# AST242 LECTURE NOTES PART 3

## Contents

## 1. Viscous flows

Up to this time we have ignored viscosity. The most common astrophysical application of viscosity is the accretion disk, so we will introduce viscosity specifically to study viscous evolution of accretion disks.

We showed earlier that conservation of momentum could be written as

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi} = -\rho \nabla \Phi \tag{1}$$

with stress tensor $\pi_{ij} = p\delta_{ij} + \rho u_i u_j$ giving the momentum flux. We now consider the momentum flux caused by viscosity and add this *viscous stress tensor* to the stress tensor above coming from bulk flow and pressure.

As we discussed earlier the $ij$ component of the stress tensor is the $i$-th component of the force per unit area on a surface with normal in the $j$ direction. The $ij$ component of the stress tensor is also the $i$-th component of the momentum density through a surface with normal in the $j$ direction.

### 1.1. **The velocity gradient tensor.**

If there is no gradient in velocity then we expect no stress. We expect viscous forces to depend on the gradient of the velocity, $\boldsymbol{\nabla}\mathbf{u}$, however this is a 2 index tensor as each component is $\frac{\partial u_i}{\partial x_j}$. It can be described as a $3 \times 3$ matrix, each index covering $xyz$. We can decompose the gradient of the velocity into three components: a traceless symmetric component, $\boldsymbol{\sigma}$, a traceless rotation component, $\mathbf{r}$, and a component that has the trace $\theta$;

$$\boldsymbol{\nabla}\mathbf{u} = \frac{1}{3}\theta \mathbf{g} + \boldsymbol{\sigma} + \mathbf{r} \tag{2}$$

where $\mathbf{g}$ is the tensor $\delta_{ij}$ and

$$\theta = \boldsymbol{\nabla} \cdot \mathbf{u} \tag{3}$$

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\theta\delta_{ij} \tag{4}$$

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

We explain why the tensor $\mathbf{r}$ is equivalent to solid body rotation of the fluid element and so should not cause any viscous stress. For a body in solid body rotation with angular velocity $\boldsymbol{\Omega}$ the velocity at each point is $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$ or using summation notation

$$u_i = \epsilon_{ijk}\Omega_j x_k \tag{6}$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol (see http://en.wikipedia.org/wiki/Levi-Civita_symbol). Let's be specific about he Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 0, & & \text{any two labels are the same} \\ 1, & \text{if} & i,j,k \text{ is an even permutation of } 1,2,3 \\ -1, & & i,j,k \text{ is an odd permutation of } 1,2,3 \end{cases} \tag{7}$$

Any cross product $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ can be written

$$A_i = \epsilon_{ijk} B_j C_k \tag{8}$$

where summation notation means any repeated index is summed over all coordinates.

Taking the derivative of equation (6)

$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ikl}\Omega_k \frac{\partial x_l}{\partial x_j} = \epsilon_{ikl}\Omega_k \delta_{lj} = \epsilon_{ikj}\Omega_k \tag{9}$$

This is antisymmetric. Let us take this expression for $u_{i,j}$ and with it create an antisymmetric tensor like $\mathbf{r}$

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) &= \frac{1}{2}\left(\epsilon_{ikj} - \epsilon_{jki}\right)\Omega_k \\ &= \frac{1}{2}\left(\epsilon_{ikj} + \epsilon_{ikj}\right)\Omega_k \\ &= \epsilon_{ikj}\Omega_k \end{aligned} \tag{10}$$

How many degrees of freedom does an antisymmetric $3 \times 3$ matrix have? The diagonal terms must be zero. There are 6 non-diagonal terms and they must come in pairs, positive and negative. Therefore there are 3 degrees of freedom. There are also three components of the vector $\Omega$. This means that *any* $\mathbf{r}$ can be described in terms of a solid body rotation with

$$r_{ij} = \epsilon_{ikj}\Omega_k \tag{11}$$

Because our antisymmetric tensor $\mathbf{r}$ describes the solid body rotational fluid motion, it can cause no viscous stress.

1.2. **Viscous stress tensor.** It has been found experimentally that the magnitude of the shear stress in viscous flows is often proportional to the symmetric components of the velocity gradient. This is an analogy to Hooke's law. In other words

$$\mathbf{T}_{visc} \quad \text{approximately} \quad \propto \boldsymbol{\nabla}\mathbf{v} \tag{12}$$

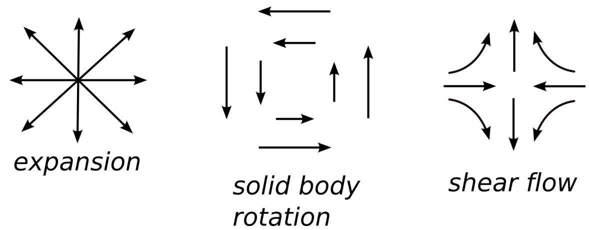


FIGURE 1. The velocity gradient tensor is decomposed into three pieces, the trace, describing compression or expansion, solid body rotation and compression-less shear.

More exactly, using two coefficients $\zeta, \eta$, the component of the stress tensor due to viscosity is written as

$$\mathbf{T}_{visc} = -\zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma} \tag{13}$$

and there is no force if the fluid is simply rotating so no dependence on $\mathbf{r}$. The first coefficient, $\zeta$, is known as the bulk viscosity and only is important in compressible flows. The second term, $\eta$, is sometimes called the shear viscosity. Bulk viscosity is often neglected in astrophysical flows except when considering the structure of shocks. Both bulk and shear viscosity are often assumed to be independent of position and temperature. Here we have assumed that the viscous strain is proportional to the velocity gradient. When this is true the fluid is said to be 'Newtonian'. Some materials have memory or behave similar to solids when there is a rapidly varying pressure gradient but behave like fluids when the forces on them are slowly changing. These would require modifications to the stress tensor. For example, see the NCFM movie on rheological properties of fluids for some examples of non-Newtonian fluids.

The gradient of the viscous stress tensor is a term that we can add to our equation for conservation of momentum;

$$\nabla\mathbf{T}_{visc} = -\nabla(\zeta\nabla\cdot\mathbf{u}) - 2\nabla\cdot(\eta\boldsymbol{\sigma}) \tag{14}$$

If we assume that the bulk and shear viscosity are independent of position then we can more easily compute the $i$-th component of the **second** term (in summation notation, summing over $j$ index)

$$\begin{aligned} -2\eta\frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij}\right] &= -\eta\left[\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} - \frac{2}{3}\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}\right] \\ &= -\eta\left[\sum_j \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3}\frac{\partial}{\partial x_i}(\nabla\cdot\mathbf{u})\right] \end{aligned} \tag{15}$$

We can then write

$$\nabla \mathbf{T}_{visc} \sim -(\zeta + \eta/3)\nabla(\nabla \cdot \mathbf{u}) - \eta\nabla^2\mathbf{u}. \tag{16}$$

Assuming incompressible flow (so $\nabla \cdot \mathbf{u} = 0$) and inserting this component of the stress tensor into our equation for conservation of momentum we find

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} \tag{17}$$

where

$$\nu \equiv \frac{\eta}{\rho} \tag{18}$$

is the **kinematic** viscosity and we have assumed that $\nu$ does not vary with position in the fluid. The above equation is known as the **Navier-Stokes** equation. Let us be specific about the components. For the $i$-th component of this equation the viscosity term looks like

$$\nu \sum_j \frac{\partial^2 u_i}{\partial x_j^2} \tag{19}$$

If the flow is compressible then more generally

$$\rho \frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla} p + (\zeta + \eta/3)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \eta \boldsymbol{\nabla}^2 \mathbf{u} \tag{20}$$

though extra terms can be added if $\eta$ and $\zeta$ are dependent on position.

Note that the addition of second order derivatives means that integrations require additional boundary conditions. Viscosity is a form of energy dissipation. When viscosity is important there is a corresponding term in the energy conservation equation that depends on the viscosity and gradients of the velocity.

### 1.3. **Navier Stokes equation – diffusion.**

Consider the Navier Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} \tag{21}$$

Consider a region with a really steep velocity gradient. The second derivatives dominate the first ones and

$$\frac{\partial \mathbf{u}}{\partial t} \sim \nu \boldsymbol{\nabla}^2 \mathbf{u} \tag{22}$$

This can be recognized as a diffusion equation with the kinematic viscosity, $\nu$, the coefficient of diffusion. Viscosity tends to reduce velocity gradients.

1.4. **Viscosity to order of magnitude.** Note that the kinematic viscosity, $\nu$, has units cm$^2$/s like a diffusion coefficient. For a gas it can be approximated as $v^2\tau$ or $\lambda^2/\tau$ or $v\lambda$ where $v$ is a mean thermal velocity, $\tau$ the collision timescale and $\lambda$ the mean free path. For a turbulent medium one can use a mean eddy velocity and length scale.

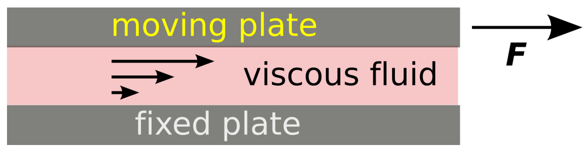


FIGURE 2. Two flat plates separated by a small distance, $h$, that is filled with a viscous fluid. A force applied to the top plate allows it to move. The force depends on the viscosity of the fluid.

1.5. **Example using the viscous stress tensor: The force on a moving plate.** A plate $h = 0.01$ cm from a fixed plate moves at a velocity of $v$ =100 cm/s. In between the two flat plates is a fluid with dynamic viscosity of water or 8.9 $\times 10^{-3}$ Pa s. In cgs 1 Poise = 0.1 Pa s. So in cgs the dynamic viscosity is 8.9 $\times 10^{-4}$ Poise. *What is the force per unit area needed to maintain this velocity?*

We take $x$ in the direction that we are pushing the plate and $z$ the direction perpendicular to the plates. We can let $z = 0$ at the bottom plate and $z = 0.01$ cm at the top plate. The fluid velocity where it touches the plate should be the same as the plate.

$$\mathbf{u} = (v\frac{z}{h}, 0, 0) \tag{23}$$

We can assume the bottom plate is not moving so has $u_x = 0$ and the top plate is moving at $u_x = 100$cm/s. The gradient of the $x$ component of velocity

$$\frac{\partial u_x}{\partial z} = \frac{v}{h} = \frac{100\text{cm/s}}{0.01\text{cm}} = 10^4\text{s}^{-1} \tag{24}$$

Let's consider the traceless symmetric part of the velocity gradient $\boldsymbol{\sigma}$. We have no motion in the $z$ or $y$ directions. The gradients in the $x$ and $y$ direction of $u_x$ are zero. The only part of $\boldsymbol{\sigma}$ that is non-zero is the part containing $\frac{\partial u_x}{\partial z}$.

The viscous stress tensor gives the components of the force per unit area through a surface. Consider component $T_{xz}$ of the viscous stress tensor. This gives the force per unit area in the $x$ direction through a surface oriented with normal along the $z$ direction. Ignoring terms containing $\nabla \cdot \mathbf{u}$ which is zero if the fluid is incompressible

$$T_{visc} = -2\eta\boldsymbol{\sigma} \tag{25}$$

As the only gradient of the velocity that is non-zero is $\frac{\partial u_x}{\partial z}$ so we know that $T_{xx} = T_{yy} = T_{zz} = T_{yz} = T_{xy} = 0$. We now compute $T_{xz}$

$$\mathbf{T}_{xz} = -2\eta\boldsymbol{\sigma}_{xz} = -2\eta\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = -\eta\frac{\partial u_x}{\partial z} \tag{26}$$

Using the dynamic viscosity of water and our velocity gradient we find that

$$\frac{F}{A} = T_{visc,xz} = -\eta\frac{\partial u}{\partial x} = -8.9 \times 10^{-4}\frac{100}{0.01} = -8.9 \text{ dynes cm}^{-2} \tag{27}$$

where the sign is in the direction opposite to the flow; the fluid opposes the velocity shear.

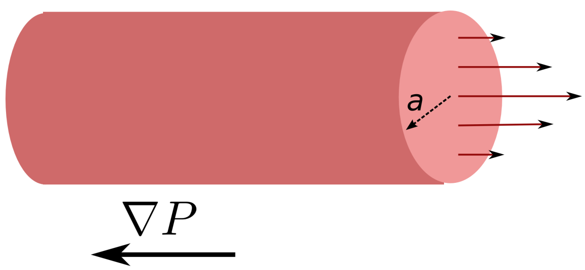


FIGURE 3. Flow of blood in a capillary that induced by a pressure gradient.

1.6. **Example using the Navier Stokes equation – Poiseuille flow or Flow in a capillary.** Consider the viscous flow of blood in a capillary. Supposing there is a constant pressure gradient $\nabla P$ along the capillary of radius $a$. We orient our coordinate system so the capillary extends along the $z$ direction. We assume cylindrical symmetry. We would like to find the velocity profile $u(r)$. We assume there is only flow in the $z$ direction and the flow is steady. We have a boundary condition $u = 0$ at $r = a$. Here is a summary of boundary conditions and assumptions

$$\nabla P = \frac{dP}{dz}\hat{\mathbf{z}} \tag{28}$$

$$\mathbf{u} = u(r)\ \hat{\mathbf{z}} \tag{29}$$

$$u(r = a) = 0 \tag{30}$$

$$\frac{\partial \mathbf{u}}{\partial z} = 0 \tag{31}$$

$$\frac{\partial \mathbf{u}}{\partial t} = 0 \tag{32}$$

$$\frac{\partial \mathbf{u}}{\partial \phi} = 0 \tag{33}$$

The Navier-Stokes equation for steady flow

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} P + \nu \boldsymbol{\nabla}^2 \mathbf{u} \tag{34}$$

As $\mathbf{u} \perp \boldsymbol{\nabla} \mathbf{u} = 0$, the term on the left is zero. There are only radial derivative terms but the only component of interest is the $z$ component as the pressure gradient is zero in other directions. Since there are only radial gradients of $u$ (where $u$ is in the $z$ direction) we need only take the radial term in the diverge term in cylindrical coordinates

$$\frac{\nabla P}{\rho \nu} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) \tag{35}$$

We multiply both sides by $r$ and integrate

$$r \frac{\nabla P}{\rho \nu} = \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) \tag{36}$$

$$r^2 \frac{\nabla P}{2 \rho \nu} + C = r \frac{\partial u}{\partial r} \tag{37}$$

with constant $C$. We divide by $r$ and integrate again

$$r \frac{\nabla P}{2 \rho \nu} + \frac{C}{r} = \frac{\partial u}{\partial r} \tag{38}$$

$$r^2 \frac{\nabla P}{4 \rho \nu} + C \ln r + D = u \tag{39}$$

with additional constant $D$. We now use our boundary conditions to set constants, $C, D$. $D$ must be zero so that $u$ is finite at $r = 0$. The velocity is equal to zero at $r = a$ and this lets us solve for $C$. We find a velocity profile

$$u(r) = \frac{\nabla P}{4 \rho \nu} (r^2 - a^2) \tag{40}$$

The sign of $u$ is such that motion is in the opposite direction as the pressure gradient. Flow is from high to low pressure.

Here we assumed that there were no gradients in the $z$ direction. At the beginning of a pipe, boundary layers caused by viscosity would grow until they meet, and afterwards we could have a smooth steady Poiseuille flow.

1.7. **Viscous Energy Dissipation.** A stress tensor, $\mathbf{T}$, gives a momentum flux or a force per unit area. For example if we have a unit vector $\hat{\mathbf{n}}$

$$\mathbf{A} = \mathbf{T} \cdot \mathbf{n} \qquad \text{or} \qquad A_j = T_{ij} n_i \tag{41}$$

gives the $j$-th force component per unit area on a surface with normal $\hat{\mathbf{n}}$ or equivalently the flux of the $j$-th component of momentum through a surface with normal $\hat{\mathbf{n}}$. Work is force times distance, so

$$\mathbf{A}ds = \mathbf{T} \cdot \mathbf{n}ds \tag{42}$$

is the work per unit area pushing on the surface as it moves a distance $ds$ and

$$\mathbf{B} = \mathbf{T} \cdot \mathbf{u} \qquad \text{with} \qquad B_j = T_{ij}u_i \tag{43}$$

is the work per unit area per unit time for the surface moving with the fluid. Thus $\mathbf{T} \cdot \mathbf{u}$ is an *energy flux*.

Our viscous stress tensor represents an additional momentum flux which can do work on the fluid at a rate $\mathbf{T}_{visc} \cdot \mathbf{u}$ per unit area. A work per unit area is also an energy flux. There is an energy flux caused by viscous dissipation

$$\mathbf{F}_{visc} = \mathbf{T}_{visc} \cdot \mathbf{u} \qquad \text{or} \qquad F_{visc,i} = T_{ij}u_j \tag{44}$$

This energy flux is a vector and has units of energy per unit area per unit time.

The gradient of this energy flux will give us the energy dissipated per unit volume per unit time. Sometimes you hear the energy flux described as stress times strain.

Our equation for energy conservation gains a flux term

$$\boldsymbol{\nabla} \cdot (\mathbf{T}_{visc} \cdot \mathbf{u}) \tag{45}$$

with $\mathbf{T}_{vis} = -\zeta\theta\mathbf{g} - 2\eta\boldsymbol{\sigma}$. Our conservation of energy equation becomes

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E + p)\mathbf{u} + \mathbf{T}_{visc} \cdot \mathbf{u}] = -\rho\dot{Q}_{cool} + \rho\frac{\partial\Phi}{\partial t} - \nabla \cdot \mathbf{h} \tag{46}$$

In the above $E$ is the total energy per unit volume. Here $\dot{Q}_{cool}$ is the cooling rate per unit mass, $\mathbf{h}$ is the heat flux due to thermal conductivity (energy per unit area per unit time).

To make it clearer how much energy is due to viscous dissipation we would like to find $TdS/dt$ due to viscous heating. Consider our conservation equation for conservation of momentum

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (p\mathbf{g} + \rho\mathbf{u} \otimes \mathbf{u} + \mathbf{T}_{visc}) = -\rho\nabla\Phi \tag{47}$$

If we take the dot product of $\mathbf{u}$ times the conservation law for momentum above and subtract it from our equation for energy conservation (equation 46) we find

$$\rho T\frac{dS}{dt} = -\mathbf{T}_{visc} : \nabla\mathbf{u} - \rho\dot{Q}_{cool} + \rho\frac{\partial\Phi}{\partial t} - \nabla \cdot \mathbf{h} \tag{48}$$

Our viscous stress tensor is negative so we get a positive heating rate. In the above form it may be clearer that the heating rate is the stress ($\mathbf{T}$) times the strain, ($\nabla\mathbf{u}$).

We have assumed that the viscous stress tensor is a constant times the strain. The rate of heating can be written then as a viscosity coefficient times a square of the strain. The form of the viscous dissipation term

$$\mathbf{T}_{visc} : \nabla \mathbf{u} = \sum_{ij} T_{ij} \frac{\partial u_i}{\partial x_j} \tag{49}$$

where I have been specific about the indices.

More specifically $\nabla \mathbf{u} = \frac{1}{3}\mathbf{g}\theta + \boldsymbol{\sigma} + \mathbf{r}$ where the trace $\theta = \nabla \cdot \mathbf{u}$, and $\boldsymbol{\sigma}$ is the symmetric traceless component. The antisymmetric component, $\mathbf{r}$, can be described in terms of the rotation or vorticity and $g_{ij} = \delta_{ij}$ for a Cartesian coordinate system. Inserting our form for the viscous stress tensor (equation 14)

$$\rho \dot{Q}_{visc} = -\mathbf{T}_{visc} : \nabla \mathbf{u} = (\zeta \theta \mathbf{g} + 2\eta \boldsymbol{\sigma}) : \left( \frac{\theta \mathbf{g}}{3} + \boldsymbol{\sigma} + \mathbf{r} \right) \tag{50}$$

The antisymmetric term with $\mathbf{r}$ drops out of the sum by symmetry. Consider the double sum of a trace term with a traceless term

$$\theta \mathbf{g} : \boldsymbol{\sigma} = \theta \sum \delta_{ij} \sigma_{ij} = \theta \sum_i \sigma_{ii} = 0 \tag{51}$$

where the last step follows because $\boldsymbol{\sigma}$ has zero trace. Because of the double dot, a trace part on the left requires a non-zero trace term on the right (and vice versa) to give a non-zero term. Consequently

$$\rho \dot{Q}_{visc} = \zeta \theta^2 + 2\eta \boldsymbol{\sigma} : \boldsymbol{\sigma} \tag{52}$$

where

$$\boldsymbol{\sigma} : \boldsymbol{\sigma} = \sum_{ij} \sigma_{ij} \sigma_{ij} = \text{trace } \boldsymbol{\sigma}^2 \tag{53}$$

As $\boldsymbol{\sigma}$ is symmetric it can be diagonalized and the trace of its square is the sum of 3 positive numbers, so $\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq 0$. This implies that $\dot{Q}_{visc} \geq 0$ as expected for dissipation. We can write $\rho \dot{Q}_{visc}$ out as

$$\rho \dot{Q}_{visc} = \zeta \left( \sum_i \frac{\partial u_i}{\partial x_i} \right)^2 + \eta \sum_{i \neq j} \left[ \left( \frac{\partial u_i}{\partial x_j} \right)^2 + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right]. \tag{54}$$

## 2. The Accretion disk

We consider the setting of a disk comprised of gas rings, each in circular rotation around a massive object such as a black hole or star or planet. We use a cylindrical coordinate system $R, \phi, z$ and assume that the disk is vertically thin. We assume

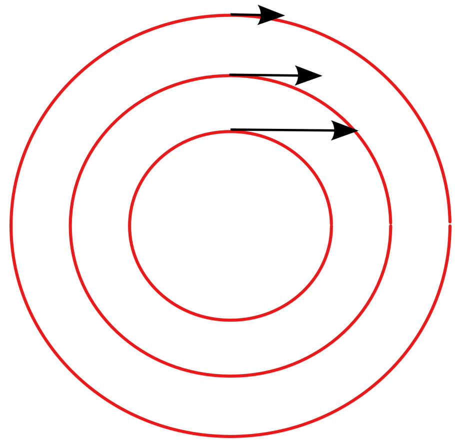

FIGURE 4. In a rotating Keplerian disk, the angular rotation rate drops with increasing radius, giving a velocity shear. This gives a viscous stress that drives radial flow.

that velocity $\mathbf{u}$, and density $\rho$ are independent of $\phi$ so the system is axisymmetric. Integrating the gas density, $\rho$, vertically

$$\Sigma(R,t) = \int_{\infty}^{\infty} \rho(R,z,t)dz \tag{55}$$

where $\Sigma(R,t)$ describe the mass surface density (mass per unit area) in the disk. With the assumption of a thin disk we can also integrate the velocity components along the vertical direction describing the velocity in two dimensions as

$$\mathbf{u} = u_R\hat{\mathbf{R}} + u_\phi\hat{\boldsymbol{\phi}} \tag{56}$$

We can start by assuming that the disk is low mass compared to a central object of mass $M$. The velocity of a particle in a circular orbit

$$v_c(R) = \sqrt{\frac{GM}{R}} \tag{57}$$

and angular rotation rate, $\Omega(R)$

$$\Omega(R) = \dot{\phi} = \frac{v_c}{R} = \sqrt{\frac{GM}{R^3}} \tag{58}$$

Because $\Omega$ depends on radius, there is **differential rotation** and gas rings experience viscous stress that can transfer angular momentum between rings, causing radial inflow or outflow of gas. We will describe the viscosity with a kinematic viscosity $\nu$.

We consider **thin** disks with rotation that is nearly Keplerian

$$u_\phi \sim v_c \tag{59}$$

and assume that

$$|u_R| \ll |u_\phi| \tag{60}$$

With similar formalism we can consider a disk embedded in a galaxy or take into account the self-gravity of the gas disk itself. In these cases $v_c$ and $\Omega$ can still be functions of radius but it would be a different function than given in equations 57 and 58.

Assuming axisymmetry (no $\phi$ dependence) the continuity equation (conservation of mass) becomes

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \tag{61}$$

The Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi + \nu\boldsymbol{\nabla}^2\mathbf{u} \tag{62}$$

We expect both gravity and pressure to have gradients proportion to $\hat{\mathbf{r}}$. The $\phi$ component of the Navier-Stokes equation in cylindrical polar coordinates becomes

$$\Sigma\left(\frac{\partial u_\phi}{\partial t} + u_R\frac{\partial u_\phi}{\partial R} + \frac{u_R u_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial u_\phi}{\partial R}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma u_\phi\right) - \frac{\nu\Sigma u_\phi}{R^2} \tag{63}$$

There is no pressure or gravity derivatives because the above is the $\phi$ component though the tangential velocity will depend on the gravitational potential. All terms on the right hand side contain the viscosity and so are due to viscous stress.

It is useful to rewrite the above equation in terms of $\Omega = u_\phi/R$. Combining the continuity equation (times $Ru_\phi$) and the above (times $R$) we find

$$\frac{\partial}{\partial t}\left(R\Sigma u_\phi\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu R^3\Sigma\frac{d\Omega}{dR}\right) \tag{64}$$

Angular momentum per unit area is

$$l = \Sigma R u_\phi \tag{65}$$

so the first term is the rate of change of the angular momentum per unit area. The second term is the radial flux of the angular momentum per unit area due to advection at velocity $u_R$. So we can write equation 64 as

$$\frac{\partial l}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(Rlu_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu R^3\Sigma\frac{d\Omega}{dR}\right) \tag{66}$$

The right hand side describes the torque due to viscosity.

Let's consider the viscous stress tensor. The only component of it that we need is the $T_{visc,R\phi}$ component that depends on the velocity shear due to differential rotation. This is the gradient of the tangential velocity component in the radial direction. If the angular rotation rate, $\Omega \equiv \frac{u_\phi}{R}$ is constant there is no shear. The gradient in the $R$ direction of the tangential velocity component must therefore be $R\frac{d\Omega}{dR}$, so we expect

$$T_{visc,r\phi} = -\nu \Sigma R \frac{d\Omega}{dR} \tag{67}$$

The viscous stress tensor gives the momentum density flux. We multiply by $R$ to estimate the angular momentum density flux. The divergence of that is what we see on the right hand side of equation (64) which we can now write as

$$\frac{\partial l}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(Rlu_R) = \frac{1}{R}\frac{\partial}{\partial R}\left(-R^2 T_{visc,r\phi}\right) \tag{68}$$

If we multiply both sides of equation (64) by $2\pi R dR$ we can consider the angular momentum in a ring of radius $R$ with width $dR$. The right hand side then gives the net torque on the annulus or the torque at $R$

$$2\pi dR \frac{\partial}{\partial R}\left(\nu R^3 \Sigma \frac{d\Omega}{dR}\right) = 2\pi \nu \Sigma R^3 \frac{d\Omega}{dR} \tag{69}$$

If there is an additional torque on the disk, for example from spiral density waves driven by a planet at a resonance, it would be inserted into equation (64) in the same position as the viscous torque term given above.

We can use a Keplerian approximation $u_\phi = \sqrt{GM/R}$ for mass $M$. It is helpful to compute

$$\frac{d\Omega}{dR} = -\frac{3}{2}\frac{\Omega}{R} \tag{70}$$

Substituting $u_\phi$ and $d\Omega/dR$ into equation (64),

$$\begin{aligned}
\frac{\partial}{\partial t}\left(R^{1/2}\Sigma\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R^{3/2}\Sigma u_R + \frac{3}{2}\nu R^{1/2}\Sigma\right) &= 0 \qquad (71)\\
R^{1/2}\frac{\partial \Sigma}{\partial t} + \frac{3}{2}R^{-1/2}\Sigma u_R + R^{1/2}(\Sigma_{,R}u_R + \Sigma u_{R,R}) + \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{3}{2}\nu R^{1/2}\Sigma\right) &= 0\\
R^{3/2}\frac{\partial \Sigma}{\partial t} + \frac{3}{2}R^{1/2}\Sigma u_R + R^{3/2}\left(u_{R,R}\Sigma + u_R\Sigma_{,R}\right) + \frac{\partial}{\partial R}\left(\frac{3}{2}\nu R^{1/2}\Sigma\right) &= 0 \qquad (72)
\end{aligned}$$

where for the second equation we have multiplied by $R$. Our continuity equation gives us

$$\frac{\partial \Sigma}{\partial t} + \frac{\Sigma u_R}{R} + \Sigma_{,R}u_R + \Sigma u_{R,R} = 0 \tag{73}$$

Subbing in to equation 72 for $\frac{\Sigma u_R}{R} + \Sigma_{,R} u_R$ we find a relation for the radial velocity for a Keplerian disk

$$u_R = -3\Sigma^{-1} R^{-1/2} \frac{\partial}{\partial R}\left(\nu R^{1/2}\Sigma\right) \tag{74}$$

Eliminating $u_R$ from equation (71) we find

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right] \tag{75}$$

which can also be written

$$\frac{\partial \Sigma}{\partial t} = -\frac{3}{R}\frac{\partial}{\partial R}\left(3R\Sigma u_R\right) \tag{76}$$

The second derivative in equation (75) implies that there are diffusive terms in the evolution of the density as well as advective or accretion terms. We might call "accretion" a flow that takes place even if the density is very smooth and diffusion a flow that depends on the gradient of the density. Looking again at equation (76) that holds for a Keplerian system. The mass flux through the disk at radius $R$ is

$$\dot{M} = 2\pi R\Sigma u_R \tag{77}$$

We see from a comparison of the mass flux with equation (76) that a disk with constant mass flux is also a steady state disk ($\dot{\Sigma} = 0$).

The more general expression for $u_R$ in a non-Keplerian setting (using the same procedure as followed above) is

$$u_R = \frac{1}{R\Sigma}(2u_\phi + R^2\Omega_{,R})^{-1}\frac{\partial}{\partial R}\left(\nu R^3 \Sigma \frac{d\Omega}{dR}\right) \tag{78}$$

The sign of the radial velocity, $u_R$, is set by the sign of the derivative. The derivative is the same one giving angular momentum flux on the right hand side of equation (64) and so determines the direction of angular momentum transport. If the derivative is positive then one could have an excretion disk rather than an accretion disk.

2.1. **Accretion to order of magnitude.** Going back to equation (64)

$$\frac{\partial}{\partial t}\left(R\Sigma u_\phi\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu R^3\Sigma\frac{d\Omega}{dR}\right) \tag{79}$$

We consider a steady solution (so drop the first term on left) and consider the typical size of each term. The second term $\sim \Sigma u_\phi u_R$. The right hand side $\sim \nu\Sigma u_\phi/R$. Equating these two we find

$$u_R \sim \nu/R \tag{80}$$

typical accretion timescale

$$t_\nu \sim R/u_R \sim R^2/\nu \tag{81}$$

and accretion rate through the disk

$$\dot{M} = 2\pi R \Sigma u_R \sim 2\pi\Sigma\nu \tag{82}$$

For Keplerian rotation with $d\Omega/dR = \frac{3}{2}u_\phi/R^2$ and equation (74) gives

$$u_R \sim -\frac{3}{2}\frac{\nu}{R} \tag{83}$$

and so

$$\dot{M} \sim 3\pi\Sigma\nu \tag{84}$$

2.2. **Hydrostatic equilibrium for an accretion disk.** We recall that hydrostatic equilibrium gives us

$$\rho\boldsymbol{\nabla}\phi = -\boldsymbol{\nabla}p \tag{85}$$

Consider a cylindrical coordinate system with

$$s = \sqrt{R^2 + z^2} \tag{86}$$

and potential $\Phi(s)$. It is convenient to compute

$$\frac{\partial s}{\partial z} = \frac{z}{s} \tag{87}$$

We can expand the potential in $z$

$$\Phi(r,z) \approx \Phi(R) + \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} z + \left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0}\frac{z^2}{2} \tag{88}$$

Evaluating derivatives

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial\Phi}{\partial s}\frac{z}{s} \tag{89}$$

$$\frac{\partial^2\Phi}{\partial z^2} = \frac{\partial\Phi}{\partial s}\left(\frac{1}{s} - \frac{z}{s^2}\frac{\partial s}{\partial z}\right) + \frac{\partial^2\Phi}{\partial s^2}\left(\frac{\partial s}{\partial z}\right)^2 = \frac{\partial\Phi}{\partial s}\left(\frac{1}{s} - \frac{z^2}{s^3}\right) + \frac{\partial^2\Phi}{\partial s^2}\frac{z^2}{s^2} \tag{90}$$

Only the second order term will contribute when $z = 0$ so

$$\Phi(R,z) \sim \Phi(R) + \frac{\partial\Phi}{\partial s}\frac{z^2}{2R} \tag{91}$$

We recall that the circular velocity for a particle in a circular orbit is

$$v_c^2 = R\frac{\partial\Phi}{\partial R} \tag{92}$$

so that

$$\Phi(R,z) \sim \Phi(R) + \frac{v_c^2 z^2}{2R^2} \tag{93}$$

and

$$\frac{d\Phi}{dz} \sim \frac{v_c^2 z}{R^2} \tag{94}$$

Using the hydrostatic equilibrium equation we find

$$-\frac{d\Phi}{dz} = \frac{1}{\rho}\frac{dp}{dz} = \frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{dz} = \frac{c_s^2}{\rho}\frac{d\rho}{dz} \tag{95}$$

So that

$$-\frac{d\rho}{\rho} = zdz\frac{v_c^2}{R^2 c_s^2} \tag{96}$$

A solution is $\rho \propto \exp(-z^2/(2h^2))$ with scale height such that

$$\frac{h}{R} \sim \frac{c_s}{v_c} \tag{97}$$

This useful equation relates the temperature of a disk to its aspect ratio or thickness. The ratio $h/r$ is often called the aspect ratio. The above relation implies that a thick disk is a hot one and that a cold disk is a thin one. A velocity dispersion $\sigma$ of an ensemble can be used in place of the sound speed in this relation for example in considering molecular clouds in a galaxy disk or planetesimals in a collisional ring system.

This relation can also be written

$$c_s \approx h\Omega \tag{98}$$

where $\Omega = v_c/R$ is the angular rotation rate of a particle in a circular orbit.

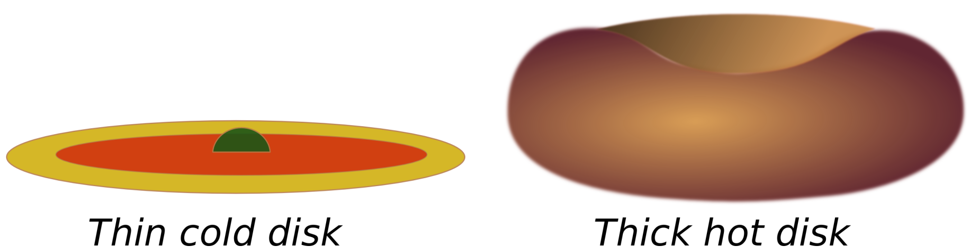


FIGURE 5. Hotter disks are thicker.

2.3. **Shakura and Sunyaev's $\alpha$-disk.** Accretion disks are suspected in young stellar systems and surrounding black holes and other compact objects. Unfortunately it is often difficult to directly observe them and so characterize their physical properties. It is common to estimate the kinematic viscosity as

$$\nu = \alpha c_s h \tag{99}$$

Here $c_s$ is the sound speed and $h$ the vertical scale height of the disk. The parameter $\alpha$ is dimensionless and we can see that we have the correct units for the kinematic viscosity. The above form for the viscosity was adopted by Shakura and Sunyaev. The parameter $\alpha$ is often unknown though it has been estimated to be small, of order 0.01 in circumstellar disks and of order 1 in some active galactic nuclei.

Let us estimate the accretion timescale for an $\alpha$ disk.

$$t_\nu \sim \frac{R^2}{\nu} \sim \frac{R^2}{\alpha c_s h} \tag{100}$$

Using our hydrostatic equilibrium equation for $c_s$

$$t_\nu \sim \alpha^{-1} \frac{R^2}{h^2} \Omega^{-1} \tag{101}$$

The angular rotation rate $\Omega = \sqrt{GM/R^3}$ so we can write

$$\Omega^{-1} = \frac{1\text{yr}}{2\pi} \left(\frac{M}{M_\odot}\right)^{-1/2} \left(\frac{R}{1\text{AU}}\right)^{3/2} \tag{102}$$

So our accretion timescale

$$t_\nu \sim \frac{1\text{yr}}{2\pi} \alpha^{-1} \left(\frac{h}{R}\right)^{-2} \left(\frac{M}{M_\odot}\right)^{-1/2} \left(\frac{R}{1\text{AU}}\right)^{3/2} \tag{103}$$

For an aspect ratio of $h/r \sim 0.05$ and $\alpha = 0.01$ we get $t_\nu \sim 10^5$ yr at 1 AU. These are "typical values" used for circumstellar disks.

2.4. **Reynolds Number.** The Reynolds number is a dimensionless number that gives the importance of viscosity in a flow. We consider a flow with $v$ a typical velocity in the problem, $L$ the length-scale and $\nu$ the kinematic viscosity. A ratio of inertial to viscous forces can be estimate from a the sizes of the $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}$ term and the $\nu \nabla^2 \mathbf{u}$ term in the Navier-Stokes equation.

$$\frac{\text{inertial force}}{\text{viscous force}} \sim \frac{v^2/L}{\nu v/L^2} \sim \frac{vL}{\nu} \tag{104}$$

We define the Reynolds number as

$$\mathcal{R} \equiv \frac{vL}{\nu} \tag{105}$$

High Reynold's number flows tend to be turbulent (flow inside a firehose). Note there is no consideration of the sound speed here. Low Reynolds number flows are flows where viscosity is important, like a bacteria swimming in water or a brick sinking in tar.

Usually the Reynolds number is estimated with $L$ the size of an object, like the diameter of a rain drop. The velocity would be the rain drop's velocity as it falls through the air. The drag force is estimated as a function of Reynold's number and shape. Equating the drag force against the force from gravity would allow one to solve for the the terminal velocity as a function of drop diameter.

What would a Reynolds number be for our accretion disk? Our typical velocity is the circular velocity. Our length scale the radius, so

$$\mathcal{R} = \frac{v_c r}{\nu} \sim \alpha^{-1} \left(\frac{r}{h}\right)^2 \tag{106}$$

and is high for low viscosity thin disks.

Watch movie on Drag III at NCFM on boundary layers, drag coefficients and Reynold's number.

2.5. **Circumstellar disk heated by stellar radiation.** In circumstellar disks the heat flux due to absorption of radiation from the central star may set the effective temperature of the disk. If the temperature is set by the radiation from the central star then

$$\frac{L_*}{4\pi R^2} \sim \sigma_{SB} T_{eff}^4 \tag{107}$$

Here $L_*$ is the luminosity of the star. The above equation should include an albedo and an emissivity and can more accurately be written

$$\frac{L_*}{4\pi R^2}(1-\beta) \ \sim \epsilon \sigma_{SB} T_{eff}^4 \tag{108}$$

where $\beta$ is an albedo and $\epsilon$ is an emissivity. Here $\beta$ and $\epsilon$ are both integrated over wavelength.

Above we have not considered the possibility that the disk itself can block radiation from the star or self-shield. If the disk is optically thick then we need to take into account the fraction of the disk per unit area that is illuminated. A correction factor to the above equation that depends on the flaring of the disk can be added. A disk with constant axis ratio ($h/R$ constant) will self-shield from the central star. So we expect only a flaring disk will absorb starlight. The difference between the ray angles from the star at a slope of $h/r$ and the local disk slope, $dh/dR$, will determine the flux absorbed. Only when $\frac{dh}{dR} > \frac{h}{R}$ will the disk be illuminated. We estimate the

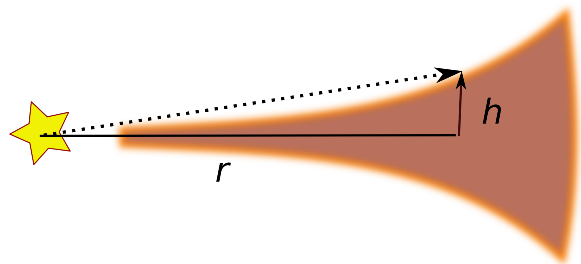


FIGURE 6. A flared disk. At each radius, $R$, the height of the disk gives an aspect ratio $h/R$. The amount of light absorbed per unit area depends on the aspect ratio and its radial gradient.

fraction of flux absorbed

$$f \sim \frac{dh}{dR} - \frac{h}{R} \tag{109}$$

which is sometimes written as

$$f \sim \frac{h}{R}\left[\frac{d\ln h}{d\ln R} - 1\right] \tag{110}$$

Our temperature balance equation becomes

$$\frac{L_*}{4\pi R^2}(1-\beta)f \ \sim \epsilon\sigma_{SB}T_{eff}^4 \tag{111}$$

Additional complications include considering that the skin of the disk could be hotter than the interior because the spectrum of starlight peaks in the optical bands but that from the thermal emission from the disk peaks in the infrared where the opacity is lower.

2.5.1. *Example of $h(R)$ for an optically thin disk with flaring set by stellar radiation.* Suppose the disk is optically thin. In this case we can ignore the disk flaring in determining the fraction of light absorbed by the disk and so setting the disk temperature. Our equation for radiation balance implies that $T_{eff} \propto R^{-1/2}$. The temperature and sound speed are related with $c_s \propto T^{1/2}$ so that $c_s \propto R^{-1/4}$. We can use our equation of hydrostatic equilibrium $h \sim c_s\Omega^{-1}$ with angular rotation rate $\Omega = \sqrt{\frac{GM}{R^3}}$ to find that

$$h \propto R^{5/4} \tag{112}$$

We note that the disk does flare as $h/R \propto R^{1/4}$ increases with radius.

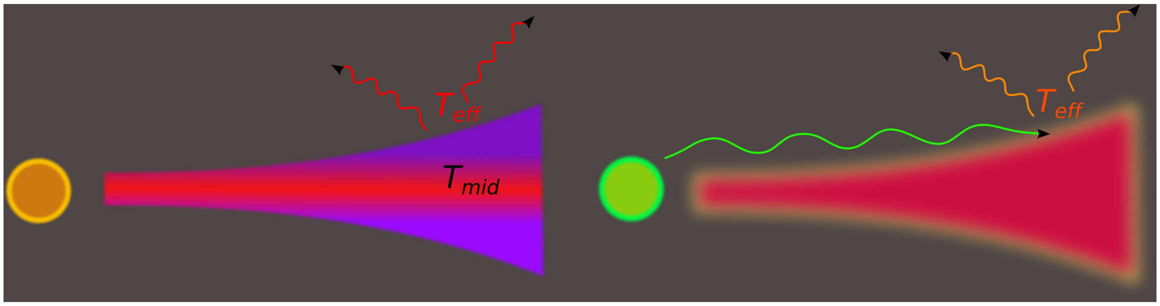


FIGURE 7. If viscous energy dissipation dominates the energy input into the disk (left) then we expect the mid plane of the disk is hotter than the surface. The effective temperature of the disk surface depends on the energy dissipated in the disk. The vertical temperature gradient depends on disk opacity. If the energy absorbed from the star dominates that dissipated viscously then the disk temperature depends on the radiation absorbed per unit area (right). The disk temperature gradient would not be large. The temperature may not be exactly uniform (as a function of height) as emission depends on opacity as a function of wavelength. For example, the center of the disk, shielded from star light might be able to radiate at longer wavelengths as the opacity is often lower at longer wavelengths. In this case the surface of the disk would be somewhat hotter than the mid-plane.

2.6. **Viscous Energy Dissipation.** We can estimate the energy dissipation rate for the accretion disk. The important component of the viscous stress tensor is $T_{visc,r\phi} \sim \nu\Sigma R\frac{d\Omega}{dR}$. From our discussion on viscous heat generation the heat generated per unit volume is the viscous stress times the strain. The only non-zero component of the viscous stress tensor is $T_{visc,r\phi}$. The strain to use is $R\frac{d\Omega}{dR}$ because it gives the velocity shear in the $r, \phi$ direction. The energy per unit area dissipated per unit time would be

$$\dot{q}_{visc} = \nu\Sigma R^2 \left(\frac{d\Omega}{dR}\right)^2 \tag{113}$$

We can evaluate this for a steadily accreting Keplerian disk using $\frac{d\Omega}{dR} = -\frac{3}{2}\frac{\Omega}{R}$,

$$\dot{q}_{visc} = \frac{9}{4}\Omega^2\nu\Sigma \tag{114}$$

We now relate $\nu\Sigma$ to $\dot{M}$ to find the dissipation rate as a function of radius. To do this we need to consider a boundary condition (and if we don't do this we will falsely find the system does not conserve energy).

Consider a disk with constant steady accretion flow

$$\dot{M} = 2\pi \Sigma R u_R = \text{constant} \tag{115}$$

Going back to equation 64 for angular momentum conservation and assuming a steady state disk

$$\begin{aligned} R^3 \Sigma \Omega u_R - \nu R^3 \Sigma \Omega_{,R} &= C \\ \frac{R^2 \Omega \dot{M}}{2\pi} - \nu R^3 \Sigma \Omega_{,R} &= C \end{aligned} \tag{116}$$

with $C$ another constant. We can solve for $\nu\Sigma$

$$\nu\Sigma = \frac{\Omega}{R\Omega_{,R}} \frac{\dot{M}}{2\pi} \left[1 - \frac{C'}{\Omega R^2}\right] \tag{117}$$

where $C'$ is a constant. For a Keplerian system $\Omega R^2 \propto R^{1/2}$ and $\frac{\Omega}{R\Omega_{,R}} = \frac{2}{3}$ so that

$$\nu\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{C''}{R}\right)^{1/2}\right] \tag{118}$$

where $C''$ is another constant — with units of radius. If we assume a boundary condition at $R_{in}$ (a truncation edge) where $\Sigma \to 0$ then

$$\nu\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{R_{in}}{R}\right)^{1/2}\right] \tag{119}$$

Note that at large radius we find $\dot{M} \sim 3\pi\nu\Sigma$ as we previously estimated.

We insert this expression in to equation (114) finding

$$\dot{q}_{visc} = \frac{3\dot{M}\Omega^2}{4\pi} \left[1 - \left(\frac{R_{in}}{R}\right)^{1/2}\right] \tag{120}$$

2.7. **Accretion Luminosity.** Consider equation (120) for the energy dissipated per unit area for a nearly Keplerian system in steady state. This viscously dissipated energy escapes as radiation from the top and bottom of the disk. Let us integrate between inner and outer disk radii $(R_{in}, R_{out})$ to estimate the total luminosity of the disk due to accretion

$$L_{disc} = \int_{Rin}^{Rout} 2\pi R \, dR \, \dot{q}_{visc} \tag{121}$$

For a Keplerian system accreting in steady state with accretion rate $\dot{M}$,

$$\begin{aligned} L_{disc} &= \int_{Rin}^{Rout} \frac{3}{2}\Omega^2 \dot{M} R \left[1 - \left(\frac{R_{in}}{R}\right)^{1/2}\right] dR \\ &= \int_{Rin}^{Rout} \frac{3}{2} G M_* \dot{M} \left[\frac{1}{R^2} - \frac{R_{in}^{1/2}}{R^{5/2}}\right] dR \qquad (122) \\ &= \frac{3}{2} G M_* \dot{M} \left(\frac{2R_{in}^{1/2}}{3R^{3/2}} - \frac{1}{R}\right)\Bigg|_{Rin}^{Rout} \\ &\sim \frac{G M_* \dot{M}}{2R_{in}} \qquad (123) \end{aligned}$$

The form and units of this are correct.

2.7.1. *Comparison to energy gained in potential energy well.* Let's consider the energy gained due to inflow into the potential well. Consider the kinetic and potential energy per unit mass for the gas in a circular orbit

$$e = \left(\frac{u_\phi^2}{2} - \frac{GM}{R}\right) = -\frac{GM}{2R} \qquad (124)$$

where I have assumed a circular velocity $u_\phi^2 = R^2\Omega^2 \sim GM/R$. The total energy per unit radius is

$$\frac{dE}{dR} = 2\pi R \Sigma e \qquad (125)$$

Flux of energy through radius $R$ would be

$$2\pi R \Sigma e u_R = e\dot{M} \approx \frac{\dot{M}\Omega^2 R}{2} \qquad (126)$$

and that per unit area (dividing by $2\pi R$) would be

$$\dot{q} \sim \frac{\dot{M}\Omega^2}{4\pi} \sim \frac{GM\dot{M}}{4\pi R^3} \qquad (127)$$

We compare this to the viscous energy dissipation. Note the units are the same and make sense but the viscous energy dissipation is larger by a factor of 3 at large radius but smaller at small radius. Probably overall it is possible to argue that energy conservation is not violated. If we integrate $\dot{q}$ over the disk we find a total energy

$$\dot{E} \sim \frac{GM\dot{M}}{2R_{in}} \qquad (128)$$

which seems to be consistent with our accretion luminosity.

2.7.2. *The temperature profile of the disk.* Assume now that the disk is primarily heated by the viscous energy dissipation. Using equation (114)

$$\dot{q}_{visc} = \frac{9}{4}\Sigma\Omega^2\nu \sim 2\epsilon\sigma_{SB}T_{eff}^4 \tag{129}$$

where $T_{eff}$ is the effective temperature at the surface of the disk and $\epsilon$ is the emissivity. We use a factor of 2 on the right hand side to include both top and bottom surfaces of the disk. By surface we mean the height at which radiation can escape (sometimes call the $\tau = 1$ or where the optical depth is 1).

If the midplane of the disk is viscous then the heating takes place in the midplane and heat must diffuse vertically through the disk before it escapes. In this case the center of the disk could be hotter than the surface. The two temperatures would be related by the disk opacity setting the diffusion coefficient of radiation through the disk. We expect

$$T_{midplane}^4 \sim \tau T_{eff}^4 \tag{130}$$

where $\tau$ is the optical depth. The above holds if $\tau \gtrsim 1$, however if $\tau < 1$ then the midplane temperature is similar to the surface temperature.

$$\tau = \kappa\Sigma/2 \tag{131}$$

where $\kappa$ is a mean opacity of the gas in the disk and the factor of two in the above comes from considering escape of radiation from the midplane or only through half of the disk. Note $\tau$ is unitless but $\kappa$ has units of the inverse of $\Sigma$ or $\mathrm{g}^{-1}\mathrm{cm}^2$. We note that the above equation relating temperatures picks up a factor of 3/8 in a plane parallel setting. A popular value for $\kappa$ might be of order 1 $\mathrm{g}^{-1}\mathrm{cm}^2$ in a dusty disk or about half that for an ionized disk. In general the averaged opacity is a function of density and temperature. Lower values for opacity are used for cold molecular clouds, $\kappa \sim 0.01\mathrm{g}^{-1}\mathrm{cm}^2$.

The above equations are sufficient to roughly estimate the thermal structure of a simple circumstellar accretion disk and are useful for exploring their physical parameters in a variety of settings. In some cases more than one heat source must be considered to compute the thermal structure of a disk. If you are lucky one heat source will dominate and you can neglect the others simplifying the calculation. The temperature of low density circumstellar disks is set by the radiation of the central star and only for the inner regions of an actively accreting system is there more heating due to viscous dissipation in which case this process sets the disk temperature. Accretion disks in Seyfert galaxies and quasars the inner hot region of the accretion disk may illuminate the outer regions. To fully compute the temperature and density structure of a disk the opacity as a function of depth and wavelength must be taken into account.

## 3. Vorticity and Rotation

Vorticity is important in planetary atmospheres, including our own. 2014's cold winter is being attributed to variations in the structure of the vortex and vortices near the north pole.

Define the vorticity as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{132}$$

To get some intuition on what vorticity is consider a solid body rotation with $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r}$. Let's compute the vorticity for this

$$\boldsymbol{\omega} = \nabla \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{133}$$

To compute this let's consider the $i$-th component and use summation notation

$$\begin{aligned} \omega_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \Omega_l x_m \\ &= \epsilon_{kij} \epsilon_{klm} \Omega_l \frac{\partial x_m}{\partial x_j} \end{aligned} \tag{134}$$

Before we evaluate this further it is useful to know the following identity

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{kn} \tag{135}$$

Using the identity

$$\omega_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l \frac{\partial x_m}{\partial x_j} \tag{136}$$

$$= \Omega_i \delta_{jm} \frac{\partial x_m}{\partial x_j} - \Omega_j \delta_{im} \frac{\partial x_m}{\partial x_j} \tag{137}$$

$$\begin{aligned} &= \Omega_i \delta_{jm}\delta_{jm} - \Omega_j \frac{\partial x_i}{\partial x_j} \\ &= \Omega_i 3 - \Omega_j \delta_{ij} \\ &= 3\Omega_i - \Omega_i \\ &= 2\Omega_i \end{aligned} \tag{138}$$

For solid body rotation we find that

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega} \tag{139}$$

Vorticity is a property of rotating flows. The above equation specifies the units for vorticity. Vorticity is in units of $\mathrm{s}^{-1}$ as it is like an angular rotation rate.

Remember our definition for the antisymmetric component of the velocity gradient **r**

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$
$$= \frac{1}{2}\epsilon_{ijk}\frac{\partial u_i}{\partial x_j} \tag{140}$$

We can write the vorticity as

$$\boldsymbol{\omega}_k = \epsilon_{kij}\frac{\partial u_i}{\partial x_j} \tag{141}$$

The similarity between these two expressions means

$$r_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k \tag{142}$$

3.1. **Helmholtz Equation.** Starting with Euler's equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{143}$$

We use the vector identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{u^2}{2}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{144}$$

and the enthalpy $\nabla h = \frac{1}{\rho}\nabla p$. The Euler equation can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) - \mathbf{u}\times\boldsymbol{\omega} = -\nabla h \tag{145}$$

Take the curl of this

$$\frac{\partial(\nabla\times\mathbf{u})}{\partial t} - \nabla\times\nabla\left(\frac{u^2}{2} + h\right) = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) \tag{146}$$

equivalent to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) \tag{147}$$

where we have dropped terms involving the curl of a gradient which are zero. This is known as **Helmholtz's equation** and is related to **Kelvin's circulation theorem**. Note that we used an enthalpy here. This is justified as long as the constant pressure and density contours are the same or the fluid is barotropic. If we had taken the curl of $\frac{1}{\rho}\nabla p$ we would have found a term proportional to $\nabla\rho\times\nabla p$ which is zero when the fluid is barotropic (or $p(\rho)$).

We did not include an additional force in Euler's equation in our derivation above but as long as only conservative forces are considered, they do not change Helmholtz's equation because curl of a grad is zero and so they drop out of the equation. Non-conservative forces, such as the Coriolis force would change the equation. The Coriolis force is rotational so that is not necessarily surprising.

It may be interesting to read Kip's notes at this point where he constructs a derivative corresponding to the rate of change of a vector with respect to a vector that is moving with the fluid. With this new derivative $D\boldsymbol{\omega}/Dt = -\boldsymbol{\omega}(\nabla \cdot \mathbf{u})$ and the vorticity evolution is parallel to itself. This may better explain the idea that vortex lines are frozen into the fluid. Shu's book on the other hand considers the rate of change of an area vector for a surface.

Instead we follow the illustration by Pringle and King which I found the clearest. We first consider the change in both length and direction of a small linear fluid element that has both ends moving with the fluid.

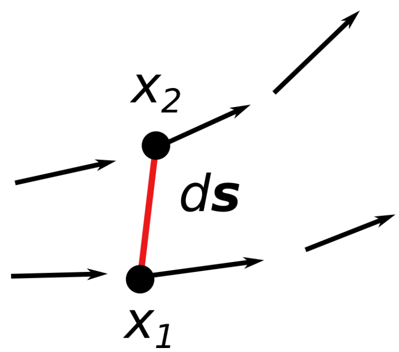


FIGURE 8. A vector element with each end that is moving with the fluid changes both in length and direction with the flow.

3.2. **Rate of Change of a vector element that is moving with the fluid.** Consider a small linear element $d\mathbf{s}$ that has both ends moving with the fluid. We can call the left hand side of $d\mathbf{s}$ a position $\mathbf{x}_1$ and the right and side $\mathbf{x}_2$. The original vector at time $t = 0$ is

$$d\mathbf{s} = \mathbf{x}_2 - \mathbf{x}_1. \tag{148}$$

After time $dt$ the vector is at

$$d\mathbf{s} + [\mathbf{u}(\mathbf{x}_2) - \mathbf{u}(\mathbf{x}_1)]dt \tag{149}$$

For any function $f(\mathbf{x})$ we can write

$$f(\mathbf{x_2}) - f(\mathbf{x_1}) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \boldsymbol{\nabla} f(\mathbf{x_2} - \mathbf{x_1}) = (\boldsymbol{\nabla} f)\cdot d\mathbf{s} = (d\mathbf{s}\cdot\boldsymbol{\nabla})f \tag{150}$$

In the same way

$$\mathbf{u}(\mathbf{x}_2) - \mathbf{u}(\mathbf{x}_1) = \nabla \mathbf{u} \cdot (\mathbf{x}_2 - \mathbf{x}_1) = (d\mathbf{s} \cdot \boldsymbol{\nabla})\mathbf{u} \tag{151}$$

so

$$d\mathbf{s} + [\mathbf{u}(\mathbf{x}_2) - \mathbf{u}(\mathbf{x}_1)]dt = d\mathbf{s} + (d\mathbf{s} \cdot \nabla)\mathbf{u}dt \tag{152}$$

where each component of $\mathbf{u}$ is expanded out in Taylor series to first order. The change in the vector element moving with the fluid is

$$\frac{Dd\mathbf{s}}{Dt} = (d\mathbf{s} \cdot \nabla)\mathbf{u}. \tag{153}$$

This implies that the length of our little linear element only changes if the gradient of the velocity in a direction parallel to $ds$ is non-zero. The little linear element does not change length if the velocity has a gradient perpendicular to the line element. If there is no velocity gradient then the line element is swept along and stays the same length and direction. Only if there is a velocity gradient does each end move separately and the element changes length and direction.

Now let's go back to our relation for vorticity or Helmholtz's equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \tag{154}$$

We use the identity

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \mathbf{u}(\nabla \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} \tag{155}$$

As $\boldsymbol{\omega}$ is a curl $\nabla \cdot \boldsymbol{\omega} = 0$. This gives us

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} \tag{156}$$

or

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) \tag{157}$$

where we have used our advective derivative. The first term on the right hand side is in the same form as equation (153) for the rate of change of a linear element moving with the fluid. The second term on the right hand side we manipulate using the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{or} \qquad \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \tag{158}$$

Inserting this into our relation for vorticity

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \boldsymbol{\omega}\frac{1}{\rho}\frac{D\rho}{Dt} \tag{159}$$

or

$$\frac{D\boldsymbol{\omega}}{Dt} + \rho\boldsymbol{\omega}\frac{D\rho^{-1}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} \tag{160}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u} \tag{161}$$

Compare this to equation (153) which I repeat here

$$\frac{Dd\mathbf{s}}{Dt} = (d\mathbf{s}\cdot\nabla)\mathbf{u} \tag{162}$$

It is clear that they are in the same form. The interpretation is that $\boldsymbol{\omega}/\rho$ moves as if it were frozen into the fluid.

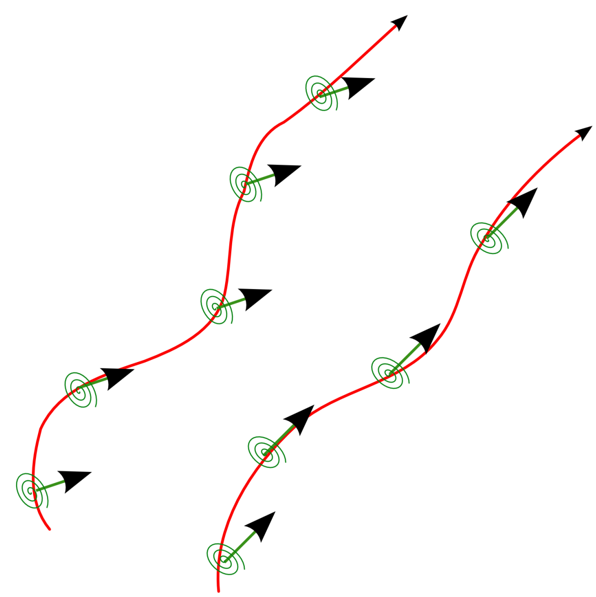

FIGURE 9. In an inviscid barotropic fluid, vorticity divided by density is transported with the fluid.

3.3. **Kelvin Circulation Theorem.** We will look at the evolution of vorticity in another way, that involving circulation. Consider the circulation around a small loop $C$ with a line integral

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{s} \tag{163}$$

Apply Stokes theorem

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = \int_S \boldsymbol{\omega} \cdot d\mathbf{A} \tag{164}$$

where $S$ is a surface inside the loop $C$. The above implies that the circulation through a small loop $C$ is the same thing as the integral of the vorticity passing through the loop.

Now we let $C$ move with the fluid and consider the rate of change of the circulation, $\Gamma$.

$$\begin{aligned} \frac{D\Gamma}{Dt} &= \frac{D}{Dt} \oint_C \mathbf{u} \cdot d\mathbf{s} \\ &= \oint_C \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{s} + \oint_C \mathbf{u} \cdot \frac{Dd\mathbf{s}}{Dt} \\ &= \oint_C \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) d\mathbf{s} + \oint_C \mathbf{u} \cdot \frac{Dd\mathbf{s}}{Dt} \end{aligned} \tag{165}$$

(166)

The term on the right hand side has the rate of change in length and direction for a linear element of $d\mathbf{s}$ where the line element is moving with the fluid. We have looked this carefully previously with equation (153). Using equation (153)

$$\oint_C \mathbf{u} \cdot \frac{Dd\mathbf{s}}{Dt} = \oint_C \mathbf{u} \cdot (d\mathbf{s} \cdot \nabla)\mathbf{u} = \oint_C \frac{1}{2} \nabla u^2 \cdot d\mathbf{s} \tag{167}$$

$$= \int_S \nabla \times \left( \frac{1}{2} \nabla u^2 \right) d\mathbf{A} = 0 \tag{168}$$

Last step again using Stokes theorem.

Going back to our rate of change of circulation

$$\frac{D\Gamma}{Dt} = \oint_C \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) d\mathbf{s} \tag{169}$$

Using Euler's equation, and assuming conservative forces and an barotropic fluid

$$\frac{D\Gamma}{Dt} = \oint_C -\frac{\nabla P}{\rho} d\mathbf{s} = \int_S -(\nabla \times \nabla h)\, d\mathbf{A} = 0 \tag{170}$$

The circulation around a loop that is moving with the fluid does not change in time. A equivalent statement is that the vorticity passing through a surface moving with the fluid remains constant.

If we take the above equation (169) and our vector identity (equation 144) on the right hand side we find

$$\frac{D\Gamma}{Dt} = \oint_C \left( \frac{\partial \mathbf{u}}{\partial t} + \frac{\nabla u^2}{2} - (\mathbf{u} \times \boldsymbol{\omega}) \right) \cdot d\mathbf{s} \tag{171}$$

now use Stokes equation

$$\frac{D\Gamma}{Dt} = \int_S d\mathbf{A} \cdot \left( \frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \right). \tag{172}$$

Thus the circulation theorem $d\Gamma/dt = 0$ implies that Helmholtz's equation (equation 147) is satisfied. The statement that vorticity is frozen into the fluid, or moves with the fluid is equivalent to Helmholtz's equation. While we considered the time derivative of the loop $C$ as it moved with the fluid we could also have considered the time derivative of the area $dA$ as the surface $S$ moves through the fluid to give the same expression.

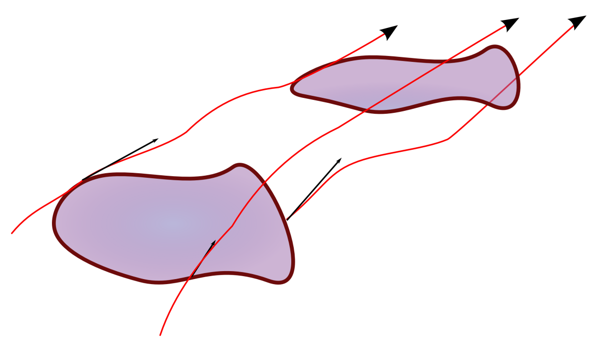

FIGURE 10. In an inviscid barotropic fluid, the circulation around a loop that is moving with the fluid remains constant. Likewise the vorticity integrated through the surface bounded by the loop remains constant.

3.4. **Vortex lines and vortex tubes.** A *vortex line* is a line connecting local vorticity vectors. At each point on a vortex line, the tangent to the line is equal to the vorticity. A *vortex tube* is a surface formed by vortex-lines passing through a loop. The *strength* of a vortex-tube, or the *vortex flux*, is the integral of the vorticity across a surface that slices a vortex tube. Using Gaus's law on a vortex tube

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{\omega} dV = \int_A \boldsymbol{\omega} \cdot d\mathbf{A} \tag{173}$$

where $A$ is a surface with top and bottom slicing a vortex tube and with sides consisting of vortex lines. The divergence of the vorticity is zero so the left hand side

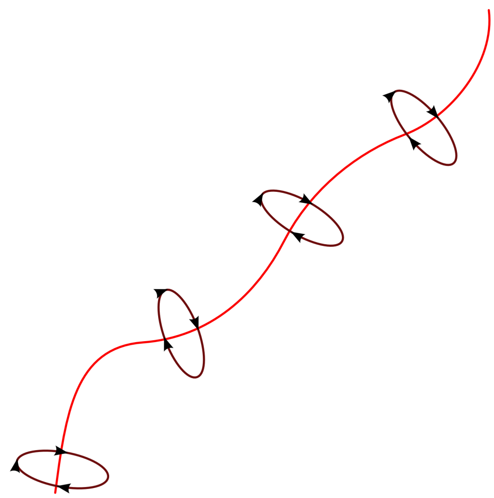

FIGURE 11. A vortex line is a line connecting local vorticity vectors.

of the above equation is zero.

$$\int_{top} \boldsymbol{\omega} \cdot d\mathbf{A} + \int_{bottom} \boldsymbol{\omega} \cdot d\mathbf{A} + \int_{sides} \boldsymbol{\omega} \cdot d\mathbf{A} = 0 \tag{174}$$

Since along the sides the surface is parallel to the vorticity, the third integral is zero. This is implies that the first two terms are equivalent and so the vortex flux is the same at the top of the tube as at the bottom.

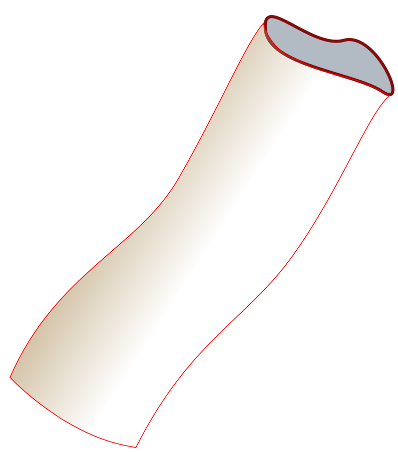

FIGURE 12. A vortex tube is a surface bounded by vortex lines. The vortex flux (integral of vorticity across a surface that slices a vortex tube) must be the same everywhere along the tube.

Using the MHD approximation we will find an equation for the magnetic field that looks similar to equation (147) and we will then conclude that magnetic field lines are frozen into the fluid.

In the above discussion we have neglected viscous forces. When viscous forces are included Helmholtz's equation equation (147) picks up an extra and nonzero term involving the viscosity. Where viscosity is not important the vorticity is frozen into the fluid and moves with the fluid. However vorticity is generated in turbulent regions and boundary layers.

To summarize: For a barotropic, inviscid fluid, the vorticity integrated through a surface moving with the fluid is constant. Equivalently the circulation around a loop moving with the fluid is constant. Equivalently the vector $\boldsymbol{\omega}/\rho$ is frozen into the fluid.

3.5. **Vortex stretching and angular momentum.** Looking at our equation that we interpreted in terms of vorticity being frozen into the fluid.

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\boldsymbol{\nabla}\right)\mathbf{u} \tag{175}$$

In an incompressible setting we can remove the density

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{u} \tag{176}$$

If the gradient of $\mathbf{u}$ is positive in the direction of $\boldsymbol{\omega}$ then the Lagrangian derivative of the vorticity is positive and the vorticity increases. If we think of the vorticity as lying between two points frozen into the fluid and the motion moves those two points further apart then the length of the vorticity vector increases. This is known as *vortex stretching* and it has to arise via conservation of momentum as we have primarily been manipulating variations of Euler's equation and this is equivalent to conservation of momentum.

But why is it that when we have a diverging flow that it spins up? If we have a diverging flow along the vorticity direction and the flow is incompressible then we must also have a converging flow in the other directions. This is like an ice skater reducing her momentum of inertia and so increasing her spin. Looking at the flux tube picture if the flux lines get close together then the vorticity inside a flux tube must increase. Because the vorticity is frozen into the fluid, this only happens when there is converging flow making the the vortex lines approach each other and stretching along the vortex lines corresponding to a diverging flow along the flux tube.

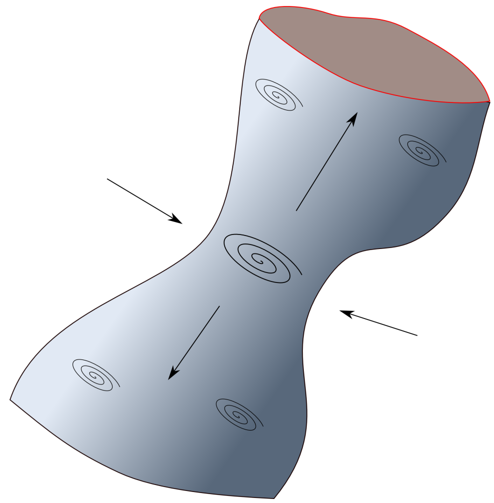

FIGURE 13. When a vortex tube is pinched and stretched, the vorticity increases in the waste of the tube.

3.6. **Bernoulli's constant in a wake.** In our derivation of Bernoulli's equation, but starting with the Navier Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u} \tag{177}$$

using the vector identity

$$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla}\left(\frac{u^2}{2}\right) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \tag{178}$$

and using enthalpy we can write

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}\left(\Phi + h + \frac{u^2}{2}\right) + \nu\boldsymbol{\nabla}^2\mathbf{u} \tag{179}$$

We dot this with $\mathbf{u}$ to consider what happens along streamlines. In a steady state flow and along a streamline

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left(\Phi + h + \frac{u^2}{2}\right) = \nu\mathbf{u} \cdot \boldsymbol{\nabla}^2\mathbf{u} \tag{180}$$

The Bernoulli constant or function $\Phi + h + \frac{u^2}{2}$ is not constant in the flow if viscosity is important. The Bernoulli function $\Phi + h + \frac{u^2}{2}$ would decrease along a streamline that is affected by viscous forces. Primarily streamlines that pass close to the surface of a body or pass through turbulent eddies would be affected.

3.7. **Diffusion of vorticity.** Recall the Navier-Stokes equation but including the possibility of compressible flow

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho}\left[-\boldsymbol{\nabla} p + (\zeta + \eta/3)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \eta\boldsymbol{\nabla}^2\mathbf{u}\right] \tag{181}$$

Taking the curl of both sides and using a vector identity

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = \frac{\eta}{\rho}\nabla^2\boldsymbol{\omega} + \frac{\nabla\rho}{\rho^2}\times\left[\nabla p - \eta\nabla^2\mathbf{u} - (\zeta + \frac{\eta}{3})\nabla(\nabla\cdot\mathbf{u})\right] \tag{182}$$

The first term on the right hand side gives the diffusion of vorticity due to kinematic viscosity. The second term on the right arises when the fluid is compressible. When the fluid is incompressible, ($\nabla\rho = 0$), then the equation is, in fact, a diffusion equation. For incompressible flow

$$\frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = \nu\nabla^2\boldsymbol{\omega} \tag{183}$$

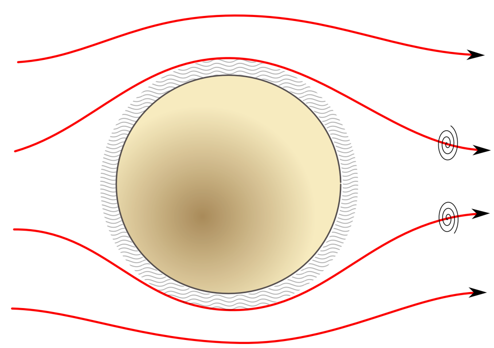

FIGURE 14. In the laminar part of the flow, Bernoulli's constant and the vorticity are conserved along stream lines. Turbulence and boundary layers and viscous stresses in them can induce vorticity into an object's wake. Bernoulli's constant is reduced in streamlines in the object's wake that pass through boundary layers.

3.8. **Potential Flow and d'Alembert's paradox.** Consider an object moving through a fluid that is at rest and extend to infinity in all directions. Where the fluid is at rest, the flow has no vorticity. As vorticity is frozen into the fluid, it should be zero everywhere. We can consider a velocity field such that the vorticity is zero everywhere;

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} = 0. \tag{184}$$

There should be a potential function, $\phi(x, y, z)$, such that

$$\nabla\phi = \mathbf{u}. \tag{185}$$

Because $\mathbf{u}$ is a gradient of a potential function, its curl is automatically zero, and the vorticity is zero everywhere. When the flow is laminar, it can be convenient to solve for a potential function instead of the entire flow field.

When the fluid is incompressible $\nabla \cdot \mathbf{u} =$ and $\nabla^2 \phi = 0$. In this case the potential function satisfies Laplace's equation and may be easier to work with.

Boundary layers and rotational flows cannot be described as a potential flow. As a consequence, though mathematically attractive, potential flow models have limited use. It is sometimes convenient to use them to describe part of a flow, and matching the boundaries of the potential flow to a flow description that is not described by a potential.

Consider the object moving through a fluid that at rest and extend to infinity in all directions. As the vorticity is zero at infinity, the vorticity should be zero everywhere. After using a single vector identify Euler's equation (equation 145) for a barotropic fluid can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{u^2}{2} \right) - \mathbf{u} \times \boldsymbol{\omega} = -\nabla h \tag{186}$$

If the vorticity is zero everywhere and the flow is steady state then

$$\boldsymbol{\nabla} \left( h + \frac{u^2}{2} \right) + \frac{\partial u}{\partial t} = 0 \tag{187}$$

If the fluid is irrotational then we can use a potential function $\boldsymbol{\nabla} \phi = \mathbf{u}$ and the above equation becomes

$$\boldsymbol{\nabla} \left( h + \frac{u^2}{2} + \frac{\partial \phi}{\partial t} \right) = 0 \tag{188}$$

This implies that

$$h + \frac{u^2}{2} + \frac{\partial \phi}{\partial t} = F(t) \tag{189}$$

only depends on time and cannot depend on position. We can define a new potential

$$\phi' = \phi - \int^t F(t) dt \tag{190}$$

so that $\boldsymbol{\nabla} \phi' = \boldsymbol{\nabla} \phi$ and

$$\frac{\partial \phi'}{\partial t} = \frac{\partial \phi}{\partial t} - F(t) \tag{191}$$

Inserting this back into equation 189 we find

$$h + \frac{u^2}{2} + \frac{\partial \phi'}{\partial t} = \text{constant} \tag{192}$$

and this implies that we can choose the potential function such that the entire function is constant. If the fluid is steady state then

$$h + \frac{u^2}{2} = \text{constant} \tag{193}$$

This is Bernoulli's function, but here the function is not only constant on streamlines but is constant *everywhere*. In short: if the flow is irrotational, barotropic and inviscid Bernoulli's function is constant *everywhere*.

For an incompressible fluid, this implies that

$$p = p_\infty - \rho \frac{u^2}{2}. \tag{194}$$

The drag on an object (or equivalently lift) in an inviscid flow is the pressure integrated over the surface of the object

$$\mathbf{F} = \int_S p d\mathbf{A} = \int_S \left( p_\infty - \rho \frac{u^2}{2} \right) d\mathbf{A} \tag{195}$$

where $S$ is the surface of the object. Using Gaus's law (and remembering that we have assumed an incompressible fluid with $\rho$ constant and $\nabla \cdot \mathbf{u} = 0$), and assuming a fixed body shape we find that

$$\mathbf{F} = \int_V \nabla \cdot (\frac{\rho}{2} u^2) dV = 0. \tag{196}$$

An irrotational flow about a body in a barotropic, inviscid, and incompressible fluid gives no drag or lift. This is d'Alembert's paradox.

The paradox is resolved by considering where our approximations fail. Flow is not inviscid near the body and vortices can be generated from boundary layers. The paradox implies that vorticity generation is necessary in order to account for (or estimate) lift and drag forces. As a consequence, the potential flow approximation cannot be used to describe the entire flow.

Vortex shedding is a necessary component to generate lift. However, it takes energy to generate vorticity, so if a plane generates excessive vorticity then it will burn more fuel. This is a particular issue for vortices generated at airplane wing tips. Fuel efficient planes can minimize what is known as "vortex drag" by having large, wide wings compared to the plane length, or with winglets – these are the little tags pointing upwards at the ends of some airplane wings.

3.9. **Burger's vortex.** Burger's vortex is one of a few known simple steady state analytical solutions to the Navier-Stokes equation that exhibit vorticity. It can be used as an analogy for how water rotates as it goes down a drain, or perhaps for a tornado.

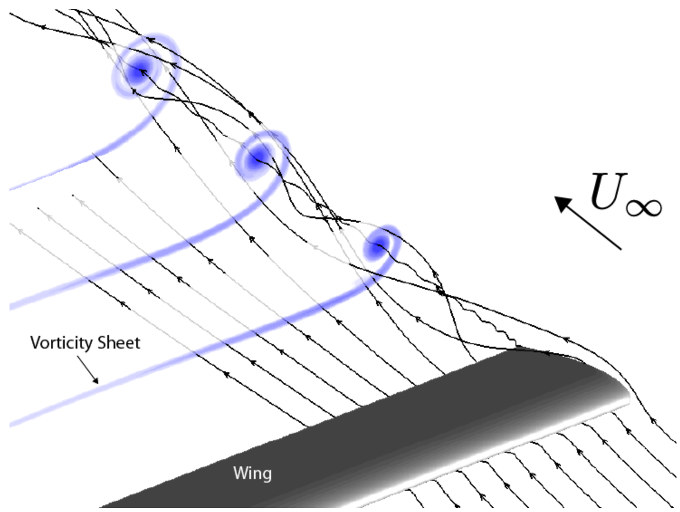


Figure 15. Vorticity generated at a planet wing tip. This figure from Wikipedia.

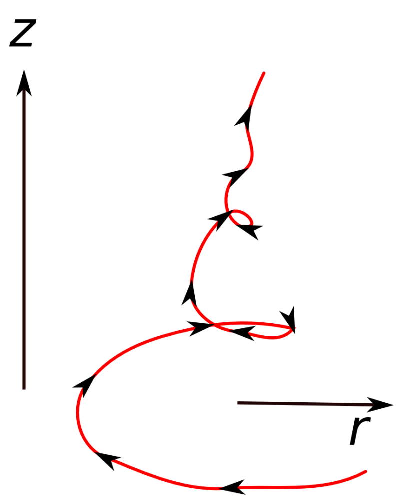


Figure 16. Streamlines for Burger's vortex. If $z$ is flipped then the flow is like water going down a drain.

Consider a steady flow in cylindrical coordinates with velocity vector

$$\mathbf{v} = v_r \hat{\mathbf{r}} + v_z \hat{\mathbf{z}} + v_\phi \hat{\boldsymbol{\phi}} \tag{197}$$

with

$$\begin{aligned} v_r &= -\frac{1}{2}\alpha r \\ v_z &= \alpha z \\ v_\phi &= v_\phi(r) \end{aligned} \tag{198}$$

and $\alpha > 0$ a constant that describes the strain or rate of shear in the flow.

The vorticity for this flow only contains a $\hat{\mathbf{z}}$ component.

$$\omega_z = \frac{1}{r}\frac{d}{dr}(r v_\phi(r)) \tag{199}$$

The $\hat{\mathbf{z}}$ component of the Navier-Stokes equation then implies that

$$\frac{D\omega_z}{Dt} = \omega_z \alpha + \nu \boldsymbol{\nabla}^2 \omega_z \tag{200}$$

The term on the left, caused by the strain $\alpha$, gives vortex line stretching and increases the vorticity. The term on the right, due to viscosity, reduces the vorticity. There is a particular radial distribution of vorticity where the two terms balance.

A steady state solution to equation 200 is

$$\omega_z = \omega_0 \exp\left(-cr^2\right) \tag{201}$$

with constant $c$, that depends on a ratio of $\nu$ and strain $\alpha$. If the strain is higher then the vorticity is concentrated in a smaller region. If the viscosity is higher then the balance is achieved with a wider vorticity distribution.

## 4. Rotating Flows

4.1. **Coriolis Force.** In a rotating frame, a particle moving at a constant velocity can actually be on a curved trajectory. The acceleration of a fluid element depends on the particle's position, with respect to the center of rotation, and on the particle's velocity. In a frame rotating with angular rotation rate or spin $\boldsymbol{\Omega}$ the acceleration is

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{202}$$

gaining two terms, a Coriolis term (depending on $\mathbf{u}$) and a centripetal term proportional to $\Omega^2$.

If $\boldsymbol{\Omega}$ is in the $\hat{\mathbf{z}}$ direction we evaluate the centripetal acceleration term $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = \Omega^2(x, y, 0)$. More generally

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \Omega^2(\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}) \tag{203}$$

where $\hat{\mathbf{n}} = \mathbf{\Omega}/|\Omega|$. In cylindrical coordinates with $R = \sqrt{x^2 + y^2}$ and with $z$ aligned with the spin the centripetal acceleration can be described as a gradient

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\nabla}\left(\frac{\Omega^2 R^2}{2}\right) \tag{204}$$

Here $R$ is the distance to the axis of rotation. We call the effective potential

$$\Phi_{eff} = \frac{\Omega^2 R^2}{2} \tag{205}$$

This can be incorporated into either gravity or pressure term (in the barotropic case) within Euler's equation.

The Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \mathbf{\nabla})\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\mathbf{\nabla} p - \mathbf{\nabla}\Phi' + \nu\nabla^2\mathbf{u} \tag{206}$$

where the gravitational potential

$$\Phi' = \Phi + \Phi_{eff} \tag{207}$$

4.2. **Rossby and Ekman numbers.** Previously we took a ratio of inertial to viscous forces to create a dimensionless number called the Reynolds number. We now have an additional free parameter $\Omega$. We can use it to create two new dimensionless numbers. Our physical objects are a velocity scale, $v$, a size scale $L$, a viscosity $\nu$ and a rotation rate $\Omega$. For the inertial force we estimate $(\mathbf{u} \cdot \mathbf{\nabla})\mathbf{u} \sim v^2/L$. For the Corolis force we estimate $\mathbf{\Omega} \times \mathbf{u} \sim \Omega v$. For the viscous force we estimate $\nu\nabla^2\mathbf{u} \sim \nu v^2/L^2$. We call the **Rossby** number the ratio of the inertial to Coriolis force

$$\mathcal{R}o \equiv \frac{\text{Inertial force}}{\text{Coriolis force}} \equiv \frac{v}{\Omega L} \tag{208}$$

We call Ekman number the ratio of viscous force to Coriolis force

$$\mathcal{E}k \equiv \frac{\text{Viscous force}}{\text{Coriolis force}} \equiv \frac{\nu}{\Omega L^2} \tag{209}$$

The Ekman number is low when viscosity is unimportant. The Rossby number is low when the Coriolis force is important. Jupiter's atmosphere is a low Rossby number setting.

4.3. **Geostrophic flows and the Taylor Proudman theorem.** We consider a setting with small Rossby and Ekman numbers. We neglect the inertial force and we neglect the viscous force. But we keep pressure force and Coriolis force and these balance (in the steady state)

$$2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\mathbf{\nabla} p - \nabla\Phi' \tag{210}$$

If the flow is incompressible we can incorporate the effective potential term within the pressure gradient. If on the surface of a planet we can ignore vertical variations in gravitational potential. Geostrophic balance is then written

$$2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p' \tag{211}$$

where $p' = p + \rho\Omega^2 R^2/2$.

The left hand side is a vector that is perpendicular to $\mathbf{u}$ and so is perpendicular to stream lines. The right hand side is in the direction of pressure gradient and so perpendicular to constant pressure contours. So the above equation is interpreted to imply that *pressure is constant along streamlines.*

Let us take the curl of the geostrophic flow equation (equation 211).

$$\boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \mathbf{u}) = 0 \tag{212}$$

We use a vector identity

$$(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\Omega} + \mathbf{u}(\boldsymbol{\nabla} \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\boldsymbol{\nabla} \cdot \mathbf{u}) = 0 \tag{213}$$

Because $\Omega$ is a constant, the second and third terms are zero. If the fluid is incompressible then the last term is zero and we find

$$(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\mathbf{u} = 0 \tag{214}$$

If we align our coordinate system so that $\hat{\mathbf{z}}$ is along the spin axis

$$\Omega\frac{\partial \mathbf{u}}{\partial z} = 0 \qquad \to \qquad \frac{\partial \mathbf{u}}{\partial z} = 0 \tag{215}$$

There cannot be any vertical variations in the velocity. This is called the *Taylor-Proudman theorem.* If we describe $\mathbf{u} = (u, v, w)$, then

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0 \tag{216}$$

The velocity components can only vary in the plane, motions can only take place in planes perpendicular to the spin axis. Often a boundary sets $w = 0$ somewhere and we find that $w = 0$ everywhere. A consequence of the Taylor-Proudman theorem is the formation of features known as *Taylor columns.*

A steady state, incompressible flow with low Rossby and Ekman numbers gives us geostrophic flow. In such a flow there can only be motions perpendicular to the axis of rotation. A rapidly spinning planet is expected to be comprised of columns. Vertical motions are surpressed.

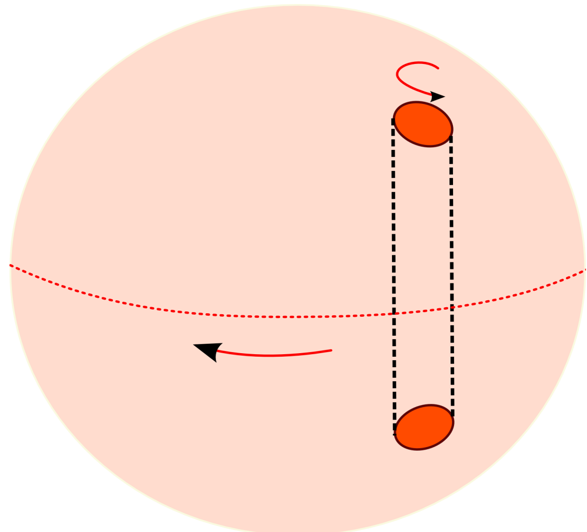

FIGURE 17. Taylor columns for low Rossby, Eckman number steady state flow in a rotating body.

4.4. **Two dimensional flows on the surface of a planet.** On the surface of a planet we can look at the velocity components only on the surface. In this case $\mathbf{u}$ has two directions (azimuthal and latitudinal). Let $\mathbf{u} = (u, v, w)$ with $u$ for the east-west direction, $v$ the north-south component and $w$ the vertical component. This is equivalent to assuming that the vertical velocity component is small compared to velocities of horizontal wind on the surface of the planet. Setting $w = 0$ we can compute the vorticity finding only a vertical component

$$\boldsymbol{\omega} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{\mathbf{z}} \tag{217}$$

Let us call $\zeta$ the vertical component of the vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{218}$$

This is in a local coordinate system with $x, y, z$ aligned with $u, v, w$. However we have not taken into account the vorticity due to the rotation of the planet. The vorticity of a rotating body is equivalent to $2\mathbf{\Omega}$ where $\mathbf{\Omega}$ is the spin vector of the body. The vertical component of $\mathbf{\Omega}$ depends on the latitude. If the neighborhood is on the equator then there is no component of the spin in the vertical direction. Altogether the vertical component of the vorticity is

$$f = 2\Omega \sin\theta \tag{219}$$

where $\theta$ is the latitude. The total vorticity (taking into account both horizontal winds and rotation of the planet) is

$$\boldsymbol{\omega} = (0, 2\Omega\cos\theta, \zeta + 2\Omega\sin\theta) \tag{220}$$

In this local coordinate system.

*Question:* How come the vorticity evolution equation only seems to depend on $\zeta + f$? What happens with the other component? A north south motion should change the $y$ component of vorticity.

4.5. **Thermal winds?** Within the context of a geostrophic flow, flow velocities can depend on height. Instead of using height, $z$, as a free variable, the pressure is used. Hydrostatic equilibrium gives us a relation between pressure and height; $dp = -\rho g dz$. Vertical gradients can be written as a derivative with respect to pressure.

$$\frac{1}{\rho} = -g\left(\frac{\partial z}{\partial p}\right)_{x,y,t} \tag{221}$$

By combining the geostrophic equations with the vertical pressure variation we can derive a relation between vertical velocity gradient and horizontal temperature gradient (at constant pressure).

$$f\frac{\partial \mathbf{v}}{\partial p} = -\frac{1}{\rho T}\boldsymbol{\nabla} T \tag{222}$$

where the temperature gradient is at constant pressure and $\mathbf{v} = (u, v)$. These are known as the *thermal wind equations.* Vertical velocity gradients are related to horizontal temperature or density gradients.

## 5. Acknowledgments

Viscous flows roughly following Thorne & Blandford, and Clarke & Carswell with some insight from Pringle and King.