AST242 LECTURE NOTES PART 9 – INFLATION AND PERTURBATIONS

Contents

1. Inflation

In the previous lecture we discussed the horizon problem. Here we now discuss the flatness problem. The inflation scenario presents solutions to these problems.

1.1. Flatness problem.

From the previous lecture we found that

$$1 = \Omega + \Omega_K \tag{1}$$

or

$$1 - \Omega = -\frac{K}{(aH)^2} \tag{2}$$

where we have used the definition for Ω_K. Here Ω contains matter, radiation and possibly a cosmological constant. Note $\Omega = 1$ requires $K = 0$ and Ω stays fixed. Suppose $K \neq 0$.
If the universe is matter dominated, then $a \propto t^{2/3}$, $\dot{a} \propto t^{-1/3}$ and $H = \dot{a}/a \propto t^{-1}$.
If the universe is radiation dominated, then $a \propto t^{1/2}$, $\dot{a} \propto t^{-1/2}$ and $H = \dot{a}/a \propto t^{-1}$.

For the matter dominated case $(aH)^{-2} \propto t^{2/3}$. For the radiation dominated case $(aH)^{-2} \propto t$. In both cases Ω_K increases with t and so rapidly dominates over all other forms of matter. If the initial spatial curvature is positive then Ω_K will get so large that expansion will stop and the universe will collapse (closed solutions). If the initial spatial curvature is negative then the expansion rate will increase, the solution will be increasingly open and empty and cold. We are currently in a nearly flat universe. But if $K \neq 0$, the nearly flat time is only short lived.

Another way to look at this problem is to ask: What size must Ω be at a previous time that would allow our current universe to be nearly flat now. This requires fine tuning and so seems less likely.

This issue is known as the **flatness** problem. It is solved by a rapid inflation period that makes the universe increasingly flat. In an expansion phase a increases exponentially and so $|\Omega - 1|$ is rapidly driven towards zero. Inflation does not require the universe to be flat prior to inflation, but does make it very flat afterwards.

1.2. **The Inflaton field.** Field equation for matter and radiation, and Einstein's field equation can be derived by minimizing an action

$$S = \int \sqrt{-g}\mathcal{L} d^4x, \tag{3}$$

where g is the determinant of the metric tensor. The product $\sqrt{-g}d^4x$ is coordinate independent. Since $\mathcal{L}$ is a scalar, so is the action.

Inflation considers the evolution of the universe including a field. Consider a scalar field that we call the *inflaton field* with Lagrangian

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi), \tag{4}$$

where the sign on the first term is set by our convention that the dt^2 part of the metric is negative. Here $\frac{\partial \varphi}{\partial x_\mu} = \varphi_{,\mu} = \partial_\mu \varphi$ are different notation for the same thing. If the field is 'free' and massive (with mass m) then the gravitational potential

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2. \tag{5}$$

By minimizing the action or using the Euler Lagrange equations the Lagrangian gives the Klein Gordon equation. The Klein Gordon equation arises from the infinite limit of low amplitude motions for a 1 dimensional array of pendulums connected by springs.

Minimizing the action gives the Euler Lagrange equations which here are

$$\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\varphi} - \partial_\mu \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial[\partial_\mu\varphi]}\right] = 0, \tag{6}$$

and this gives

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) - V'(\varphi) = 0 \tag{7}$$

where $V'(\varphi) = \partial V(\varphi)/\partial\varphi$. For flat space time (Minkowski space) $g^{\mu\nu}$ = diag(-1,1,1,1), so

$$\partial_\mu\partial^\mu\varphi = -\ddot{\varphi} + \nabla^2\varphi = V'(\varphi). \tag{8}$$

For the FRW metric $g^{\mu\nu} = \text{diag}(-1, a^{-2}, a^{-2}, a^{-2})$ (in Cartesian not spherical coordinates). Using this metric the previous equation becomes

$$\ddot{\varphi} + 3H\dot{\varphi} - a^{-2}\nabla^2\varphi = -V'(\varphi) \tag{9}$$

Often one assumes that the field is homogenous and $\nabla\phi = 0$. (If the field inflates then it will rapidly become homogenous.) In this case

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi). \tag{10}$$

If we neglect the term with $\dot{\varphi}$ and use $V' = m^2\varphi$ then this gives

$$\ddot{\varphi} = -m^2\varphi \tag{11}$$

which is just the equation of a harmonic oscillator with frequency m. The term $3H\dot{\varphi}$ in equation 10 acts like a damping or **friction** term as it is proportional to $\dot{\varphi}$. Equation 10 is the motion of a point particle moving in φ space with a potential $V(\varphi)$ with friction. If the potential has a minimum the damping term lets φ settle to the potential minimum. During this time oscillations decay. If the initial value of φ is large then the field can coast or move slowly down the potential curve.

Using variational principles (but with the metric rather than φ, and requiring that $T^{\mu\nu}_{;\nu} = 0$), the Lagrangian also gives us an energy momentum tensor that must be in the form

$$T_{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial_\nu\varphi + g_{\mu\nu}\mathcal{L} \tag{12}$$

For the Lagrangian given in equation 4

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left[\frac{1}{2}\partial_p\varphi\partial^p\varphi + V(\varphi)\right] \tag{13}$$

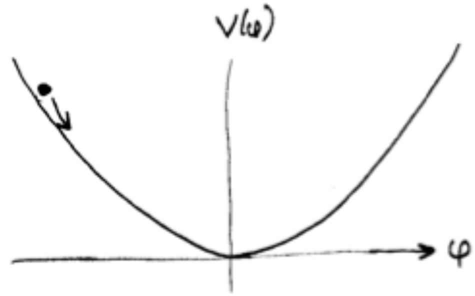

FIGURE 1. The evolution of the inflaton field. This figure comes from Syksy Räsänen's lecture notes.

Using the FRW metric

$$\begin{aligned} \rho &= T_{00} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}a^{-2}\nabla^2\varphi + V(\varphi) \\ p &= \frac{1}{3}T^i_{,i} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{6}a^{-2}(\nabla\varphi)^2 - V(\varphi) \end{aligned} \tag{14}$$

(in the above sum i only ranges from 1-3). Setting $\nabla\varphi = 0$ (taking the field to be homogenous)

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \\ p &= \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \end{aligned} \tag{15}$$

and equation of state parameter

$$w = \frac{p}{\rho} = \frac{1 - 2V(\varphi)/\dot{\varphi}^2}{1 + 2V(\varphi)/\dot{\varphi}^2} \tag{16}$$

The parameter w is in the range $[-1, 1]$. If $\dot{\phi}$ is large then $w \sim 1$, but if V is large then $w \sim -1$. We can combine equations 14 to give

$$\begin{aligned} \rho + p &= \dot{\varphi}^2 \\ \rho + 3p &= 2(\dot{\varphi}^2 - V(\varphi)) \end{aligned} \tag{17}$$

The condition for inflation, $\rho + 3p < 0$ (discussed in the last set of lecture notes, and giving acceleration and exponential behavior for a) is satisfied if

$$\dot{\varphi}^2 < V(\varphi). \tag{18}$$

We now combine our evolution equations for φ with those for the metric in an isotropic, homogeneous flat universe. Friedmann's equation

$$H^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2} = \frac{\rho}{3M_{Pl}^2} - \frac{K}{a^2} \tag{19}$$

where I have rewritten the constants in terms of the reduced Planck length. Assuming that the dominant contribution to ρ is that associated with our scalar field, setting $K = 0$, and using equation 15

$$H^2 = \frac{1}{3M_{Pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]. \tag{20}$$

It is possible to add the two Friedmann's equations together (with $K = 0$) to find

$$\dot{H} = -4\pi G(\rho + p) = -\frac{1}{2M_{Pl}^2}\dot{\varphi}^2 \tag{21}$$

The matter and radiation density components decrease as a increases, $\propto a^{-3}, a^{-4}$, respectively. As soon as the inflaton field begins to dominate, the other components quickly become negligible.

1.3. **Slow-Roll inflation.** I restate equation 10

$$\ddot{\varphi} + 3H\dot{\varphi} = -V'(\varphi). \tag{22}$$

The friction term, $3H\dot{\varphi}$, can cause φ to vary slowly. The situation where time derivatives are small or

$$\dot{\varphi}^2 \ll V(\varphi) \tag{23}$$

$$|\ddot{\varphi}| \ll |3H\dot{\varphi}| \tag{24}$$

is known as **slow-roll** and the above two equations are **slow-roll conditions**. Using these two conditions (known as the *slow-roll approximation*) we derive the *slow-roll equations* below. From equation 15 and using the slow-roll conditions we find that

$$\rho \sim V(\varphi) \qquad p \sim -V(\varphi). \tag{25}$$

In equation 20 we neglect the $\dot{\varphi}^2$ term and find

$$H^2 = \frac{V(\varphi)}{3M_{Pl}^2}, \tag{26}$$

and in equation 22, neglecting the $\ddot{\varphi}$ term gives

$$3H\dot{\varphi} = -V'(\varphi). \tag{27}$$

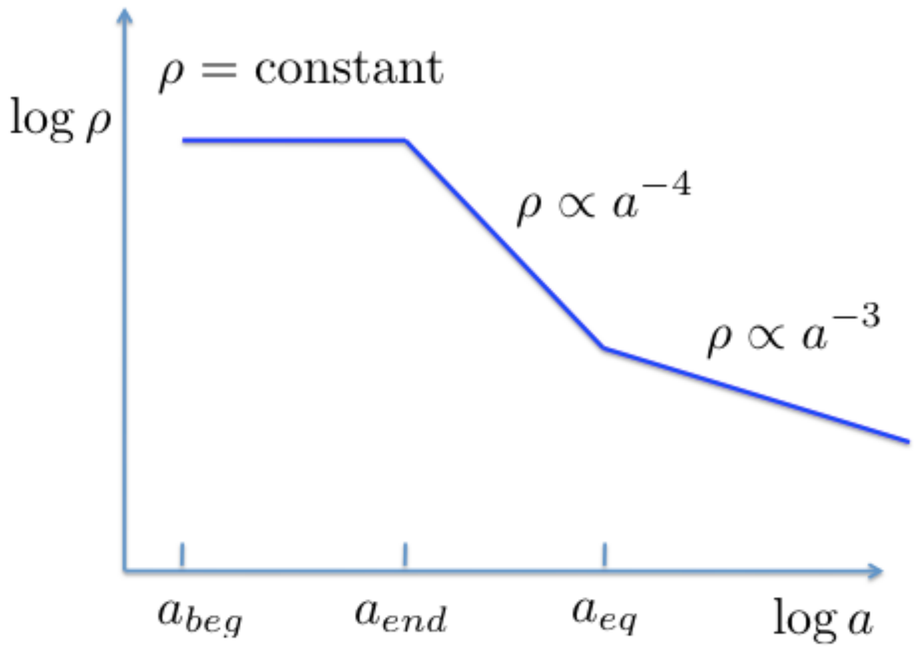

FIGURE 2. The energy density in an inflationary scenario.

Equations 26 and 27 are known as the *slow-roll equations*. Note that equation 17 and 25 imply that $\rho + 3p < 0$ (as long as V is positive). This and the Friedmann equation

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) = -\frac{1}{2M_{Pl}^2}(\rho + 3p) \tag{28}$$

implies inflation (or acceleration; $\ddot{a} > 0$). When the slow-roll conditions are satisfied, then *inflation is guaranteed*. The opposite is not true; inflation does *not* imply that the slow-roll conditions are satisfied.

Given initial conditions, $\varphi, \dot{\varphi}, \ddot{\varphi}$ then $\varphi(t)$ can be determined. Initially the evolution might not satisfy the slow-roll conditions. However the evolution may approach a solution that satisfies the slow-roll conditions, independent of the initial conditions. If this happens then the universe inflates. Once the universe inflates, it looses all memory of its initial conditions.

It is sometimes useful to describe slow-roll in terms of two parameters that depend only on the local shape of the potential

$$\epsilon(\varphi) \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \tag{29}$$

$$\eta(\varphi) \equiv M_{Pl}^2 \frac{V''(\varphi)}{V(\varphi)} \tag{30}$$

The slow roll conditions (equations 24) imply that $\epsilon \ll 1$ and $|\eta| \ll 1$ (but the opposite is not necessarily true).

Taking the time derivative of the slow-roll approximation equation 26

$$2H\dot{H} = \frac{V'(\varphi)\dot{\varphi}}{3M_{Pl}^2} \tag{31}$$

Multiply by $H/2$ and use the second slow-roll equation 27

$$H^2\dot{H} = \frac{V'H\dot{\varphi}}{6M_{Pl}^2} = -\frac{V'^2(\varphi)}{18M_{Pl}^2} \tag{32}$$

Some manipulation leads to

$$-\frac{\dot{H}}{H^2} = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \epsilon \ll 1 \tag{33}$$

During inflation, the Hubble parameter varies slowly ($|\dot{H}|$ is small), while the scale factor increases exponentially.

1.4. **Number of e-foldings.** During inflation the scale factor grows by a huge factor. The number of e-foldings from time t to t_{end}

$$N(t) \equiv \ln \frac{a(t_{end})}{a(t)} \tag{34}$$

This can be calculated using $N(\varphi(t))$

(35)

$$N(\varphi) = \ln \frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} \frac{da}{a(t)} = \int_t^{t_{end}} H(t)dt = \int_\varphi^{\varphi_{end}} \frac{Hd\varphi}{\dot{\varphi}} \approx \frac{-1}{M_{Pl}^2}\int_\varphi^{\varphi_{end}} \frac{V}{V'}d\varphi$$

and the last expression used the slow-roll equations.

A scalar field model for inflation consists of

(1) A potential $V(\phi)$, and
(2) A way of ending inflation.

To end inflation

(1) The slow roll approximation stops being valid, or
(2) Coupling of the scalar field to other fields stops inflation.

Inflation models can be classified into two classes depending upon the change in φ during the slow-roll period ($\Delta_{sr}\varphi$)

(1) small-field inflation; $\Delta_{sr}\varphi < M_{Pl}$.
(2) large-field inflation; $\Delta_{sr}\varphi > M_{Pl}$.

It is interesting to discuss initial conditions that allow inflation to take place even though, inflation itself erases any memory of them. Inflation could begin at the Planck scale $\rho \sim M_{Pl}^4$ which might be based on a quantum space foam type of era. In this case space itself might experience large quantum fluctuations and one of these could inflate, in one of those φ could drop so that classical behavior dominates and becoming the universe we live in.

1.5. **Reheating and Inflaton Decay.** If the potential contains a minimum then φ oscillates at the bottom of the potential while decaying to the minimum value. Additional fields can be coupled to the inflaton. At this time some of the inflaton energy density can be transferred into a thermal bath of particles. If the coupling is strong then the decay time would be shorter. The temperature of this bath is determined by the energy density ρ_{reh} at the end of the reheating temperature. Often it is assumed all particles reach this temperature even though weakly interacting particles might not have time to reach thermal equilibrium.

1.6. **Solving the Horizon Problem.** In the previous lecture notes we defined a comoving Hubble length as

$$l_H^c = \frac{1}{aH} = \mathcal{H}^{-1} = \dot{a}^{-1} \tag{36}$$

We now discuss a *comoving wave number*, $k = 2\pi/\lambda$ where λ is a comoving wavelength. A scale is

- superhorizon, when $k < \mathcal{H}$ (or $k^{-1} > \mathcal{H}^{-1}$)
- subhorizon, when $k > \mathcal{H}$ (or $k^{-1} < \mathcal{H}^{-1}$)

When $k = \mathcal{H}$ then the scale is entering or leaving the horizon.

Consider

$$\dot{\mathcal{H}}^{-1} = -\frac{\ddot{a}}{\dot{a}^2} \tag{37}$$

Inflation is when $\ddot{a} > 0$ or when $\mathcal{H}^{-1}$ is shrinking ($\dot{\mathcal{H}}^{-1} < 0$). Otherwise $\ddot{a} < 0$ and $\mathcal{H}^{-1}$ is growing.

On a plot of k^{-1} vs time we can draw the comoving Hubble length as a curve. Values of k^{-1} that lie above the curve are superhorizon scales, and those below the curve are subhorizon scales. Some scales will exit the horizon during inflation and then reenter the horizon afterwards.

Our current comoving Hubble length is $(a_0 H_0)^{-1}$. During inflation H was constant and related to the density by Friedmann's equation. The comoving Hubble length is largest at the beginning of inflation. We describe inflation from the beginning and end scales a_{beg}, a_{end}. We require that the horizon at the beginning of inflation be at least as big as the current horizon so

$$(a_{beg} H_i)^{-1} > (a_0 H_0)^{-1} \tag{38}$$

where H_i is the Hubble constant during inflation (approximately constant) and $H_i \sim \sqrt{V(\phi)/3} M_{pl}^{-1}$ by Friedmann's equation. Based on an assumed density scale for inflation (say assuming associated with a grand unified field theory or GUT scale), then the ratio of a_{beg} to a_{end} and so the number of e-foldings can be constrained. A number often given is 60 e-foldings! This number of e-foldings also solves the flatness problem.

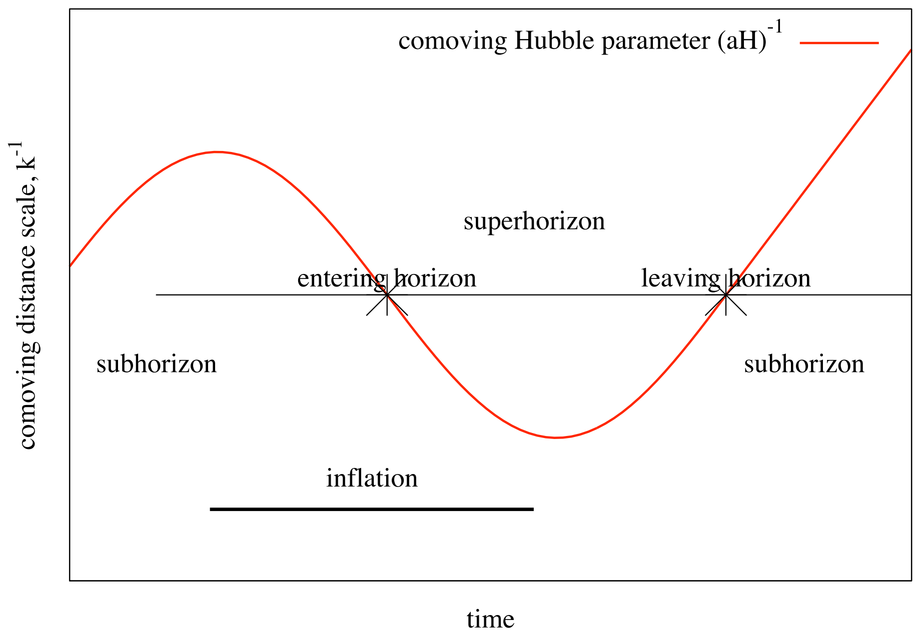

FIGURE 3. The evolution of the Hubble length as seen in comoving coordinates.

2. Linear Metric and Density Perturbations

Previously we considered a background homogenous and isotropic universe described with a FRW model. We now discuss the period of time when there are small perturbations on this background model. We will use first order perturbation theory to explore the evolution of small perturbations. The contributions to the stress energy tensor

$$\begin{aligned} \rho(t,\mathbf{x}) &= \bar{\rho}(t) + \delta\rho(t,\mathbf{x}) \\ p(t,\mathbf{x}) &= \bar{p}(t) + \delta p(t,\mathbf{x}) \end{aligned} \tag{39}$$

where $\mathbf{x}$ are the comoving spatial coordinates. Energy fluctuations are coupled to perturbations in the metric

$$ds^2 = (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta})dx^\alpha dx^\beta \tag{40}$$

where

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{41}$$

is our isotropic background metric. It is nontrivial to require that the metric perturbations are small compared to the background as the metric functions are dependent on the coordinates. Really we require that curvature perturbations are small compared to that of the background. In the linear approximation the metric perturbations do not influence the evolution of the background.

The perturbations can be split into scalar, vector and tensor components. In the linear approximation each of these evolve independently. Physically, the tensor components correspond to gravity waves and only the scalar component is related to the density perturbations. So far (as of 2011) tensor perturbations have not been identified in the cosmic microwave background.

The most general linear perturbation around the FRW metric can be written

$$\begin{aligned} ds^2 &= -(1+\Phi)dt^2 + 2a(t)(B_{,i} - S_i)dx^i dt \\ &\quad + a^2(t)\left[(1-\Psi)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right]dx^i dx^j \end{aligned} \tag{42}$$

where Φ, Ψ, B, E are scalars, S_i, F_i are vectors and h_{ij} is a tensor. Commas refer to derivatives $f_{,i} = \partial f/\partial x^i$ and indices are raised and lowered with the background metric tensor.

The metric tensor is symmetric and each index has 4 possibilities. This gives 10 degrees of freedom. However 4 of these degrees of freedom correspond to ways of choosing the four coordinates. The different coordinate systems are called different *gauges*. The choice of coordinate is called a choice of *gauge*. It happens that setting $E = B = 0$ fixes the coordinate system completely. This choice is known as the 'longitudinal gauge' or the 'conformal Newtonian gauge'. It happens that the vector perturbations decay in time so we neglect them, $S_i = F_i = 0$. In the longitudinal gauge and neglecting the vector degrees of freedom,

$$ds^2 = -(1+\Phi)dt^2 + a^2(t)\left[(1-\Psi)\delta_{ij} + h_{ij}\right]dx^i dx^j. \tag{43}$$

The functions Φ, Ψ are known as the *Bardeen potentials*. The function Φ is also called the Newtonian potential since in the Newtonian limit it is equal to the Newtonian gravitational potential.

Previously it made sense to consider the stress energy tensor diagonal in a frame comoving with the comoving coordinate system. However our perturbations may induce off-diagonal terms in the stress energy tensor. In additions to perturbations in ρ, p (as given in equation 40) we can consider perturbations in a four velocity

$$u^a(t, \mathbf{x}) = \delta^{\alpha 0} + \delta u^a(t, \mathbf{x}) \tag{44}$$

with stress energy tensor

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta} \tag{45}$$

As a four velocity is normalized, $g_{\alpha\beta}u^\alpha u^\beta = 1$, it follows that $\delta u^0 = \Phi$.

The perturbed metric can be used to calculate the tensor G_{ij}. Using the Einstein field equation the following first order equations appear

$$4\pi G\delta\rho = \frac{1}{a^2}\nabla^2\Psi - 3H(\dot{\Psi} + H\Phi) \tag{46}$$

$$4\pi G(\bar{\rho} + \bar{p})\delta u_i = -(\dot{\Psi} + H\Phi)_{,i} \tag{47}$$

$$4\pi G\delta p\delta_{ij} = \left[(2\dot{H} + 3H^2)\Phi + H\dot{\Phi} + \ddot{\Psi} + 3H\dot{\Psi} + \frac{1}{2a^2}\nabla^2 D\right]\delta_{ij} - \frac{1}{2a^2}D_{,ij} \tag{48}$$

$$0 = \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} \tag{49}$$

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$ and $D \equiv \Phi - \Psi$. The non-diagonal parts of equation 48 imply that $D_{,ij} = 0$ for $i \neq j$. That means that D is separable in x, y, z or $D = A(x) + B(y) + C(z)$. So that there is no preferred coordinate axis, we may assume that D is a constant. Since the mean of the perturbations gives a zero-th order term, this implies that $D = 0$ and so $\Phi = \Psi$. Only equation 49 involves h_{ij} and so can henceforth be considered separately.

Before we rewrite the perturbation equations we define

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}} \tag{50}$$

$$w \equiv \frac{\bar{p}}{\bar{\rho}} \tag{51}$$

$$v^2 \equiv \frac{\delta p}{\delta\rho} \tag{52}$$

The first order equations become

$$0 = \ddot{\Phi} + H(4+3v^2)\dot{\Phi} - \frac{v^2}{a^2}\nabla^2\Phi + [2\dot{H} + 3(1+v^2)H^2]\Phi \tag{53}$$

$$\delta = \frac{2}{3}\frac{1}{(aH)^2}\nabla^2\Phi - \frac{2}{H}\dot{\Phi} - 2\Phi \tag{54}$$

$$\delta u^i = \frac{1}{a^2\dot{H}}\partial_i(\dot{\Phi} + H\Phi). \tag{55}$$

Only equation 55 involves δu^i so we can consider this perturbation as derived from the others. Equation 55 gives δ in terms of Φ so if we have a solution for Φ then we can find δ. Note the resemblance of equation 55 to Poisson's equation. The density perturbations arise from the scale perturbation.

A procedure for solving for the perturbations is the following:

(1) Give w, v for the matter model.
(2) Solve the background cosmology to determine $a(t)$.
(3) Solve equation 53 for $\Phi(t)$. Use equation 54 to find $\delta(t)$. Use equation 55 to find $\delta u^i(t)$.

2.1. **Adiabatic perturbations.** When

$$c_s^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} = \frac{\delta p}{\delta \rho}, \tag{56}$$

then the perturbations are said to be **adiabatic**. The background universe is expected to expand adiabatically (no entropy production). Currently observations of cosmic microwave background are consistent with adiabatic perturbations.

The density and pressure may each contain more than one component.

$$\rho = \sum_i \rho_i \qquad p = \textstyle\sum_i p_i \tag{57}$$

$$\delta\rho = \sum_i \delta\rho_i \qquad \delta p = \textstyle\sum_i \delta p_i \tag{58}$$

We can define a contrast parameter for each component

$$\delta_i \equiv \frac{\delta\rho_i}{\bar{\rho}_i}. \tag{59}$$

The density parameter

$$\delta = \frac{\delta\rho}{\bar{\rho}} = \frac{\sum_i \delta\rho_i}{\sum_i \bar{\rho}_i} \neq \sum_i \delta_i \tag{60}$$

Recall the continuity equation

$$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p}) = -3H(1+w)\bar{\rho}. \tag{61}$$

As long as there is no interaction between components this is satisfied for each component

$$\dot{\bar{\rho}}_i = -3H(1 + w_i)\bar{\rho}_i \quad (62)$$

Using the continuity equation

$$\frac{\dot{\bar{\rho}}_i}{\bar{\rho}_i}\frac{1}{1 + w_i} = -3H. \quad (63)$$

We relate the perturbation $\delta\rho(x)$ to conditions at a slightly different time $\dot{\bar{\rho}}\delta t$. If the perturbations are adiabatic then for the same δt, we can also relate $\delta p = \dot{\bar{p}}\delta t$. We assume that for the same δt that for each component[1]

$$\delta\rho_i = \dot{\bar{\rho}}_i\delta t \quad (64)$$

Adiabatic perturbations have the property that the local state of matter at some point t, x is the perturbed universe is the same as the background universe at a slightly different time $t + \delta t$. If there is a single fluid, perturbations are automatically adiabatic. Using the above assumption, and the continuity equation for each component, we find that

$$\frac{\delta_i}{1 + w_i} = \frac{\delta_j}{1 + w_j} \quad (65)$$

These ratios are the same for each component, and for any number of components. For radiation $w_r = 1/3$ and for matter $w_m = 0$, consequently

$$\delta_m = \frac{3}{4}\delta_r. \quad (66)$$

Adiabatic perturbations vary the density, $\delta\rho \neq 0$. However, **isocurvature** perturbations (previously called *isothermal* perturbations) fix $\delta\rho = 0$ and allow differing perturbations in the individual fluids such that $\sum \rho_i = 0$

2.2. **Fourier Transform.** Since the equations are linear we can solve the perturbation equations in Fourier space. Let

$$\Phi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \Phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (67)$$

and similar expansions for $\delta_{\mathbf{k}}, u^i_{\mathbf{k}}$. Only in flat space do plane waves form a complete basis function, so the above assumes we have a flat background model. Here $\mathbf{k}$ is called the *comoving momentum* and is not the physical momentum which is $\mathbf{k}/a$.

[1]If this is true then $\frac{\delta\rho_i}{\delta p_i} = \frac{\dot{\bar{\rho}}_i}{\dot{\bar{p}}_i} = \frac{\dot{\bar{\rho}}}{\dot{\bar{p}}} = \frac{\delta\rho}{\delta p}$ is true for all components. The last bit of equation 12.26 in Rasanen's notes is not clearly justified to me.

Using the Fourier components

$$0 = \ddot{\Phi}_{\mathbf{k}} + H(4+3v^2)\dot{\Phi}_{\mathbf{k}} + \frac{v^2k^2}{a^2}\Phi_{\mathbf{k}} + [2\dot{H} + 3(1+v^2)H^2]\Phi_{\mathbf{k}} \tag{68}$$

$$\delta_{\mathbf{k}} = -\frac{2}{3}\frac{k^2}{(aH)^2}\Phi_{\mathbf{k}} - \frac{2}{H}\dot{\Phi}_{\mathbf{k}} - 2\Phi_{\mathbf{k}} \tag{69}$$

$$0 = \ddot{h}_{\mathbf{k}ij} + 3H\dot{h}_{\mathbf{k}ij} + \frac{k^2}{a^2}h_{\mathbf{k}ij} \tag{70}$$

where $k \equiv |\mathbf{k}|$. Each Fourier mode decouples from the others and each k gives a second order differential equation for $\Phi_{\mathbf{k}}(t)$.

Consider equation 68 at large k and for slowly varying solutions $\dot{\Phi}, \ddot{\Phi}$ small. Long wavelength, $k < AH$, constant solutions exist. Perturbations larger than the Hubble scale can be frozen in. In the limit of $\dot{\Phi}$ small and k/aH small, equation 69 becomes

$$\delta_k = -2\Phi_k \tag{71}$$

providing a direct relation between density and potential superhorizon scales. The term proportional to $\dot{\Phi}_k$ can be considered a decaying mode, implying that solutions approach the steady state solution.

The solutions to these equations depend on initial conditions at early times. The inflation scenario provides a way of relating perturbations at early times to quantum fluctuations.

2.3. **Gaussian perturbations.** The assumption that the perturbations are Gaussian implies that the $g_{\mathbf{k}}$ are drawn from a Gaussian distribution with probability

$$\frac{1}{2\pi s_{\mathbf{k}}^2}\exp\left(-\frac{|g_{\mathbf{k}}|^2}{2s_{\mathbf{k}}^2}\right) \tag{72}$$

that depends only on one parameter $s_{\mathbf{k}}$. The mean of the distribution is zero

$$\langle g_{\mathbf{k}}\rangle = 0 \tag{73}$$

and the variance

$$\langle |g_{\mathbf{k}}|^2\rangle = 2s_{\mathbf{k}}^2. \tag{74}$$

Cosmological perturbations are real which means that $g_{-\mathbf{k}} = g^*$. The Fourier component $g_{\mathbf{k}}$ can be complex and we can chose real and imaginary parts from the same Gaussian distribution. The probabilities of different $\mathbf{k}$ modes are independent. This implies that

$$\langle g_{\mathbf{k}}g_{\mathbf{k}'}^*\rangle = 0 \quad \text{for} \quad \mathbf{k} \neq \mathbf{k}' \tag{75}$$

$$= 2\delta_{\mathbf{k}\mathbf{k}'}s_{\mathbf{k}}^2 = \delta_{\mathbf{k}\mathbf{k}'}\langle |g_{\mathbf{k}}|^2\rangle. \tag{76}$$

So that the distribution is independent of direction

$$s_{\mathbf{k}} = s(k). \tag{77}$$

Convention used for integrals is

$$g(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^k \tag{78}$$

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^x. \tag{79}$$

For a series in the limit of box size $L^3 \to \infty$

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{k}} \to \int d^3k \tag{80}$$

$$\left(\frac{L}{2\pi}\right)^3 g_{\mathbf{k}} \to \frac{1}{(2\pi)^{3/2}} g(\mathbf{k}) \tag{81}$$

$$\left(\frac{L}{2\pi}\right)^3 \delta_{\mathbf{k}\mathbf{k}'} \to \delta^3(\mathbf{k} - \mathbf{k}') \tag{82}$$

Often calculations are done in series notation and converted to integrals at the end.

2.4. **The Power Spectrum.** All statistical information about Gaussian perturbations are encoded in a single function $s(k)$ or $\langle |g_{\mathbf{k}}|^2 \rangle$. This function is usually discussed in terms of the power spectrum defined as follows

$$P_g(k) = \left(\frac{L}{2\pi}\right)^3 4\pi k^3 \langle |g_{\mathbf{k}}|^2 \rangle = \frac{L^3}{2\pi^2} k^3 \langle |g_{\mathbf{k}}|^2 \rangle. \tag{83}$$

It is not clear to me why the convention involves k^3.

3. Inflation perturbations

3.1. **Evolution of inflaton perturbations.** When we considered the evolution of perturbations of the metric we found the following first order equations (in a suitable choice of gauge),

$$0 = \ddot{\Phi} + H(4 + 3v^2)\dot{\Phi} - \frac{v^2}{a^2}\nabla^2\Phi + [2\dot{H} + 3(1 + v^2)H^2]\Phi \tag{84}$$

$$\delta = \frac{2}{3}\frac{1}{(aH)^2}\nabla^2\Phi - \frac{2}{H}\dot{\Phi} - 2\Phi \tag{85}$$

$$\delta u^i = \frac{1}{a^2\dot{H}}\partial_i(\dot{\Phi} + H\Phi). \tag{86}$$

with

$$\delta \equiv \frac{\delta\rho}{\bar\rho} \tag{87}$$

$$w \equiv \frac{\bar p}{\bar\rho} \tag{88}$$

$$v^2 \equiv \frac{\delta p}{\delta\rho} \tag{89}$$

We now need to add to these an equation for the evolution of perturbations of the scalar field. We split φ into two terms

$$\varphi(t, \mathbf{x}) = \bar\varphi(t) + \delta\varphi(t, \mathbf{x}) \tag{90}$$

with $\bar\varphi$ the background zero-th order term, and $\delta\varphi$ a first order perturbation that now depends on position.

Recall equation 9 that we found from varying φ in the action that involved the scalar field Lagrangian, that I repeat here

$$\ddot\varphi + 3H\dot\varphi - a^{-2}\nabla^2\varphi + V'(\varphi) = 0 \tag{91}$$

Because we are considering perturbations that depend on position, we have not dropped the $\nabla^2\varphi$ term above. We derived this equation with the FRW metric. But the equations above Equation 84, 85, 86 we used a metric with a first order perturbation (that depended on Bardeen potentials). So we need to re-derive equation 91 from the action, but now using the perturbed metric. To first order in φ and metric perturbations the result is

$$\delta\ddot\varphi + 3H\delta\dot\varphi - \left(a^{-2}\nabla^2\varphi + V''(\bar\varphi)\right)\delta\varphi = -2\Phi V'(\bar\varphi) + 3(\dot\Phi + \dot\Psi)\dot{\bar\varphi} \tag{92}$$

Taking the Fourier transform of this

$$\delta\ddot\varphi_k + 3H\delta\dot\varphi_k - \left(\frac{k^2}{a^2} + m^2(\bar\varphi)\right)\delta\varphi_k = -2\Phi_k V'(\bar\varphi) + 3(\dot\Phi_k + \dot\Psi_k)\dot{\bar\varphi} \tag{93}$$

where

$$m^2(\bar\varphi) \equiv V''(\bar\varphi). \tag{94}$$

(Note we previously argued that we could set $\Phi = \Psi$.) Equation 93 could be solved together with equations 68, 69, 70 and using density and pressure appropriate for the inflaton field. Recall that we had some gauge freedom in how we chose to describe the metric perturbation. It turns out that it is possible to chose the coordinate system such that the metric perturbations make a negligible contribution to the equation of the inflaton perturbations and so the right hand side of Equation 93 can be neglected, giving

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k - \left(\frac{k^2}{a^2} + m^2(\bar{\varphi})\right)\delta\varphi_k = 0 \tag{95}$$

3.2. **Solutions for perturbations during inflation.** During inflation H and m^2 can change slowly with t. We can make an approximation where we treat them as constant. Assuming H is constant is equivalent to $a \propto e^{Ht}$. The solution to equation 95 can be put in the form

$$\delta\varphi_k(t) = a^{-3/2}\left[A_k J_{-\nu}\left(\frac{k}{aH}\right) + B_k J_\nu\left(\frac{k}{aH}\right)\right] \tag{96}$$

where J_ν is the Bessel function of order ν with

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \approx \frac{3}{2} \tag{97}$$

where we have at the last step assumed slow-roll conditions $m^2 = V'' \ll H^2$. With half integer Bessel functions we can describe the solutions as

$$\delta\varphi_{\mathbf{k}}(t) = A_k w_{\mathbf{k}}(t) + B_k w^*_{\mathbf{k}}(t) \qquad \text{with} \qquad w_{\mathbf{k}}(t) = \left(i + \frac{k}{aH}\right)e^{\frac{ik}{aH}} \tag{98}$$

Recall in the above section we discussed a comoving wavenumber in comparison to the horizon? In the above equation we can see the ratio

$$\frac{k}{aH} = \frac{k}{\mathcal{H}} \tag{99}$$

which is exactly the comoving wavenumber.

- For $k > aH$ the argument in the exponent in equation 98 is large. Solutions oscillate. For wavenumbers smaller than the horizon scale, solutions oscillate.
- For $k < aH$, the solutions are fixed and approach a constant value of $i(A_k - B_k)$. For wavenumbers larger than the horizon scale, solutions are frozen.

3.3. **Spectrum generated by the inflaton.** The quantum mechanical field operator for the inflaton perturbations is

$$\delta\hat{\varphi}_k(t) = w_k(t)\hat{a}_k + w^*_k(t)\hat{a}^\dagger_{-k} \tag{100}$$

and this gives a power spectrum of inflaton perturbations of

$$\mathcal{P}_\varphi(k) = \frac{L^3k^3}{2\pi^2}|w_k|^2. \tag{101}$$

After a perturbation exits the horizon $k < aH$ and the mode function approaches a constant value

$$w_k(t) \to L^{-3/2}\frac{iH}{\sqrt{2k^3}} \qquad \text{and} \qquad \mathcal{P}_\varphi(k) = \frac{L^3k^3}{2\pi^2}|w_k|^2 = \left(\frac{H}{2\pi}\right)^2 \tag{102}$$

We calculate the power spectrum of the inflaton field perturbations by using the quantum mechanical expectation value of the square of the field perturbation. Then this is identified with the expectation value of a probability distribution of a classical variable. We replace an expectation value of a quantum state with the ensemble average of a classical distribution. Quantum mechanics generates the initial perturbations and solves the problem of how perturbations can emerge from a state which is homogenous and isotropic. As a remnant of the indeterministic origin of the perturbations we cannot predict any specific member of the distribution but can calculate the ensemble average. All Fourier modes $\delta\varphi_k$ acquire their values as independent variables with a Gaussian probability.

Equation 102 was estimated assuming H is constant. However H varies slowly during inflation. Really the power spectrum would be set by the value when a particle scale exits the horizon and so becomes fixed.

$$\mathcal{P}_\varphi(k) = \left(\frac{H}{2\pi}\right)^2_{aH=k} \tag{103}$$

where for each comoving k we take H when that wavelength exits the horizon. It is convenient to describe perturbations in terms of a gauge invarient quantity $\mathcal{R}$, the comoving curvature perturbation.

In terms of comoving curvature perturbations

$$\mathcal{R}_k = -H\frac{\delta\varphi_k}{\dot{\bar{\varphi}}} \tag{104}$$

so that

$$\mathcal{P}_\mathcal{R}(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\right)^2 \mathcal{P}_\varphi(k) = \left(\frac{H}{\dot{\bar{\varphi}}}\frac{H}{2\pi}\right)^2_{aH=k} \tag{105}$$

Assuming slow-roll, $H^2 = V/(3M_{Pl}^2)$, and $3H\dot{\varphi} = -V'$ the above becomes

$$\mathcal{P}_\mathcal{R} = \frac{1}{12\pi^2 M_{Pl}^6}\frac{V^3}{V'^2} = \frac{1}{24\pi^2 M_{Pl}^4}\frac{V}{\epsilon} \tag{106}$$

The amplitude of the ‘primordial’ power spectrum measured from the CMB gives

$$\mathcal{P}_\mathcal{R} \approx 5 \times 10^{-5} \tag{107}$$

and is nearly independent of k. This leads to constraint on the size of V during inflation (with the assumption $\epsilon \ll 1$).

Note equation 105 does not assume that the power spectrum is independent of k. We can define an index

$$\mathcal{P}_\mathcal{R} = A^2 \left(\frac{k}{k_p}\right)^{n-1} \tag{108}$$

where A is the amplitude at k_p. If the power spectrum is constant then $n = 1$ and the spectrum is said to be *scale invariant* or a Harrison-Zeldovich spectrum. The power $\mathcal{P}_\mathcal{R}(k)$ is measured at $k = aH$ during inflation (where H is nearly constant). With some manipulation it can be shown that $n-1$ depends on slow roll parameters η, ϵ as so that we expect that n is nearly 1. Based on the CMB observations (and in the context of ΛCDM) WMAP observations are used to estimate $n = 0.97 \pm 0.01$.

4. Perturbations After Inflation

4.1. **Evolution of Perturbations during the radiation dominated era.** During the radiation dominated era,

$$a \propto t^{1/2}, \qquad H = \frac{1}{2t}, \qquad \rho \approx \rho_r \propto a^{-4}, \qquad v^2 = c_s^2 = \frac{1}{3} \tag{109}$$

Define

$$y \equiv \frac{k}{\sqrt{3}aH}. \tag{110}$$

Using these parameters the solution to equation 68 and 69 for $y \ll 1$ (superhorizon scales)

$$\Phi_k(t) = -\frac{1}{9\sqrt{3}}\left(\frac{k}{H_0}\right)^3 A_{1k} = \text{constant} \tag{111}$$

$$\delta_k(t) = \frac{2}{9\sqrt{3}}\left(\frac{k}{H_0}\right)^3 A_{1k} = \text{constant} \tag{112}$$

and for $y \gg 1$ (dropping a decaying mode) on subhorizon scales

$$\Phi_k(t) = \frac{1}{\sqrt{3}}\left(\frac{k}{H_0}\right) a^{-2} A_{1k} \cos y \tag{113}$$

$$\delta_k(t) = -\frac{2}{3\sqrt{3}}\left(\frac{k}{H_0}\right)^3 A_{1k} \cos y \tag{114}$$

where A_{1k} is a constant. As was true for the inflaton, superhorizon scales are constant and subhorizon ones oscillate.

During the radiation dominated era, cold dark matter does not dominate ρ however it is not coupled to the radiation the way that baryons are. At subhorizon scales cold dark matter perturbations can grow. They are influenced by the existing potential perturbations. It turns out that perturbations in cold dark matter grow logarithmically during this era.

4.2. **Evolution of Perturbations during the matter dominated era.** During the matter dominated era for cold dark matter (the dominant component)

$$a \propto t^{2/3}, \qquad H = \frac{2}{3t}, \qquad \rho \approx \rho_m \propto a^{-3}, \qquad v^2 = c_s^2 = 0 \tag{115}$$

Using these parameters the solution of equation 68 and 69 is

$$\Phi_k(t) = B_{1k} + a^{-5/2} B_{2k} \tag{116}$$

$$\delta_k(t) = -(2y^2 + 2)B_{1k} - (2y^2 - 3)a^{-5/2} B_{2k} \tag{117}$$

The terms $\propto a^{-5/2}$ decay and so can be ignored. The potential perturbation is independent of k and a constant. The density perturbation depends on y^2. As $y^2 \propto a \propto t^{2/3}$ the density perturbation grows. The initial B_{1k} coefficient should be set from the A_{1k} coefficient for perturbations from the radiation dominated era.

Question: How is this consistent with the Jean's equation?

4.3. **Baryon density perturbations and the Jean's equation.** Baryons move with the radiation before decoupling. However after decoupling perturbations in the baryons are decoupled and so can grow on their own. Pressure might be important so the equation of motion differs from that in cold dark matter.

Recall that in previous lectures we derived the Jean's instability from three equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{118}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\nabla p}{\rho} + \nabla \Phi = 0 \tag{119}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{120}$$

By considering perturbations for these three equations we found a dispersion relation

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0 \tag{121}$$

where ρ_0, c_s^2 refer to the density and sound speed in the background unperturbed medium. For this derivation we assumed that the background has a constant density.

In an expanding universe the background density $\bar{\rho}$ is decreasing. In the matter dominated era $\rho \propto a^{-3}$. Taking the derivative

$$\dot{\rho} = \partial_t(\bar{\rho}_0 a^{-3}) = -3\bar{\rho}_0 \dot{a} a^{-4} = -3H\rho \tag{122}$$

The mass continuity equation (equation 118) then implies that

$$\nabla \cdot \mathbf{u} = 3H \tag{123}$$

This implies that

$$\mathbf{u} = H\mathbf{r}, \tag{124}$$

as if the background velocity is equivalent to geometric expansion and $\mathbf{r} = (x, y, z)$. Equation 120 can be written

$$\nabla \cdot (\nabla \Phi) = 4\pi G \bar{\rho} \tag{125}$$

or is consistent with

$$\nabla \Phi = \frac{4\pi G}{3} \bar{\rho} \mathbf{r} \tag{126}$$

We now consider first order perturbations for equation 118 with background $\bar{\rho}$ and $\mathbf{u} = H\mathbf{r}$. Equation 118 becomes

$$\frac{\partial \delta\rho}{\partial t} + (\nabla \delta\rho) \cdot \mathbf{u} + \delta\rho(\nabla \cdot \mathbf{u}) + \bar{\rho}(\nabla \cdot \delta u) = 0 \tag{127}$$

$$\frac{\partial \delta\rho}{\partial t} + \nabla \delta\rho \cdot H\mathbf{r} + 3H\delta\rho + \bar{\rho}\nabla \cdot \delta u = 0 \tag{128}$$

$$\frac{\partial \delta}{\partial t} + H\mathbf{r} \cdot \nabla \delta + \nabla \cdot \delta u = 0 \tag{129}$$

with $\delta = \frac{\delta\rho}{\rho}$. Euler's equation and Poisson's equation are not changed. Hence the only major change between this setting and that lacking background expansion is the term proportional to H in the above equation.

Going into Fourier space the three equations (to first order) give

$$\ddot{\delta}_k + 2H\dot{\delta}_k + (c_s^2 k_{phys}^2 - 4\pi G\bar{\rho})\delta_k = 0 \tag{130}$$

where k_{phys} is a physical wavenumber and $k = k_{phys}/a$ is the comoving wavenumber. Using the comoving wavenumber we can write

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(c_s^2 \frac{k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta_k = 0 \tag{131}$$

We have a damped version of the original Jean's equation. Using Friedmann's equation this an be written

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left(c_s^2 \frac{k^2}{a^2} - \frac{3}{2}H^2\right)\delta_k = 0 \tag{132}$$

There is a time dependent Jean's length

$$\lambda_J = \sqrt{\frac{c_s^2 \pi}{G\bar{\rho}a^2}} \tag{133}$$

As the density drops, the Jean's length increases, so larger objects collapse later. The Jean's wavenumber can also be written

$$k_J = \sqrt{\frac{3}{2}}\frac{aH}{c} \tag{134}$$

For scales much smaller than the Jeans length we essential have damped sound waves. There is no growth of structure. For subhorizon wavelengths that are larger than the Jean's length solutions growth as $t^{2/3}$. Baryon perturbations grow just like CDM perturbations during this era.

During the radiation dominated era, the damping term and second derivative terms of equation 132 dominate,

$$\ddot{\delta}_k + \frac{\ddot{\delta}}{t} \approx 0 \tag{135}$$

giving the logarithmic solution for cold dark matter perturbations prior to z_{eq} that we discussed previously.

4.4. **The Transfer function.** During both matter and dominated eras, perturbations larger than the horizon remain frozen. After perturbations enter the horizon they can vary. Modifications to the spectrum can be described in terms of a transfer function, T,

$$g_k(t) = T_g(t,k)\mathcal{R}_k \tag{136}$$

where $T_g(t,k)$ is the transfer function for perturbation g. The power spectrum

$$\mathcal{P}_g(t,k) = T_g(t,k)^2\mathcal{P}_\mathcal{R}. \tag{137}$$

Density perturbations can be related to the spectrum of curvature perturbations with

$$\delta_k = \frac{2}{5}\left(\frac{k}{aH}\right)^2 \mathcal{R}_k T(k) \tag{138}$$

and a modified transfer function in the form

$$T(k) \approx \left\{ \begin{array}{c} 1 \\ \left(\frac{k_{eq}}{k}\right)^2 \ln\frac{k}{k_{eq}} \end{array} \right. \quad \text{for} \quad \begin{array}{c} k < k_{eq} \\ k > k_{eq} \end{array} \tag{139}$$

where k_{eq} is the comoving wavenumber entering the horizon at z_{eq} (radiation matter equality). For $k < k_{eq}$ these are wavelengths that are longer than k_{eq} and enter the horizon after z_{eq} during the matter dominated era. During matter domination they grow according to $1/(aH)^2 \propto a$. For $k > k_{eq}$ the wavelengths enter the horizon before z_{eq} during the radiation dominated era. Previously during radiation dominated era the cold dark matter perturbations grow logarithmically and afterwards they grow according to $1/(aH)^2$ during the matter dominated era. The transfer function has a bend at k_{eq}.

Question Is there a simple way to relate $\mathcal{R}_k$ to Φ_k? And that doesn't seem wacky for $w = -1$?

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Komatsu, E. et al. 2011, ApJS, 192, 18, Seven-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation