PROBLEM SET #2 **AST242**

1. The Rocket nozzle

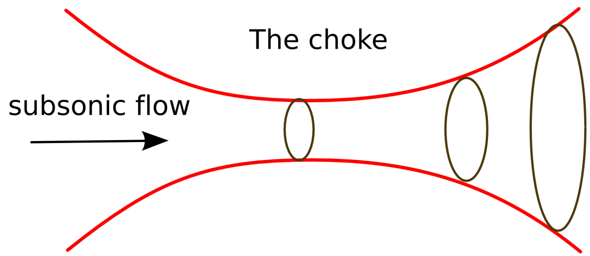

FIGURE 1. Flow through a nozzle can make the transition to supersonic flow where the cross sectional area of the nozzle is a minimum. Here $S(x)$ is the cross sectional area of the nozzle as a function of position along the flow.

In a model of a rocket, a polytropic gas of index $\gamma = 2$ flows steadily and adiabatically with velocity $u(x)$ along a smooth nozzle. The nozzle has slowly varying cross sectional area $S(x)$, where x measures the distance along the nozzle. At $x = 0$ the area is large so the flow is subsonic with sound speed c_0. We assume a steady state flow with density, pressure and sound speed functions only of position along the nozzle or $\rho(x), p(x), c_s(x)$.

(a) Find a form for conservation of mass that involves ρ, S, and u. What is $\dot{M}$ in terms of these quantities?

(b) Using Bernoulli's equation show that the velocity $u(x)$ and sound speed, $c_s(x)$ are related as follows

$$\frac{u^2}{2} + c_s^2 = c_0^2 \tag{1}$$

Assume that the flow is very slow, $u \ll c_0$, at the entrance of the nozzle.

(c) Show that Euler's equation implies that

$$u u_{,x} + c_s^2 \frac{\rho_{,x}}{\rho} = 0 \tag{2}$$

(d) Using the above form of Euler's equation, a differential form for conservation of mass and the equation of state show that

$$\frac{u_{,x}}{u}\left[u^2 - c_s^2\right] = c_s^2 \frac{S_{,x}}{S} \tag{3}$$

and so that the sonic transition point (where $u = c_s$) must occur at the point where the nozzle has minimum area.

(e) Assume that the nozzle has a miminum area S_{min} at x_{min} and then increases afterwards. Show that

$$u(x_{min}) = c_0\sqrt{2/3} \tag{4}$$

for the transonic solution.

(f) Show that the velocity leaving the nozzle is accelerated to the supersonic velocity

$$\lim_{x\to\infty} u(x) = c_0\sqrt{2} \tag{5}$$

for the transonic solution.

Sketch the velocity, sound speed and pressure for the transonic solution paying attention to the sonic point.

Based on problem 3.4.1 from Pringle & King's book "Astrophysical flows"

2. Bondi flow in supersonic limit.

(a) Using Bernoulli's equation

$$\frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} \tag{6}$$

find an expression for u as a function of r in the supersonic limit $u \gg c_s$. Using mass conservation

$$\dot{M} = 4\pi\rho u r^2 \tag{7}$$

how does ρ depend on radius? The equation for ρ can depend on $\dot{M}$.

(b) What is the velocity of a particle as a function of r that started from rest at large radius and is falling freely towards a point mass?

(c) Interpret your result in part a in terms of the velocity you find in part b.

3. Bondi accretion rates

The Galactic center harbors a Black Hole of size about a million solar masses.

(a) Estimate the accretion rate (Bondi accretion rate) in solar masses per year, and in g/s, if the ambient gas has density $n \sim 1$ cm^{-3} and a temperature of $T \sim 10^7$K appropriate for an X-ray emitting gas.

(b) Assume a luminosity $L = \epsilon \dot{M} c^2$ with an efficiency $\epsilon = 0.01$. What luminosity, in erg/s, and in units of the Sun's luminosity, does this accretion rate correspond to?

(c) Giant stars with luminosities of $\sim 10^3 L_\odot$ are detectable at the Galactic center. They are observed in the near infrared wavelength bands as light at optical wavelengths does not penetrate through the dust in the Galactic plane. How does your estimated Bondi accretion rate compare with the luminosities of giant stars?

Note: the black hole at the Galactic center and many elliptical galaxies have lower luminosity than would be predicted with Bondi accretion. Accreting flows may be poor radiators or accretion may be prevented by heating and outflows associated with previous episodes of activity.

4. Characteristics and Burger's equation

Burger's equation (lacking viscosity) is

$$u_{,t} + u u_{,x} = 0 \tag{8}$$

Consider initial condition

$$u(x, t = 0) = u_0 + A\cos(kx)$$

and $A < u_0$.

(a) Using characteristics, estimate the time at which the velocity becomes so steep that a shock will develop.

Now consider a new differential equation with constants c, d

$$u_{,t} + c(u^2 + d)u_{,x} = 0 \tag{9}$$

but with the same initial condition $u(x, t = 0) = u_0 + A\cos(kx)$.

(b) Estimate the time at which a shock develops but for the new equation. Is the time dependent on d and if not why not? What are the units of constants c, d?

5. An example using the Method of Characteristics

Consider the equation

$$u_{,t} + x u_{,x} = 0 \tag{10}$$

with initial condition

$$u(x, t = 0) = \exp(-x^2) \tag{11}$$

(a) Solve the characteristic equations. On a plot of x vs t plot the characteristics.

(b) Use the method of characteristics to find $u(x,t)$ for $t > 0$. Discuss your solution in terms of characteristics. How does u change along them?

Now consider the equation

$$u_{,t} + x u_{,x} = u \tag{12}$$

with the same initial condition

$$u(x, t = 0) = \exp(-x^2)$$

(c) Use the method of characteristics to find $u(x,t)$ for $t > 0$.

(d) Describe how u changes along the characteristics. Compare the two different solutions, that for equation 10 and that for equation 12.

The Initial condition is a Gaussian profile. At later times is the profile a wider or narrower Gaussian? How does the peak density depend on time?

6. Linearized version of isentropic fluid flow in one dimension

Isentropic fluid flow in one dimension obeys the evolution equation

$$\mathbf{y}_{,t} + \mathbf{A}(\mathbf{y})\mathbf{y}_{,x} = 0 \tag{13}$$

where the state vector

$$\mathbf{y} = \begin{pmatrix} \rho \\ j \end{pmatrix}$$

Here ρ is the linear density and $j = \rho u$ is the mass flux, and u is the velocity. The matrix

$$\mathbf{A}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ c_s^2 - u^2 & 2u \end{pmatrix} \tag{14}$$

and c_s is the sound speed.

Let consider small perturbations about a flow with velocity u and c_s. Working in units of the sound speed, we can make a linear approximation with

$$\mathbf{y}_{,t} + \bar{\mathbf{A}}\mathbf{y}_{,x} = 0 \tag{15}$$

and

$$\bar{\mathbf{A}} = \begin{pmatrix} 0 & 1 \\ 1 - M^2 & 2M \end{pmatrix} \tag{16}$$

and the Mach number $M = u/c_s$ a constant of the bulk flow. Here we have assumed that the derivatives of $\mathbf{y}$ are very small.

The eigenvectors and eigenvalues of $\bar{\mathbf{A}}$ are

$$\mathbf{y}_+ = \begin{pmatrix} 1 \\ M+1 \end{pmatrix} \qquad M+1$$
$$\mathbf{y}_- = \begin{pmatrix} 1 \\ M-1 \end{pmatrix} \qquad M-1$$

The right shows the characteristic velocities $1 + M, 1 - M$.

We assume an initial condition at $t = 0$

$$\mathbf{y}(x, t = 0) = \mathbf{y}_L = \begin{pmatrix} 1 \\ M+\delta \end{pmatrix} \quad \text{for} \quad x < 0$$
$$\mathbf{y}(x, t = 0) = \mathbf{y}_R = \begin{pmatrix} 1 \\ M-\delta \end{pmatrix} \quad \text{for} \quad x > 0$$

(a) Find the coefficients a_L, b_L, a_R, b_R such that

$$\mathbf{y}_L = a_L\mathbf{y}_+ + b_L\mathbf{y}_-$$
$$\mathbf{y}_R = a_R\mathbf{y}_+ + b_R\mathbf{y}_-$$

This is writing our initial state vector in terms of our two eigenvectors.

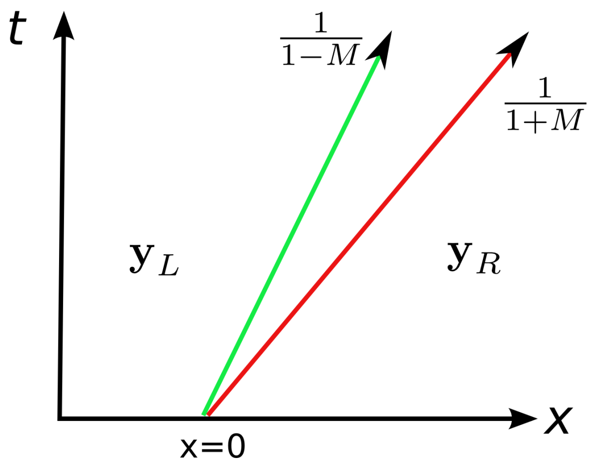

FIGURE 2. Two characteristic slopes.

(b) Find the solution at all later times. Using Figure 2 divide up the solution into three regions and each one should have its own constant solution.