuity in one dimension

Consider a non-linear equation

$$u_{,t} + 3u^2 u_{,x} = 0$$

We have an initial condition $u = 1$ for $x < 0$ and $u = 1/2$ for $x > 0$.

At what speed does the discontinuity propagate?

6. Components of the velocity gradient tensor

Consider the velocity gradient tensor $T_{ij} = \frac{\partial u_i}{\partial x_j}$ in Cartesian coordinates with $\mathbf{u}$ the velocity.

(a) Construct and draw streamlines for a velocity field such that the trace of $\mathbf{T}$ is non-zero but the antisymmetric and trace-less symmetric parts are zero.

(b) Construct and draw streamlines for a velocity field such that the symmetric components of $\mathbf{T}$ are zero but the antisymmetric component is not.

(c) Construct and draw streamlines for a velocity field such that only the traceless symmetric component of $\mathbf{T}$ is non-zero.

Consider the traceless symmetric tensor $\sigma_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) - \frac{1}{3}u_{k,k}\delta_{ij}$ where $u_{k,k}$ is the trace of $\mathbf{T}$.

(d) Compute σ_{xy} for fluid flow near a fixed surface with velocity $\mathbf{u} = (ay, 0, 0)$ that approaches zero near a surface at $y = 0$.

(e) Compute σ_{xy} for rotation in a Keplerian disk with $\mathbf{u} = v_c(r)(-\frac{y}{r}, \frac{x}{r}, 0)$ and $v_c(r) = \sqrt{\frac{GM}{r}}$. Here the radius $r = \sqrt{x^2 + y^2 + z^2}$.