1. Eulerian and Lagrangian view points with Traffic flow

Consider traffic flow described by a density of cars per unit length of one side of a divided highway, ρ, (numbers of cars per kilometer) and velocity of cars, u, (kilometers per hour). Let x describe distance along a road. For $x < 0$ the speed limit is u_a and assume that the cars are driving at this speed limit. At $x = 0$ there is a sign letting the drivers know that there is a change in speed limit. They begin to decelerate to the new speed limit, u_b. At $x = L$ the cars have reached the new speed limit u_b.

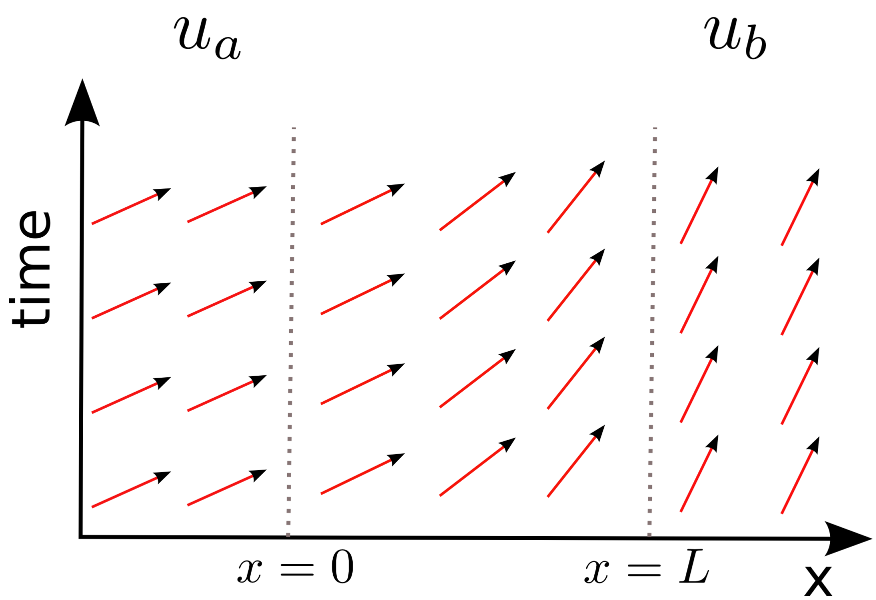

FIGURE 1. Note the x axis is space and the y axis is time. On this figure an arrow pointing upwards corresponds to a low velocity, and an array pointing nearly horizontally corresponds to a high velocity. Car trajectories $x(t)$ connect the arrows. Even though the velocity between $x = 0$ and L decreases linearly between u_a and u_b, the trajectory of a car in the flow, $x(t)$, are not linear.

Assume that the car speed is described by

$$u(x) = \begin{cases} u_a \\ u_a + (u_b - u_a)\frac{x}{L} \\ u_b \end{cases} \qquad \text{for} \qquad \begin{cases} x < 0 \\ 0 < x < L \\ x > L \end{cases}$$

The Eulerian view point describes the flow in terms of $\rho(x)$, $u(x)$. This flow is steady-state so ρ and u are only functions of x (and not also of t). The Lagrangian viewpoint concerns the velocity and density from the view point of the driver, or moving with the flow.

(a) What is $\rho(x)$ if the density of cars is ρ_a for $x < 0$?

(b) What is the position of a car as a function of time that is at $x = 0$ at time $t = 0$? Hint: $dx/dt = u(x)$, solve for $x(t)$.

(c) What acceleration does a car in the traffic flow experience as a function of position x in the flow? Compute this using a Lagrangian derivative.

(d) Check that your answer in part b is consistent with that in part c.

2. Non-linearity of the Lagrangian derivative

Consider again the setting of traffic flow with the linear density of cars $\rho(x,t)$ and the velocity of cars $u(x,t)$ as seen from an outside observer. Assume that the velocity of the cars is a traveling wave with wave vector k and angular frequency ω

$$u(x,t) = u_0\left(1 + A\sin(kx - \omega t)\right)$$

and with unit less amplitude $A < 1$ so that the cars are always traveling in one direction down the road. Unlike the previous problem, this is not a steady-state flow.

(a) Compute the instantaneous acceleration Du/Dt of a car at position x and time t.

(b) In the limit of small A qualitatively explain why the acceleration is not exactly 90° out of phase with the velocity.

(c) Qualitatively explain why the acceleration contains a second order term proportional to A^2.

3. Practice Index Gymnastics

(a) Use summation notation and the permutation tensor to show the vector identity is true

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

(b) Show that

$$\det(\mathbf{I} + \delta\mathbf{A}) \sim 1 + \delta\ \mathrm{tr}\mathbf{A}$$

for small δ (to first order in δ) where $\mathbf{I}$ is the identity matrix and matrix $\mathbf{A}$. It may be useful to write the determinant of an $N \times N$ matrix in terms of permutation tensors

$$\det \mathbf{A} = \frac{1}{N!}\epsilon_{i_1,i_2,i_3....}\epsilon_{j_1,j_2,j_3,...}A^{i_1j_1}A^{i_2j_2}A^{i_3j_3}...$$

4. Streamlines

Assume that the density at time zero is constant, ρ_0, for the following flows.

(a) Draw streamlines for a velocity field with non-zero trace $\mathbf{u} = (ax, by, 0)$ where a, b are constants and in Cartesian coordinates. Consider the case with $a, b > 0$. What happens if $a > 0, b < 0$?

Using an equation describing conservation of mass, how does the density change in time?

(b) Draw streamlines for the velocity field $\mathbf{u} = (-y, x, 0)$. How does the density change in time?

(c) Draw streamlines for the velocity field near a fixed surface $\mathbf{u} = (ay, 0, 0)$. The velocity at $y = 0$ is zero and is everywhere in the x direction. You might have such a velocity field near a river bank where the bank is at $y = 0$ and the river is flowing in the x direction. How does the density change in time?

(d) Compute the vorticity of the previous velocity fields $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$. When the vorticity is non-zero we say the flow is rotational. Which of the above velocity fields are rotational?

5. Conservation Law form for conservation of momentum

Our relation for conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$$

can be said to be in conservation law form. The term $\rho\mathbf{u}$ is the mass flux. However Euler's equation describing conservation of momentum

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p$$

is not in conservation law form. Conservation of momentum can be put in conservation law form using the stress tensor $\boldsymbol{\pi}$,

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\pi} = 0$$

Using Euler's equation and that for conservation of mass to show that conservation of momentum can be put in conservation law form.

6. Ram pressure and Drag

Consider a thin slightly stretchy wire with diameter d and linear mass density λ held at each end by a pole (like a power cable strung between two telephone poles). Because of gravity, the wire sags and is bowed. Assume that a strong wind is blowing at a speed u with respect to the cable. The density of air we denote ρ_{air} and the gravitational acceleration, g.

(a) Using ram pressure, estimate the force per unit area on the wire due to the wind.

(b) Estimate how fast the wind needs to blow to overcome sagging due to gravity.

This problem need only be done to order of magnitude. Don't worry about the vector direction of the wind.