AST242 LECTURE NOTES PART 8 – INTRODUCTION TO COSMOLOGY

Contents

1. Background cosmology

1.1. **Units.** In cosmology calculations and equations are often done with the following *natural* units,

$$c = k_B = \hbar = 1, \tag{1}$$

where c is the speed of light, k_B is Boltzmann's constant, and $\hbar = \frac{h}{2\pi}$ and h is Plank's constant.

First let us consider

$$c = 1. \tag{2}$$

With this choice of units, velocity is unitless and ranges from 0 to 1 with photons or relativistic particles having velocities of 1. If cm is kept for the unit of distance and then time can be given in units of how long it takes light to travel a cm. Alternately if we keep seconds as a unit for time, then distance is given in light seconds. With $c = 1$, energy, mass and momentum have the *same* units. $E^2 = m^2c^4 + p^2c^2$ becomes $E^2 = m^2 + p^2$ where E, p and m are the energy, momentum and mass of a particle, respectively.

$$k_B = 1 \tag{3}$$

The occupation number for a state with energy E is $\propto e^{E/k_BT}$. With $k_B = 1$, the occupation number $\propto e^{E/T}$. Temperature is in the same unit as energy.

$$\hbar = 1 \tag{4}$$

For a photon $E = \hbar\omega$ where ω is the angular frequency. Using seconds as a unit for time, and $\hbar = 1$, ω is in units of inverse seconds. This means that Energy is also in units of inverse seconds. Alternately if are using cm as our distance unit, then time is in the same unit and Energy in units of inverse cm.

To summarize, for $c = k_B = \hbar = 1$, if we chose seconds as our common unit, then distance is in units of light seconds, energy, momentum, mass and temperature are in units of inverse light seconds. Alternately if we chose cm as our common unit, then energy, momentum, mass and temperature are in units of cm^{-1}.

1.2. **Units examples.**

(1) **The gravitational Constant.** What is the gravitational constant G in units with $\hbar = c = 1$? G does not depend on temperature so we should not need k_B. G is in units of $\text{g}^{-1}\ \text{cm}^3\ \text{s}^{-2}$ and $\hbar$ in units of g cm^2/s or erg s. $G\hbar$ is in units of $\text{cm}^5\ \text{s}^{-3}$.
If I would like to work in seconds then $G\hbar c^{-5}$ is in units of s^2.
If I would like to work in cm then $G\hbar c^{-3}$ is in units of cm^2.

Sometimes the gravitational constant is written G_N referring to the Newtonian gravitational constant.

(2) **Energy in eV.** Given an energy in eV how do I convert this to seconds or cm?
$1\text{eV} = 1.6 \times 10^{-12}$ erg and $\hbar = 1.054 \times 10^{-27}$ erg s.
$1\text{eV}/\hbar = 1.52 \times 10^{15}\text{s}^{-1}$. $1\text{eV}/(\hbar c) = 5.06 \times 10^4\ \text{cm}^{-1}$.

(3) **The Planck mass.** There is only one way to combine $G, \hbar, c$ to get a mass

$$m_P \equiv \sqrt{\frac{\hbar c}{G}} = 1.2209 \times 10^{19} \mathrm{GeV/c^2} = 2.17651(13) \times 10^{-8} \mathrm{kg} \tag{5}$$

This is called the **Planck mass**. Often used is a **reduced Planck mass**

$$M_{Pl} \equiv \sqrt{\frac{\hbar c}{8\pi G}} \tag{6}$$

where the factor of 8π is that in Einstein's field equation. In units with $\hbar = c = 1$ we can replace G with the reduced Planck mass or write

$$8\pi G = \frac{1}{M_{Pl}^2}. \tag{7}$$

1.3. **Space Time and General Relativity.** The tool used to compute trajectories of particles is the metric, which gives distances between two points on the space time manifold. To compute these distances we often use a coordinate system. The choice of the coordinate system used to describe the metric is not unique. The metric for Minkowski space of special relativity is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{8}$$

in Cartesian coordinates. In spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{9}$$

Minkowski space is flat. Light moves along curves with $ds^2 = 0$. A massive particle must move along a trajectory with $ds^2 < 0$.

Curves that minimize distance between two points are called *geodesics*,

$$\delta \int_{\text{path}} ds = 0. \tag{10}$$

A guiding assumption for general relativity is that particles move along geodesics. Because the manifold (with metric) is not necessarily flat, a particle can appear to be accelerating in a local coordinate system. Paths are not straight because of the effect of gravitation which is described with a metric rather than a force law.

All equations are described in terms of tensor equations. A tensor is an object that transforms as follows when coordinates are changed from x^i to x'^i

$$A'^{kl..}_{pq} = \frac{\partial x'^k}{\partial x^m}\frac{\partial x'^l}{\partial x^n}...\frac{\partial x^r}{\partial x'^p}\frac{\partial x^s}{\partial x'^q} A^{mn}_{rs} \tag{11}$$

and we have used summation notation. Upper indices are *contravariant* and lower indices are *covariant*. A vector that is described in terms of an orthonormal basis of tangents directions $(\hat{\mathbf{e}}_t, \hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) = (\partial_t, \partial_x, \partial_y, \partial_z)$ is covariant and is written A_i. The metric can be used to transfer a covariant tensor into a contravariant tensor.

$$A_\mu = g_{\mu\nu} A^\nu, \tag{12}$$

and $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$. Physical relations are written in terms of covariant and contravariant tensors so as to be coordinate invariant.

For a curved space time, the coefficients in the metric can be non-trivial functions,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

where we have implicitly summed over μ, ν.

We can chose a coordinate system in which to write the metric. This global coordinate system can then be used to compute distances between nearby points in our space time manifold. However each point on the manifold is also consistent with a local Minkowski space. The metric can be used to transport tensors between nearby points on the manifold via covariant derivatives. A covariant derivative can be defined in such a way that derivatives of an invariant give a covariant vector. Similarly covariant derivatives of the metric component should vanish as distances should not be dependent on the coordinate system used. These assumptions be used to derive a prescription for evaluating covariant derivatives in any coordinate system using derivatives of the metric components in that coordinate system. Covariant derivatives are often written

$$V^\mu_{;\alpha} = V^\nu_{,\alpha} + \Gamma^\mu_{\rho\alpha} V^\rho \tag{14}$$

where $V^\nu_{,\alpha} = \partial_\alpha V^\nu$ and $\Gamma^\mu_{\rho\alpha}$ is a Christoffel symbol and only depends on derivatives of the metric coefficients. The Christoffel symbols are also called the *affine connection.*

If a clock is carried by a person, then the time measured by this person would be $\int d\tau$ where $d\tau = \sqrt{-ds^2}$. The four velocity of the person (or object) is defined as

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{15}$$

and corresponds to the motion of the person with respect to his proper time. The relation between proper time and coordinate time (that we used to describe the metric) is $u^0 = \frac{dt}{d\tau}$.

Einstein's field equation relates curvature in space time to mass and energy,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{16}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, depends on the metric and its derivatives, $T_{\mu\nu}$ is the **stress-energy tensor** or energy-momentum tensor and G is the gravitational constant. Here $R_{\mu\nu}$ is the **Ricci** tensor and is dependent upon derivatives of the metric tensor and the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{17}$$

The field equation is analogous to Poisson's equation $\nabla^2\Phi = 4\pi G\rho = 4\pi G T_{00}$ where Φ is the gravitational potential. Einstein's field equation can be derived from an action integral

$$I = \int d^4x\sqrt{-g}\left(\frac{R(x)}{16\pi G} + \frac{1}{2}T^{\mu\nu}g_{\mu\nu}\right) \tag{18}$$

by varying $g^{\mu\nu}$. Here g is the determinant of the metric tensor. The rightmost term could be replaced with a Lagrangian density appropriate for a matter or particle field.

1.4. **Stress-Energy Tensor.** The stress energy tensor has components
T_{00} which is the energy density (including rest mass) in the coordinate frame,
T_{0j} gives the i-th component of the momentum density,
T_{i0} gives the energy flux through a surface with normal in the i-th direction, and
T_{ij} gives the flux of momentum i-component in j-direction.

In the frame of an observer co-moving with the fluid

$$T = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{19}$$

For an observer with 4 velocity u^μ

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{20}$$

It should make sense that $\mathbf{u} = (1, 0, 0, 0)$ for an observer co-moving with the fluid gives the previous result with $g_{\mu\nu}$ for local Minkowski space. The above form for the stress-energy tensor arises because it is the only covariant form consistent with equation 19.

1.5. **Conservation law for the Stress-Energy Tensor.** In fluid mechanics mass conservation had the form

$$\frac{d\rho}{dt} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{21}$$

that can also be written

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial x^i} = 0 \tag{22}$$

In Minkowski space t is another coordinate. In the absence of gravity any set of particles and/or fields will have a symmetric energy-momentum tensor that is conserved in the sense that

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = T^{\alpha\beta}_{,\beta} = 0. \tag{23}$$

In the presence of a gravitational field

$$T^{\alpha\beta}_{;\beta} = 0, \tag{24}$$

where we make use of the covariant derivative.

1.6. **Friedmann-Robertson-Walker Model.** If we require that the manifold have spacelike slices that are homogenous and isotropic then the metric can be put in a simple form;

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{25}$$

where K is a constant. If $K \neq 0$ then we can rescale r by $|K|^{1/2}$ (replace r with $\sqrt{r/|K|}$) and let $k = K/|k|$). In this case K is replaced in the metric by k and k has values $-1, 0, 1$. For a "flat" universe $k = 0$.

1.7. **Redshift.** Light travels along paths with $ds^2 = 0$. Along a light ray path (null geodesic) that goes from t_1 to t_2 and from $r = 0$ to R, we find that

$$\int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_0^R \frac{dr}{\sqrt{1-kr^2}}. \tag{26}$$

Consider two photons, both starting from $r = 0$ and going to $r = R$ and so fixed comoving coordinates, but one leaves at t_1 and the other leaves at $t_1 + \delta t_1$. One arrives at t_2, the other arrives at $t_2 + \delta t_2$. The difference between travel distances

$$\int_{t_1}^{t_2} \frac{dt}{a(t)} - \int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{dt}{a(t)} = \int_{t_1}^{t_1+\delta t_1} \frac{dt}{a(t)} - \int_{t_2}^{t_2+\delta t_2} \frac{a}{a(t)} = \frac{\delta t_1}{a(t_1)} - \frac{\delta t_2}{a(t_2)} = 0. \tag{27}$$

This difference is zero because both photons are traveling from $r = 0$ to R. We rewrite this as

$$\frac{\delta t_1}{\delta t_2} = \frac{a(t_1)}{a(t_2)}. \tag{28}$$

If we set δt_1 to be one wavelength then the photon arrives time dilated by a ratio of $a(t_1)/a(t_2)$. We define redshift in terms of this time dilation

$$1 + z = \frac{\lambda_2}{\lambda_1} = \frac{a(t_2)}{a(t_1)}. \tag{29}$$

for an observer at R and t_2. Usually we see this as

$$\frac{a(t)}{a_0} \equiv \frac{1}{1+z} \tag{30}$$

where a_0 is the scale factor at $z = 0$.

The Hubble parameter describing the expansion rate of the universe

$$H = \frac{\dot{a}}{a}. \tag{31}$$

Taking the derivative of the inverse of equation 30

$$dz = -\frac{a_0}{a}\frac{\dot{a}}{a}dt = -(1+z)Hdt \tag{32}$$

or more commonly

$$dt = -\frac{dz}{1+z}H^{-1}. \tag{33}$$

Angular and luminosity distances can be written in terms of integrals that involve the above expression.

1.8. **Distance Measure.** Recall that for a null geodesic in flat space time

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{r_e}^{r_o} dr \tag{34}$$

where r_e, t_e are coordinates for the emitted and r_o, t_o are those for the observer. We can write

$$\frac{dt}{a} = \frac{dt}{a}\frac{da}{da} = \frac{da}{\dot{a}a} = \frac{da}{Ha^2} \tag{35}$$

where we have used $H = \dot{a}/a$. The relation for redshift in terms of the current coordinate

$$\frac{a_0}{a} = (1+z) \tag{36}$$

Taking the derivative of this

$$dz = -\frac{a_0}{a^2}da. \tag{37}$$

Inserting this into equation 35,

$$\frac{dt}{a} = -\frac{dz}{H(z)a_0} \tag{38}$$

Setting the time of the observer to zero and assuming that $t_e < 0$

$$r_e(z) \equiv r_o - r_e = \int_0^z \frac{dz'}{H(z')a_0}. \tag{39}$$

This is sometimes called the **distance measure** since it is related to coordinate distance as defined by the metric.

It is convenient to define an integral

$$D_0(z) = \int_0^z \frac{dz' H_0}{H(z')} \tag{40}$$

and write a number of things in terms of this integral.

- Distance measure $r_e(z) = \frac{D_0(z)}{a_0 H_0}$.
- Angular distance $d_A(z) = D_0(z)(1+z)^{-1}H_0^{-1}$.
- Luminosity distance $d_L(z) = D_0(z)(1+z)H_0^{-1}$.

and we have assumed that $k = 0$ for these. (If $k \neq 0$ then factors of sin or sinh $\Omega_k D_0$ are required).

1.9. **Angular and Luminosity Distances.** Using the coordinate system of the FRW metric (equation 25), consider two positions on the manifold separated by angle θ but with the same r, t, ϕ. The proper distance between these two points is $s^2 = a(t)^2 r^2 \theta^2$ or $s = a(t) r\theta$. Now consider an observer who detects light from both positions. We let the observer be at $r = 0$. Let the emitter be at z. There is a difference between the coordinate time of emitter and observer. The angular distance is that giving $d_A = s/\theta$. Because s depends on z so does d_A. We need to relate r to z to determine how $d_A(z)$ depends on z of the emitter.

$$d_A = \frac{s}{\theta} = a(t)r \tag{41}$$

For light (and using the FRW metric; equation 25)

$$dt = a(t)\frac{dr}{\sqrt{1-kr^2}}. \tag{42}$$

Integrating for $k = 0$

$$\int_{-t_e}^0 \frac{dt}{a(t)} = r \tag{43}$$

and so

$$d_A = -a(t)\int_0^{t_e} \frac{dt'}{a(t')} \tag{44}$$

In terms of the redshift (equation 33), the angular distance

$$d_A(z) = (1+z)^{-1}\int_0^z \frac{dz'}{H(z')}. \tag{45}$$

It is possible to modify this expression to take into account non-zero k using sin or sinh of r.

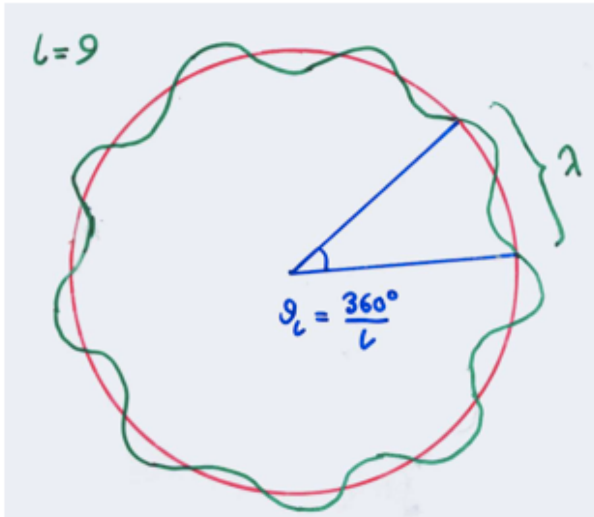

FIGURE 1. Relation between multipole number, l, and angular scale; here for $l = 9$ (corresponding to 9 wavelengths). The context here is the microwave background. The physical scale $\lambda = d_A\theta = d_A 2\pi/l$ where $d_A(z)$ is the angular distance. Figure from Räsänen's lecture notes, chap 13.

Flux is proportional to the inverse square of distance. We define a luminosity distance d_L such that the observed flux

$$F = \frac{L}{4\pi d_L^2} \tag{46}$$

where L is the actual luminosity of the emitting object.

Luminosity is numbers of photons per unit time per unit area times the average energy of a photon. Consider a source emitting photons. Photons will arrive at lower energies and will arrive further apart due to time dilation. Both effects depend on $1+z$, so the observed flux depends on $(1+z)^2$. The luminosity distance to take into account these factors. Because the flux depends on the inverse square of the distance

$$d_L = (1+z)r \tag{47}$$

where r is the coordinate distance between observer and emitter. This gives

$$d_L(z) = (1+z)^2 d_A(z). \tag{48}$$

1.10. **Comoving and proper distance.** Sometimes it is useful to discuss the distance between two points that have the *same* coordinate time or that are on the same slice of constant time. This is sometimes called **proper distance**. The proper distance assumes that between points $dt = 0$ so that $ds = a(t)dr$. Assume that the two points are separated by ΔR, then the proper distance

$$d_P = a(t) \int_R^{R+\Delta R} dr = a(t)\Delta R. \tag{49}$$

The proper distance between two points expands as the universe expands. One can use comoving coordinates that also expand as the universe expands. The **comoving proper distance** weights the proper distance by the ratio of the scale factors between emitted and observer. Putting the observer at the present time

$$d_P^c = \frac{a_0}{a_1} d_P = (1+z)d_P. \tag{50}$$

Consider two points that have no peculiar velocity with respect to the comoving coordinate frame and are separated by ΔR. At t_1 they would have proper distance $d_P(t_1) = a_1 \Delta R$. At t_0 they would have proper distance $d_P(t_0) = a_0 \Delta R$. However their comoving proper distance would be equal to the proper distance at t_0 and would not change.

We can ask a subtle question: Supposing we see a galaxy at redshift z. The light must travel to reach us so the galaxy is not nearby. Assuming that neither we or the galaxy have a significant peculiar velocity what would the *proper distance* to this galaxy be at the current cosmological time? We know that $R = \int \frac{dt}{a} = r_e(z)$ because light travels along null geodesics. This is the radial coordinate and is equivalent to the *comoving proper distance*. In units of a_0

$$d_P^c(z) = r a_0 = a_0 \int \frac{dt}{a} = \int \frac{da a_0}{\dot{a} a} = \int_0^z \frac{dz'}{H(z')}. \tag{51}$$

The maximum distance that we can see corresponds to

$$d_{hor}^c = \lim_{z\to\infty} d_P^c(z) = \int_0^\infty \frac{dz'}{H(z')} \tag{52}$$

This is a *horizon* distance and is called the **particle horizon**. It corresponds to the maximum coordinate r distance (comoving) from which we could have received information. It may be convenient to write the above equation in terms of da

$$d_{hor}^c = \int_0^{a_0} \frac{da\ a_0}{a^2 H(a)} \tag{53}$$

Comoving distances are given with respect to the r coordinate so that they correspond to points on the underlying manifold that is expanding. A particle with no peculiar velocity would keep the same r coordinate as the manifold expands. These should be compared to distances that involve integrating with the metric, or integrated dt for null geodesics.

1.11. **Age of the universe.** We can ask how much coordinate time it takes to expand from a_1 to a_2 (or from z_1 to z_2)?

$$dt = \frac{da}{\dot{a}} = -\frac{dz}{1+z}H^{-1} \tag{54}$$

where I have used equations 36 and 37. An age

$$t_2 - t_1 = \int_{z_1}^{z_2} \frac{dz'}{(1+z')H(z')} \tag{55}$$

The age of the universe at z

$$t(z) = \int_{z}^{\infty} \frac{dz'}{(1+z')H(z')} \tag{56}$$

and the present age of the universe

$$t_0 = \int_{0}^{\infty} \frac{dz'}{(1+z')H(z')}. \tag{57}$$

These can also be done as integrals with respect to da.

$$t_0 = \int_0^{a_0} \frac{da}{Ha} \sim \int \frac{d\ln a}{H} \tag{58}$$

The above motivates using H_0^{-1} as the approximate age of the universe.

1.12. **Hubble and Horizon lengths.** For a null geodesic $dr = dt/a$. Consider the distance in r that could be travelled in a Hubble time. The Hubble time is H^{-1} so this distance is $\frac{1}{Ha}$. A horizon distance is

$$d_{hor}(t) \sim \frac{1}{aH} = \mathcal{H}^{-1} \tag{59}$$

where $\mathcal{H} \equiv aH$. d_{hor} is approximately how far (in r) light can get during the entire age of the universe.

Consider the time of photon-decoupling t_{dec}. A problem arises when

$$\frac{d_{hor}(t_{dec})}{d_{hor}(t_0)} \sim \frac{a_0 H_0}{a_{dec} H_{dec}} \ll 1 \tag{60}$$

This condition implies that the current horizon scale is much larger than that at decoupling. In other words regions on opposite sides of the sky that we can see at our current time would never have detected light from each other during previous times. The problem occurs when

$$\frac{d}{dt}(aH) = \ddot{a} < 0 \tag{61}$$

or when the universe is decelerating. A period of time with $\ddot{a} > 0$, with acceleration can make it possible for previous even horizons to be larger than current ones.

We can define a **Hubble length** as

$$l_H \equiv H^{-1}. \tag{62}$$

Note if H is in units of inverse time, then we can also consider it in units of inverse length, as $c = 1$. As H^{-1} is approximately the age of the universe, then it is also the distance that light can travel during the age of the universe.

The **comoving Hubble length** is defined as

$$l_H^c \equiv \frac{l_H}{a} = \frac{1}{Ha} = \mathcal{H}^{-1} \tag{63}$$

weighting the Hubble length by the current expansion factor a. The comoving Hubble length gives the comoving distance that light travels in a cosmological timescale. This gives the coordinate distance r that light travels in a cosmological timescale.

Consider a time t and light paths beginning at time t. The coordinate (or comoving) distance that photons can travel from t to t_{max} is

$$r = \int_t^{t_{max}} \frac{dt'}{a(t')} \tag{64}$$

Remembering that null geodesics have $dt = a(t)dr$ this corresponds to a difference in time coordinate (metric distance) of

$$d_e \equiv a(t)r \equiv a(t) \int_t^{t_{max}} \frac{dt'}{a(t')} \tag{65}$$

This distance is called the **event horizon**. The event horizon d_e might be finite as $t_{max} \to \infty$, in which case the event horizon gives a maximal distance over which information can be communicated in the future.

We can define the **particle horizon** as

$$d_p(t) \equiv a(t) \int_{t_{min}}^{t} \frac{dt'}{a(t')} \tag{66}$$

the extent of regions admitting causal connection (from the past) at time t. If the integral converges as $t_{min} \to 0$ we say there exists a particle horizon.

2. Friedmann equations

Components of Einstein's equation for a Friedmann-Robertson-Walker metric can be reduced to

$$3\frac{\dot{a}^2}{a^2} + 3\frac{K}{a^2} = 8\pi G\rho \tag{67}$$

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) \tag{68}$$

An additional equation that arises (called the energy continuity equation)

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \tag{69}$$

These equation are often called "Friedmann equations." The first one comes from the T_{00} component and Einstein's field equation. The second one from T_{ii} diagonal components. The third one follows from the conservation law $T^{ij}_{;j} = 0$.

For acceleration or $\ddot{a} > 0$, equation 68 implies that $\rho + 3p < 0$ must be true. This condition is satisfied during *inflation*.

2.1. Critical density.

Rewriting equation 67 in terms of $H = \dot{a}/a$

$$H^2 + \frac{K}{a^2} = \frac{8\pi G\rho}{3}. \tag{70}$$

The Hubble constant

$$H_0 = \frac{\dot{a}|_0}{a_0}. \tag{71}$$

Taking equation 67 or 70 at the current time

$$H_0^2 + \frac{K}{a_0^2} = \frac{8\pi G\rho_0}{3} \tag{72}$$

where ρ_0 is the current energy density. Solving for ρ_0

$$\rho_0 = \frac{3H_0^2}{8\pi G} + \frac{3K}{8\pi G a_0^2} \tag{73}$$

We define the current critical density

$$\rho_{c0} \equiv \frac{3H_0^2}{8\pi G} \tag{74}$$

that which gives a flat universe (when $K = 0$). The critical density can be defined as a function of coordinate time

$$\rho_c(t) = \frac{3H^2}{8\pi G}. \tag{75}$$

A density parameter, $\Omega(t)$, is defined in terms of the critical density

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \tag{76}$$

A spatial curvature parameter

$$\Omega_K(t) = -\frac{K}{a^2 H^2} \tag{77}$$

allows equation 67 or 70 to be written as

$$\Omega + \Omega_K = 1. \tag{78}$$

2.2. **Matter components.** Equation 69 I write again here

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

can be rewritten as

$$p = -\rho - \frac{\dot{\rho} a}{3\dot{a}} = -\frac{1}{3Ha^3}\frac{d(\rho a^3)}{dt}. \tag{79}$$

This equation lets us determine how a varies with t for various types of fluids. As long as there are no strong interactions this equation is satisfied separately for each fluid.

- **Matter.** If the fluid contains matter that has mass and is non-relativistic then the pressure can be neglected. In this case equation 79 implies that $\rho_m a^3 =$ constant and $\rho_m \propto a^{-3}$.
- **Radiation.** Photons or relativistic particles satisfy $p_r = \rho_r/3$. Using this assumption equation 69 can be rewritten as

$$\dot{\rho}_r = -\frac{4\rho\dot{a}}{a} \tag{80}$$

  This equation can be manipulated to show that $\rho_r a^4$ is constant or $\rho_r \propto a^{-4}$. We note that the energy density in radiation drops faster than matter as a increases.
- **Vacuum Energy.** For vacuum energy the density is constant. In this case equation 79 implies that $p_{vac} = -\rho_{vac}$. This type of matter is also called the 'cosmological constant' with

$$\Lambda \equiv 8\pi G \rho_{vac}. \tag{81}$$

  This constant is sometimes directly inserted into Einstein's field equation; $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$.

We can divide ρ into three components $\rho_m, \rho_r, \rho_{vac}$. For each component we can define a density or Ω parameter. Equation 67 becomes

$$1 = \Omega_K + \Omega_m + \Omega_r + \Omega_{vac}. \tag{82}$$

As long as there are no interactions between matter components then equation 69 and equation 79 are satisfied by each matter component separately. Consequently even when there are multiple matter components $\rho_m \propto a^{-3}, \rho_r \propto a^{-4}, \rho_{vac} =$ constant.

2.3. **How the Hubble parameter varies with z; $H(z)$.** The Friedmann equation (equation 67) with matter, radiation and vacuum energy

$$H^2 = -\frac{K}{a^2} + \frac{8\pi G}{3}\left(\rho_m + \rho_r + \rho_{vac}\right) \tag{83}$$

Using how each matter component depends on a and scaling from a_0

$$H^2 = -\frac{K}{a^2} + \frac{8\pi G}{3}\left(\frac{\rho_{m0}a_0^3}{a^3} + \frac{\rho_r a_0^4}{a^4} + \rho_{vac}\right) \tag{84}$$

Dividing by H_0^2

$$\frac{H(z)}{H_0} = \sqrt{\Omega_{K0}(1+z)^2 + \Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0}} \tag{85}$$

where the density parameters are evaluated at $z = 0$. This convenient form gives us $H(z)$ which is useful for calculating sizescales of various things.

2.4. **Expansion laws.** Here we assume that one component of the density dominates. We then determine how a depends on t. Let us define a parameter that is related to the equation of state

$$w = \frac{p}{\rho}. \tag{86}$$

Assuming that w is a constant (as for our three above discussed types of matter components) and using equation 69 repeated here

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \tag{87}$$

we find that

$$\rho \propto a^{-3(1+w)}. \tag{88}$$

Friedmann's equation with $K = 0$ (equation 67) can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3}. \tag{89}$$

Inserting equation 88 into the above and we find that

$$(\dot{a})^2 \propto a^{-(1+3w)} \tag{90}$$

and as long as $w > -1$,

$$a \propto (t - t_0)^{\frac{2}{3(1+w)}}. \tag{91}$$

Our three types of matter have

- **Matter.** Here $p = 0$ so $w = 0$, and $a \propto t^{\frac{2}{3}}$.
- **Radiation.** Here $p = \rho/3$ so $w = 1/3$ and $a \propto t^{\frac{1}{2}}$.
- **Vacuum energy.** Here $p = -\rho$ so $w = -1$. Friedmann's equation with $K = 0$ (equation 89) gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_{vac}}{3} \tag{92}$$

with solution

$$a \propto e^{\left(\frac{8\pi G\rho_{vac}}{3}\right)^{1/2} t}. \tag{93}$$

The exponential form is related to *inflation*. Looking back at equation 68 ($3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p)$), accelerating expansion (aka inflation), $\ddot{a} > 0$, only can happen when $\rho + 3p < 0$.

Above we have considered expansion laws when a single component dominates ρ. In general ρ contains multiple components. At different times, different components dominate. At early times, a cosmological constant (or something like it) may dominate and we have inflation. Then radiation dominates. And finally in the current era, mass dominates.

3. References and Acknowledgments

We have primarily been following Syksy Räsänen's lovely lecture notes from 2011.