PROBLEM SET #1B **AST242**

1. Validity of continuum or fluid approximation in a hydrostatic planetary atmosphere

Consider the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \mathbf{g} \tag{1}$$

with an additional force from $\mathbf{g}$, the gravitational acceleration. In *hydrostatic equilibrium* we can assume that the velocity $\mathbf{u} = 0$ and remains that way.

(a) Assume a planet's atmosphere is isothermal and has temperature T and sound speed c_s. Show that a solution for the density as a function of height above the surface z is $\rho \propto \exp(-z/h)$ and find an expression for the scale height h in terms of the gas temperature.

(b) The velocity, v_K, of an object in circular orbit of radius r around a planet is

$$v_K = \sqrt{\frac{GM_p}{r}}$$

where M_p is the planet's mass and G the gravitational constant. Compare this to the sound speed of the gas. For what sound speed would the atmosphere scale height be the same order as the planet's radius, R_p? For what scale height (in units of the planet's radius) is the mean thermal velocity equal to or below the escape velocity? When the scale height is of order R_p or greater, a constant gravitational acceleration is a bad assumption.

(c) High above a planet's surface, the atmosphere becomes more and more rarified. Assume a density at the planet's surface of ρ_0, the atmosphere is comprised of molecules of mass m and a collision cross section for the molecules of σ. At what height above the planet's surface does a fluid approximation fail? (Where the mean free path is greater than the scale height). When the fluid approximate fails, equilibrium is no longer maintained by collisions. This height is called the *exobase*.

This problem is based on one posted by Eugene Chiang.

2. Destabilizing influence of radiation pressure.

The most massive stars that can form are those in which radiation pressure and the non-relativistic kinetic pressure are approximately equal. The total pressure is the sum of that from gas and radiation,

$$P = \frac{k_B \rho T}{\mu m_p} + \frac{4\sigma_{SB} T^4}{3c} \tag{2}$$

where k_B is Boltzmann's constant, σ_{SB} is the Stefan-Boltzmann constant, c, the speed of light, m_p the mass of a proton, μ the mean molecular weight, ρ the gas density, and T the temperature.

Assume the the gravitational binding energy of a star of mass M and radius R is

$$E_g \sim GM^2/R$$

The virial theorem (see the problems below) can be used to relate the mean pressure to the gravitational binding energy

$$\bar{P} = \frac{E_g}{3V} \tag{3}$$

where V is the volume of the star. The gravitational binding energy can be integrated as

$$E_g = \int_0^R 4\pi r^2 \rho \Phi(r) dr$$

where $\Phi(r)$ is the gravitational potential at r. The mean pressure is also an integral

$$\bar{P} = \frac{1}{V} \int_0^R 4\pi r^2 P(r) dr$$

(a) Using an equation for hydrostatic equilibrium and the integral expressions for E_g and $\bar{P}$, show that equation 3 is approximately correct.

-To do this first show that $\int 4\pi r^3 \frac{dP}{dr} dr = -3\bar{P}V$. Then use hydrostatic equilibrium to write the same integral in terms of E_g. It may be useful to know that $\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2}$ and $\Phi(r) = \frac{GM(r)}{r}$ where $M(r)$ is the mass inside radius r.

(b) Using relation 3 for mean pressure show that

$$\bar{P} \sim GM^{2/3} \rho^{4/3} \tag{4}$$

(Remove V, R from the equation by replacing with expressions that depend on M, ρ)

(c) Assume that the radiation pressure is approximately equal to the gas kinetic pressure (as is true in very bright stars). Making this assumption solve for T (using equation 2). Show that the total pressure is then

$$P \sim \left(\frac{\sigma_{SB}}{c}\right)^{-1/3} \left(\frac{k_B \rho}{\mu m_p}\right)^{4/3} \tag{5}$$

(d) Equate the two expressions for pressure (equation 4 and 5) and solve for mass in solar masses assuming a fully ionized hydrogen composition. Put your mass in solar masses! This gives a very approximate estimate for the maximum mass of a bright star.

Based on a problem in *Astrophysics in a Nutshell* by Dani Maoz but without some of the factors.

3. The Potential Energy Tensor

The gravitational potential, $\Phi(\mathbf{x})$ generated by a mass distribution with density $\rho(\mathbf{x})$ is

$$\Phi(\mathbf{x}) = \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \tag{6}$$

We define a **potential energy tensor** (attributed to Chandrasekkar)

$$W_{ij} \equiv -\int \rho(\mathbf{x}) x_i \frac{\partial \Phi}{\partial x_j} d^3\mathbf{x} \tag{7}$$

Apparently, this integral has been called *the virial* by R. Clausius.

(a) Show that the potential energy tensor can be written

$$W_{ij} = -\frac{G}{2} \iint \rho(\mathbf{x})\rho(\mathbf{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{x}-\mathbf{x}'|^3} d^3\mathbf{x} d^3\mathbf{x}'$$

It is a symmetric tensor.

Hint: After writing W_{ij} as a double integral, swap indices of integration and add to itself.

(b) Show that

$$\text{trace } \mathbf{W} = W_{ii} = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x}) d^3\mathbf{x}$$

is the total potential energy integrated over all space.

(c) Show that if the mass distribution is spherically symmetric, then W_{ij} contains terms only on the diagonal and all three diagonal terms are identical.

4. The Tensor Virial Equations

Consider a distribution of stars in phase space with function $f(\mathbf{x}, \mathbf{v}, t)$ describing numbers of stars per unit volume $d\mathbf{x}^3$ and per unit velocity volume $d^3\mathbf{v}$. The number of stars per unit volume $n(\mathbf{x}, t)$ is the integral of the distribution function over velocity space. By integrating the distribution function times v_i over velocity space and then dividing by n, we can define a mean velocity $u_i \equiv n^{-1} \int f(\mathbf{x}, \mathbf{v}, t) v_i d^3\mathbf{v}$. We can integrate $f(\mathbf{x}, \mathbf{v}, t)(v_i - u_i)(v_j - u_j)$ over velocity space giving us the velocity dispersion tensor. It is useful to define a density $\rho = mn$ where m is the average mass of the stars. The average value of the x, y, z velocity dispersions or $1/3$ the trace of the velocity dispersion tensor w_{ij} is σ_a and we can define a pressure as $P = \rho\sigma_a^2$. We can also subtract the trace from the velocity dispersion tensor giving us the traceless component y_{ij} that describes the anisotropy of the velocity distribution.

By multiplying the Collisionless Boltzmann equation by a velocity component v_i and integrating over velocity space we find

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + P\delta_{ij} + \rho y_{ij}) + \rho \frac{\partial \Phi}{\partial x_i} = 0 \tag{8}$$

In this problem we will integrate the above equation over space to derive a tensor virial theorem.

Recall

$$\langle v_i v_j \rangle = n^{-1} \int f(\mathbf{x}, \mathbf{v}, t) v_i v_j d^3\mathbf{v} = w_{ij} + u_i u_j$$

where the velocity dispersion tensor

$$w_{ij} = \langle (v_i - u_i)(v_j - u_j) \rangle = \frac{1}{\rho} P\delta_{ij} + y_{ij}$$

using the definition for P.

Define a **kinetic energy tensor K** as

$$K_{ij} \equiv \int \frac{1}{2}\rho \langle v_i v_j \rangle d^3\mathbf{x} = \int \frac{1}{2}\rho(w_{ij} + u_i u_j) d^3\mathbf{x}$$

We can divide the kinetic energy tensor into two parts, one dependent on the integral of mean velocities and other on the integral of the velocity dispersion tensor

$$K_{ij} = T_{ij} + \frac{1}{2}\Pi_{ij}$$

with

$$\begin{aligned} T_{ij} &= \int \frac{1}{2}\rho u_i u_j d^3\mathbf{x} \\ \Pi_{ij} &= \int \rho w_{ij} d^3\mathbf{x} \end{aligned}$$

The tensor $\mathbf{T}$ can be associated with the kinetic energy in bulk motions like rotation, whereas $\mathbf{\Pi}$ is associated with the kinetic energy in random motions.

Define a **moment of inertia tensor**

$$I_{ij} \equiv \int \rho(\mathbf{x}) x_i x_j d^3\mathbf{x} \tag{9}$$

In the previous problem we defined a **potential energy tensor**

$$\begin{aligned} W_{ij} &\equiv -\int \rho x_i \frac{\partial \Phi}{\partial x_j} d^3\mathbf{x} \\ &= -\frac{G}{2} \iint \rho(\mathbf{x})\rho(\mathbf{x}') \frac{(x_i - x'_i)(x_j - x'_j)}{|\mathbf{x}-\mathbf{x}'|^3} d^3\mathbf{x} d^3\mathbf{x}' \end{aligned}$$

We take equation 8 and multiply by x_k and then integrate over all space

$$\int d^3\mathbf{x}\ x_k \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + P\delta_{ij} + \rho y_{ij}) + \rho \frac{\partial \Phi}{\partial x_i} \right] = 0 \tag{10}$$

The kinetic energy, dispersion, moment of inertia and potential energy tensors are all symmetric. We switch the indices of equation 10

$$\int d^3\mathbf{x}\ x_i \left[\frac{\partial}{\partial t}(\rho u_k) + \frac{\partial}{\partial x_j}(\rho u_k u_j + P\delta_{kj} + \rho y_{kj}) + \rho \frac{\partial \Phi}{\partial x_k} \right] = 0$$

and then average the two equations

$$\begin{aligned} \frac{1}{2} \int d^3\mathbf{x} \Bigg[& x_i \frac{\partial}{\partial t}(\rho u_k) + x_k \frac{\partial}{\partial t}(\rho u_i) + \\ & x_k \frac{\partial}{\partial x_j}(\rho u_i u_j + P\delta_{ij} + \rho y_{ij}) + x_i \frac{\partial}{\partial x_j}(\rho u_k u_j + P\delta_{kj} + \rho y_{kj}) + \\ & x_k \rho \frac{\partial \Phi}{\partial x_i} + x_i \rho \frac{\partial \Phi}{\partial x_k} \Bigg] \quad = 0 \end{aligned} \tag{11}$$

(a) Show that

$$\frac{1}{2} \int \left[x_k \frac{\partial}{\partial x_j} \left(\rho u_i u_j + P\delta_{ij} + \rho y_{ij} \right) + x_i \frac{\partial}{\partial x_j} \left(\rho u_k u_j + P\delta_{kj} + \rho y_{kj} \right) \right] d^3\mathbf{x} = -2T_{ik} - \Pi_{ik}$$

This is the middle term in equation 11.

(b) Check that the two last terms in equation 11 (on the last line) give $-W_{ik}$.

(c) Using the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

show that

$$\frac{dI_{ik}}{dt} = \int (x_i u_k + x_k u_i) \rho d^3\mathbf{x}$$

and

$$\frac{1}{2}\int d^3\mathbf{x}\ \left(x_k \frac{\partial(\rho u_i)}{\partial t} + x_i \frac{\partial(\rho u_k)}{\partial t} \right) = \frac{1}{2}\frac{d^2 I_{ik}}{dt^2}$$

(d) Put these all these terms together to show that

$$\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = 2T_{ij} + \Pi_{ij} + W_{ij} \tag{12}$$

Equation 12 is known as the **Tensor Virial Theorem**. It has been used to study the relationship between velocity dispersion and anisotropy (contributing to $\mathbf{\Pi}$), rotational support (contributing to the bulk motions, $\mathbf{T}$), and the observed ellipticity of the isophotes (affective the potential energy tensor, $\mathbf{W}$) in elliptical galaxies and galactic bulges.

The total kinetic energy, K, of the system can be computed from the diagonal terms of the kinetic energy tensor $\mathbf{K}$ (this is because the total kinetic energy only involves $\rho\mathbf{v}^2$).

$$\text{trace } \mathbf{K} = \text{trace } \mathbf{T} + \frac{1}{2}\text{trace } \mathbf{\Pi}$$

Taking the trace of the tensor virial theorem and assuming that the system is in steady state (or equilibrium) so $\frac{d^2\mathbf{I}}{dt^2} = 0$ we find

$$2K + W = 0, \tag{13}$$

where W is the total potential energy and equivalent to trace $\mathbf{W}$.

Equation 13 is known as the **scalar virial theorem**.