AST242 LECTURE NOTES PART 2

Contents

1. Inviscid Barotropic flow

1.1. Enthalpy and the Bernoulli equation. Recall Euler's equation with gravity

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi \tag{1}$$

We define a quantity h, that we denote the enthalpy that is $\int dp/\rho$ so that $\nabla h = \frac{1}{\rho}\nabla p$. For an ideal gas with $p = K\rho^\gamma$ we find that

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

Using the enthalpy we can write Euler's equation as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(h + \Phi)$$

Using the vector identity

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left(\frac{u^2}{2}\right) - \mathbf{u} \times (\nabla \times \mathbf{u})$$

Euler's equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \nabla \times \mathbf{u} = -\nabla \left(\frac{u^2}{2} + h + \Phi\right) \tag{2}$$

Note the vorticy, $\boldsymbol{\omega}$, is defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

so we could write equation 2 as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{u^2}{2} + h + \Phi\right)$$

Consider a steady flow with $\frac{\partial u}{\partial t} = 0$. We consider how the previous equation varies along the direction of $\mathbf{u}$ or along streamlines. We dot the equation 2 with $\mathbf{u}$ (and dropping $u_{,t}$) finding a scalar equation

$$\mathbf{u} \cdot \nabla \left(\frac{u^2}{2} + h + \Phi\right) = 0 \tag{3}$$

This shows that the function (Bernoulli's function)

$$\frac{u^2}{2} + h + \Phi \tag{4}$$

is constant along streamlines.

$$\frac{u^2}{2} + h + \Phi = \text{constant} \tag{5}$$

is known as Bernoulli's equation.

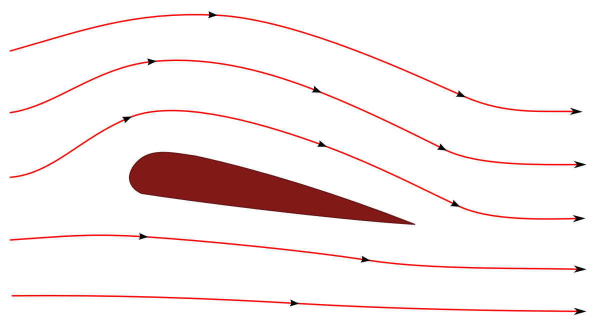

FIGURE 1. Along stream lines in a steady state flow, $\frac{u^2}{2} + h + \Phi$ is conserved. In an incompressible setting enthalpy $h \propto p$ pressure. Bernoulli's equation then implies that pressure drops where velocity increases. The gradient in pressure across an airplane wind gives a force known as **lift**.

What is meant by stream lines? We can define them locally in terms of a direction $\mathbf{dl} = (dx, dy, dz)$ such that

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \tag{6}$$

so that $\mathbf{dl} \propto \mathbf{u}$ lies in the same direction as the velocity and $\mathbf{u} \times \mathbf{dl} = 0$.

1.2. **Dimensional analysis for Bondi accretion.** We consider radial flows near a central star (or other compact object) with mass M. Parameters describing the flow are the density and sound speed at infinity ρ_∞ and c_∞, and the mass of the central object, M. We assume the ambient medium is at rest with respect to M. We can combine M along with the gravitational constant G to form a gravitational radius

$$r_G = \frac{GM}{c_\infty^2} \tag{7}$$

Inside this radius we expect the flow to be strongly dependent on the gravity of the central object and outside this we expect the flow to only be slightly influenced by the central mass.

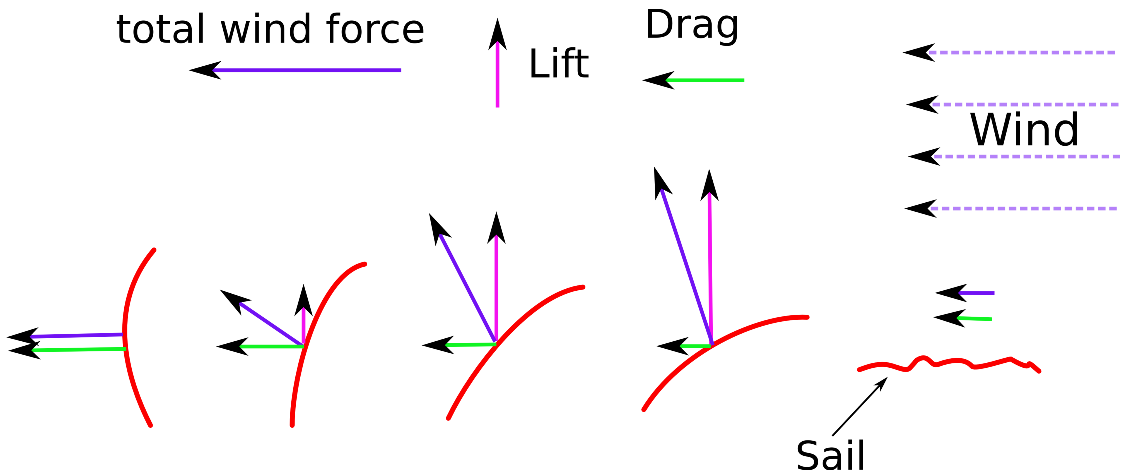

Figure 2. Lift and drag forces on a sail.

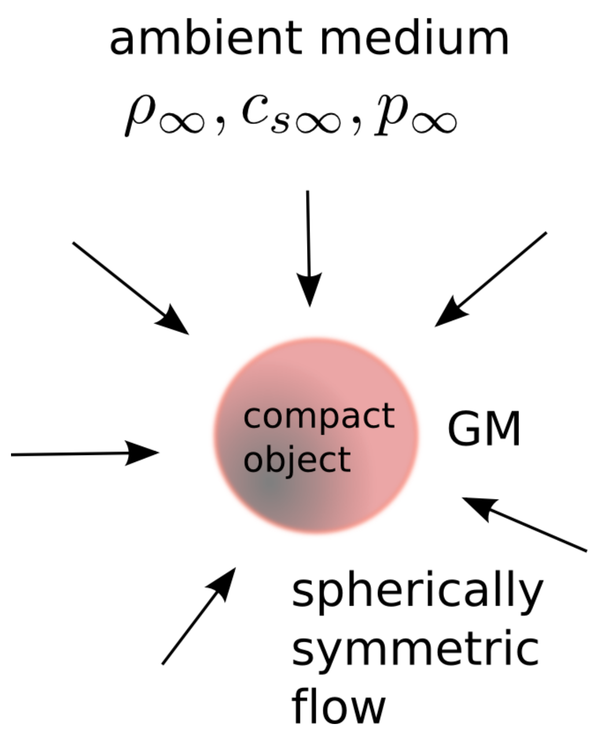

Figure 3. Setting for Bondi accretion.

Using this gravitational radius we can roughly estimate an expected accretion rate. We can assume a cross section r_G^2 and a typical velocity c_∞ and so would estimate a rough accretion rate

$$\dot{M} \sim \rho_\infty c_\infty r_G^2 \sim \rho_\infty (GM)^2 c_\infty^{-3} \tag{8}$$

1.3. **Bondi accretion.** We consider radial flows near a central star with mass M. In spherical coordinates the mass conservation equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u) = 0 \tag{9}$$

where u now refers to the velocity in the radial direction. If the flow is steady we can set $\frac{\partial \rho}{\partial t} = 0$ and integrate the previous equation finding that the outflow or inflow rate is constant or

$$\dot{M} = 4\pi \rho r^2 u \tag{10}$$

Using our expression for enthalpy Bernoulli's equation becomes

$$-\frac{GM}{r} + \frac{u^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = B \tag{11}$$

with constant B.

Consider the value of the previous equation at large radius. At large radius the gravitational term is negligible. We set the speed at large radius to be 0 so that $u \to 0$ as $r \to \infty$. The sound speed distant from the star $c_\infty^2 = \gamma p_\infty/\rho_\infty$. So

$$B = \frac{c_\infty^2}{\gamma - 1} \tag{12}$$

We have three equations that relate our variables ρ, c_s and r: the equation for $\dot{M}$, Bernoulli's equation and the equation of state. Using our boundary condition at ∞ these can be solved numerically.

Writing Bernoulli's equation in terms of u versus c_s

$$\frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} = \frac{c_\infty^2}{\gamma - 1} + \frac{GM}{r} \tag{13}$$

we see that solutions must be on an ellipse on the u versus c_s plot. At each r we get a different ellipse for solutions. The axis ratio of the ellipse is $\sqrt{\frac{\gamma-1}{2}}$.

Our equation for mass conservation allows us to relate the velocity to the accretion rate and density at infinity. We first write the speed of sound in terms of the density

$$c_s^2 = \frac{\gamma p}{\rho} = \gamma K \rho^{\gamma - 1}$$

for constant K so that

$$\left(\frac{c_s}{c_\infty}\right)^2 = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma - 1}$$

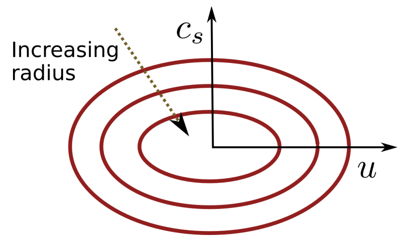

FIGURE 4. Solutions lie on ellipses with semi-major axis dependent on radius and the sound speed at infinity (equation 13).

We now use our relation for $\dot{M}$ to find

$$u = \left(\frac{\dot{M}}{4\pi\rho_\infty r^2}\right)\left(\frac{c_s}{c_\infty}\right)^{-2/(\gamma-1)} \tag{14}$$

The intersection of this curve with an ellipse on the u vs c_s plane give solutions or u, c_s. At each r we get a different curve for u vc c_s which must intersect the appropriate ellipse defined by Bernoulli's equation at that radius (equation 13).

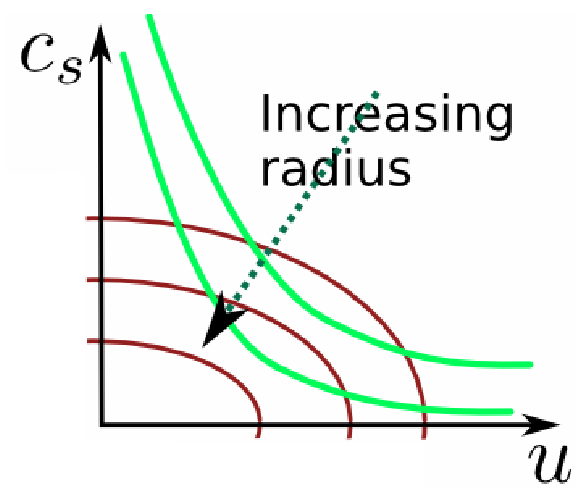

FIGURE 5. Solutions are where ellipses intersect the hyperbolic-like curves from equation 14. At each radius there is one hyperbolic-like curve (but dependent on $\dot{M}$), and one ellipse.

1.4. **The transonic point.** It is useful to classify solutions by whether they have a sonic transition point or whether there is a point where the velocity is equal to the sound speed. It is easiest to do this using the derivative forms for the mass and momentum conservation law. The radial component of Euler's equation in spherical coordinates (and ignoring derivatives with respect to θ, ϕ

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \Phi}{\partial r}$$

Using

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{1}{\rho}\frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial r} = \frac{1}{\rho}c_s^2\rho_{,r},$$

We can write the Euler equation in steady state as

$$uu_{,r} + c_s^2\frac{\rho_{,r}}{\rho} = -\frac{GM}{r^2} \tag{15}$$

where derivatives are denoted with commas. Mass conservation in spherical coordinates (equation 9 taking the steady state and dividing by ρu) can be written

$$\frac{\rho_{,r}}{\rho} + \frac{u_{,r}}{u} + \frac{2}{r} = 0 \tag{16}$$

Subbing this into equation 15

$$uu_{,r} - c_s^2\frac{u_{,r}}{u} = \frac{2c_s^2}{r} - \frac{GM}{r^2}$$

or

$$\frac{u_{,r}}{u}(u^2 - c_s^2) = \frac{2c_s^2}{r} - \frac{GM}{r^2} \tag{17}$$

The sonic transition point where $u = c_s$ occurs at a radius

$$r_B = \frac{GM}{2c_{st}^2} \tag{18}$$

where c_{st} is the sound speed at the transonic point. Putting this into Bernoulli's equation (equation 13) finding a relation between c_{st} at the sonic point and c_∞ or

$$\left(\frac{c_{st}}{c_\infty}\right)^2 = \frac{2}{5-3\gamma} \tag{19}$$

and

$$r_B = \left(\frac{5-3\gamma}{4}\right)\frac{GM}{c_\infty^2}. \tag{20}$$

The sonic transition point can be used to find the accretion rate as we know the velocity at a particular radius and we have a relation between the velocity and the accretion rate. For the solution that goes through the sonic point,

$$\dot{M}_B = \left(\frac{2}{5-3\gamma}\right)^{\frac{\gamma+1}{2(\gamma-1)}} 4\pi r_B^2\rho_\infty c_\infty = \frac{1}{4}\left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}} 4\pi (GM)^2\rho_\infty c_\infty^{-3} \tag{21}$$

where r_B is defined above as the sonic transition point.

Note r_B becomes large when $\gamma = 5/3$. When $\gamma = 5/3$ there is no trans-sonic transition point for a smooth flow. In most astrophysical situations this possibility is often ignored as it is considered too idealized.

We note that the condition that the flow contain a sonic transition point specified the location of the transonic point and the accretion rate. This implies that there is only one possible smooth transonic solution. Other solutions with discontinuities or shocks could exist. For smooth solutions the flow can only pass through a transonic point at one radius, r_B. This means that smooth solutions that don't pass through a sonic point at r_B must remain either subsonic at all times or supersonic at all times.

If the solution does not have a sonic transition the left hand side of equation 17 is only zero when u is a maximum or minimum. In this case the radius where

$$r = \frac{GM}{2c_s^2} \tag{22}$$

corresponds to the radius where the velocity is a maximum or minimum of the flow. Equations 19, 20 and 21 do not apply as we have assumed that $u = c_s$ at this radius.

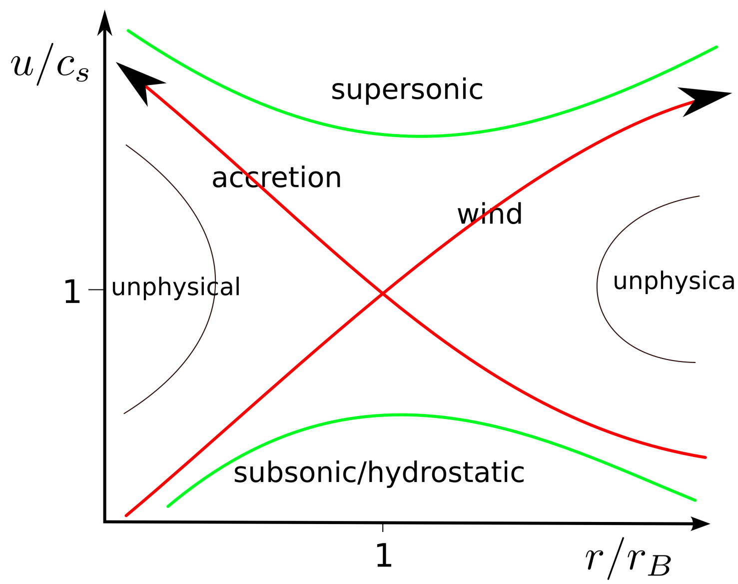

FIGURE 6. Steady state spherical flows. Only the wind and accretion flows have a transonic point and that point must happen at $r = r_B$.

There are 4 different types of solutions.

(1) Those starting subsonic at small radius and crossing at r_B to supersonic outflow (stellar winds).

(2) Those starting subsonic at large radius and crossing at r_B to supersonic inflow (Bondi accretion).
(3) Those starting subsonic and remaining subsonic everywhere. These are nearly hydrostatic solutions.
(4) And those remaining supersonic. The last two classes of solutions can be approximated by dropping either the u^2 term or the c_s^2 in equation (17).

We can also parametrize the solutions in terms of the accretion rate with

$$\dot{M} = \lambda 4\pi \rho_\infty c_\infty r_G^2 = \lambda 4\pi\rho_\infty (GM)^2 c_\infty^{-3} \tag{23}$$

We define λ_c as the value that allows a solution with a transonic point (see the coefficient in equation 21 for λ_c). The wind solutions (Parker winds) can be similarly described with a parameter λ.

1.4.1. *Maximal accretion rate solution.* One last thing that it is maybe useful to know is that the accretion rate is maximized by the solution with the sonic transition. You can show that the accretion rate for the transonic solution is an extremum by considering the velocity and speed of sound for flows that are not transonic at r_B. Let us rewrite the parameter λ giving the accretion rate in equation 23

$$\lambda = \left(\frac{u}{c_\infty}\right)\left(\frac{r}{r_G}\right)^2\left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\gamma-1}} \tag{24}$$

Now let us look at the radius where the velocity is an extremum or equal to the sound speed (where the left hand side of equation 17 is zero) and $r = r_B = GM/(2c_s^2)$. Inserting this in to the previous equation

$$\lambda = \left(\frac{u}{c_\infty}\right)\frac{1}{4}\left(\frac{c_s}{c_\infty}\right)^{\frac{2}{\gamma-1}-4} \tag{25}$$

where u, c_s are those for each flow at r_B. Now consider Bernoulli's equation (equation 13) at this radius

$$u^2\frac{(\gamma-1)}{2} + c_s^2(3-2\gamma) = c_\infty^2 \tag{26}$$

Solve for c_s^2/c_∞^2 and sub this into our equation for λ. Then take the derivative with respect to u

$$\frac{1}{\lambda}\frac{d\lambda}{du} = \left(1 - \frac{u^2}{c_\infty^2}\frac{\gamma-1}{2}\right) - \frac{u^2}{c_\infty^2}(3-2\gamma). \tag{27}$$

Set the above equation to zero to find the extremum for λ and solve for u (the velocity for the extreme solution at r_B) finding

$$\frac{u^2}{c_\infty^2} = \frac{5-3\gamma}{2} \tag{28}$$

But this is the sound speed at r_B for the transonic solution. The transonic solutions turn out to be maximum flow rate solutions.

It is interesting to watch the video on nozzle flow after thinking about Bondi flow as the equations are similar. Instead of a gravitational potential there is a nozzle surface area function. For a flow entering the nozzle under sufficient pressure, flow is regulated. The narrowest part of the nozzle sets the sonic transition point. If the pressure outside the nozzle is not consistent with that predicted using Bernoulli's equation there is a shock after the narrowest part of the nozzle. Our analysis here has not considered the possibility of shocks in the flow.

There is a nice review of Bondi-Hoyle-Lyttleton accretion at `http://nedwww.ipac.caltech.edu/level5/March09/Edgar/Edgar_contents.html` including its limitations, what happens with the addition of more physics (moving object, opacity, cooling) and some applications.

Here we have used only 3 equations to discuss the flow, conservation of mass, Euler's equation and an equation of state. If radiation or conductivity or cooling were considered as part of the flow then an equation for energy transport would be required.

1.5. **Subsonic and Supersonic limits.** Consider Bernoulli's equation

$$\frac{u^2}{2} + \frac{c_s^2}{\gamma - 1} = \frac{GM}{r} + \frac{c_\infty^2}{\gamma - 1} \tag{29}$$

When $u \gg c_s$ (supersonic limit) we can drop the terms with sound speeds and we find a relation between velocity and radius that is equivalent to gas in freefall. Pressure in the gas fails to slow the flow down at all. Because we have a solution for $u(r)$ we can write the density in terms of $\dot{M}$ and radius. We can't write our solution in terms of density at infinity because at some point at large radius the flow must drop below the sound speed and violate our assumption that $u \gg c_s$.

When $u \ll c_s$ (the subsonic limit) we can drop the term depending on u in Bernoulli's equation and we find

$$c_s = \sqrt{v_c^2(\gamma - 1) + c_\infty^2} \tag{30}$$

where $v_c^2 = GM/r$ is the velocity of a particle in a circular orbit. This solution is the same as we would have found with a hydrostatic solution. In other words if we go back to Euler's equation

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p - \nabla \Phi \tag{31}$$

Taking the limit $u \to 0$ is the same as dropping the terms on the left or letting $u \ll c_s$. And this is the same as assuming hydrostatic equilibrium. As $\frac{1}{\rho}\nabla p = \nabla h$ hydrostatic equilibrium can also be written $h = \Phi +$ constant, where h is the enthalpy. This and our definition for enthalpy gives us the above radial form for the sound speed. As we have taken the $u \to 0$ limit we have no constraint on the velocity as a function of radius. However we could solve for the density as a function of radius using our radial formula for sound speed and integrating the equation for hydrostatic equilibrium. Equivalently as $h \propto p/\rho \propto \rho^{\gamma-1}$ we can solve for ρ in terms of v_c using Bernoulli's equation. It is useful to remember that $\gamma K = c_\infty^2 \rho_\infty^{\gamma-1}$ giving

$$\frac{\rho^{\gamma-1}}{\rho_\infty^{\gamma-1}} = c_\infty^{-2}((\gamma-1)v_c^2 + c_\infty^2) \tag{32}$$

and allowing us to solve for ρ as a function of r.

1.6. **Notes on astrophysical applications of Bondi flow.** The Galactic center harbors a million solar mass black hole that lies in an X-ray emitting medium. There are massive stars in the vicinity driving winds. These winds provide a medium that can accrete onto the black hole. The Bondi accretion rate is estimated as $10^{-5} - 10^{-6} M_\odot$ yr^{-1} however the total luminosity from the black hole is less than 10^{36} erg/s which is 10^{-9} times the Eddington luminosity. The black hole at the Galactic Center, as in many elliptical galaxies, is extremely **dim**. Thin disk accretion models are ruled out at accretion rates as low as $\dot{M} \lesssim 10^{-10} M_\odot$ yr^{-1}. This problem has inspired new accretion models including radiatively inefficient forms of accretion or flows that efficiently drive winds so that not much material makes it into the black hole. There is also the possibility that accretion is episodic.

1.7. **Analogy with the De Laval Nozzle.**

Now is a good time to watch *Channel Flow of a Compression Fluid* at http://www.youtube.com/watch?v=JhlEkEk7igs (Note some of the You-tube videos are screwed up but this one seemed OK.) Maybe the lack of sync between voice and video has to do with the internet connection. I watched this one okay in my office and then it screwed up for the second (and never again) time in class.

In the above section we introduced Bernoulli's equation and immediately applied it to the astrophysically motivated problem of accretion onto massive bodies (Bondi accretion) and stellar winds. More traditionally one would have introduced the rocket nozzle. Here the nozzle cross sectional area serves the role of the gravitational potential in Bondi-flow.

The De Laval nozzle is a one dimensional flow into a pipe that has varying cross sectional area. We describe the flow as a function of time t and position x along the nozzle. This gives us pressure, density and velocity $p(x,t), u(x,t), \rho(x,t)$. These functions will be determined from the properties of our fluid at the input ρ_0, u_0 and sound speed c_0 and will depend on the shape of the pipe or nozzle. Assuming an equation of state we can use the sound speed as variable instead of the pressure as

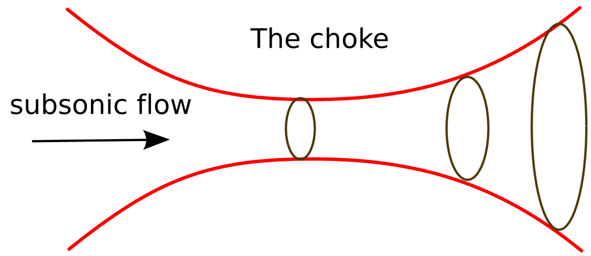

Figure 7. Flow through a nozzle can make the transition to supersonic flow where the cross sectional area of the nozzle is a minimum.

we did for our discussion on Bondi flow. Bernoulli's equation is particularly simple

$$\frac{u^2}{2} + h = \text{constant} \tag{33}$$

When we discussed Bondi flow Bernoulli's equation contains a term depend on gravity. Here instead of a gravity term we have the cross sectional area of our nozzle that depends on distance; $S(x)$. Conservation of mass becomes

$$\rho u S(x) = \dot{M} = \text{constant} \tag{34}$$

and its differential form

$$\frac{\rho_{,x}}{\rho} + \frac{u_{,x}}{u} + \frac{S_{,x}}{S} = 0 \tag{35}$$

As we will show in our problem set, the transonic point must occur at an extremum of S, where $dS/dx = 0$. This is not only an extremum but a minimum. The transonic point here is set by the nozzle minimum rather than set by the Bondi radius.

2. Riemann Invariants and the Method of Characteristics

The method of characteristics is a method used to solve the initial value problem for general first order partial differential equations. By first order we mean only containing first order derivatives. Our conservation laws fit into this category. Consider

$$a(x,t)u_{,x} + b(x,t)u_{,t} + c(x,t)u = 0 \tag{36}$$

along with an initial condition

$$u(x,0) = f(x) \tag{37}$$

We find a change of coordinate system from (x,t) to a new coordinate system (x_0, s). In the new coordinate system the partial differential equation becomes an ordinary differential equation (with only derivatives with s) along certain curves in the x,t plane. The new variable x_0 is constant along these curves. These curves are called

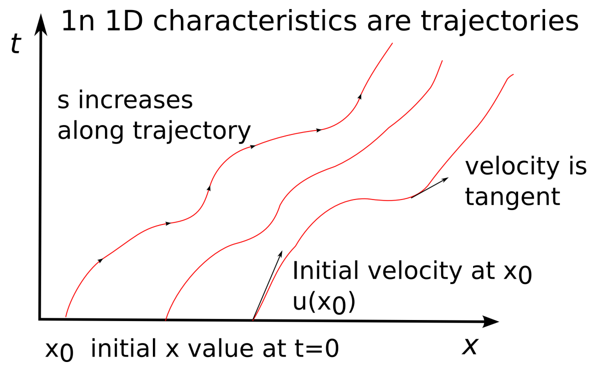

FIGURE 8. Trajectories can be specified by characteristic curves as a function of space and time where the velocity is the inverse of the slope of the curve. Along each curve (trajectory) s increases. Each curve is specified by its initial x_0 value or x position value at time $t = 0$ or with $s = 0$. Solutions can be described by how u changes along s, for each initial condition x_0 rather than $u(x, t)$.

characteristics. The variable s varies along the characteristics but x_0 does not. However x_0 changes along the initial curve set by our initial condition with $t = 0$ in the x, t plane. In 1 dimension characteristics are the trajectories of particles starting at x_0 and with velocity u as a function of position and time satisfying equation 36.

Choose

$$\frac{dx}{ds} = a(x(s), t(s))$$
$$\frac{dt}{ds} = b(x(s), t(s))$$

so that

$$\begin{aligned}\frac{du}{ds} &= \frac{dx}{ds}\frac{du}{dx} + \frac{dt}{ds}\frac{du}{dt} \\ &= \frac{dx}{ds}u_{,x} + \frac{dt}{ds}u_{,t} = a(x, t)u_{,x} + b(x, t)u_{,t}\end{aligned}$$

Our partial differential equation can be written

$$\frac{du}{ds} + c(x, t)u = 0 \qquad \frac{du}{ds} + c(x(s, x_0), t(s, x_0))u = 0 \tag{38}$$

The above is an ordinary differential equation. We solve it for $u(s)$ and use x_0 for constants of integration.

$$\frac{1}{u}\frac{du}{ds} = -c(x_0, s)$$

$$d \ln u = c(x_0, s) ds$$

The equations

$$\frac{dx}{ds} = a(x(s), t(s)) \tag{39}$$

$$\frac{dt}{ds} = b(x(s), t(s)) \tag{40}$$

are called the *characteristic equations*. They don't depend on the initial conditions x_0. However, after you integrate these equations our formulas for $x(s)$ and $t(s)$ will have constants of integration that do depend x_0. The characteristic equations have slope (velocity) a/b so that a particle trajectory depends only on x_0.

Let us divide one characteristic equation by another

$$\frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{dt} = \frac{a}{b} \tag{41}$$

This implies that a/b is a slope on a x, t plot.

Our original variations are x, t. We are considering new variables x_0, s. Our original variables $x(s, x_0)$ and $t(s, x_0)$. We specify that $t(s = 0, x_0) = 0$ and that $x(s = 0, x_0) = x_0$. We can think of the above characteristic equations as partial derivatives

$$\frac{\partial x(s, x_0)}{\partial s} = a \tag{42}$$

$$\frac{\partial t(s, x_0)}{\partial s} = b \tag{43}$$

Once these are integrated they depend on the constant of the integration or functions that depend on x_0.

2.1. **General strategy.** Here is a recipe

1) Solve the two characteristic equations for a relation between s and $x(s)$ and $t(s)$. Assume that $t = 0$ for $s = 0$.

2) Find constants of integration for the integrated characteristic equations by setting $x(s = 0) = x_0$.

3) Solve the ordinary differential equation for $\frac{du}{ds}$ with initial condition $u(0) = f(x_0)$. We now have a solution $u(x_0, s)$.

4) Solve for s and x_0 in terms of x, t using our solution for the characteristic equations.

5) Sub these values into $u(x_0, s)$ to get a solution to the original partial differential equation as a function of (x, t).

2.1.1. *Constant coefficient example.* Let $a(x,t)$ be a constant, $b = 1$ and $c = 0$. Our partial differential equation and initial condition are

$$au_{,x} + u_{,t} = 0 \tag{44}$$

$$u(x, t = 0) = f(x) \tag{45}$$

Our characteristic equations are

$$\frac{dt}{ds} = 1 \quad \to \quad t = s + \text{constant}(x_0) \tag{46}$$

$$\frac{dx}{ds} = a \quad \to \quad x = as + \text{constant}(x_0) \tag{47}$$

We find that $t = s+$ a constant that depends on x_0. Letting $t(0) = 0$ we find that $s = t$. Our second characteristic equation gives $x = as + g(x_0)$. Letting $x(0) = x_0$ we find $x = as + x_0 = at + x_0$. Altogether

$$t = s \tag{48}$$

$$x = as + x_0 \tag{49}$$

This gives us

$$x_0 = x - as = x - at \tag{50}$$

Our ordinary differential equation becomes

$$\frac{du}{ds} = 0 \to u(s, x_0) = \text{constant}(x_0) \tag{51}$$

So that u does not vary with s. This means that $u(s) = h(x_0)$ for some function $h()$. Our initial condition is $u(s = 0) = f(x_0)$. The solution is

$$u(s) = f(x_0) = f(x - at)$$

We find that the velocity is constant along lines of constant $x - at$ as expected.

Note that our constant coefficient example

$$au_{,x} + u_{,t} = 0 \tag{52}$$

could be written as

$$\frac{Du}{Dt} = \left[\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right] u = 0 \tag{53}$$

if we think about u as a quantity moving with a velocity a. The characteristics with slope (or velocity) a are streamlines or particle trajectories. Our variable s is a variable that changes along streamlines. Our characteristics velocities (or slopes) are the speed that things move along these streamlines. Each streamline is defined or fixed by its initial condition or x_0.

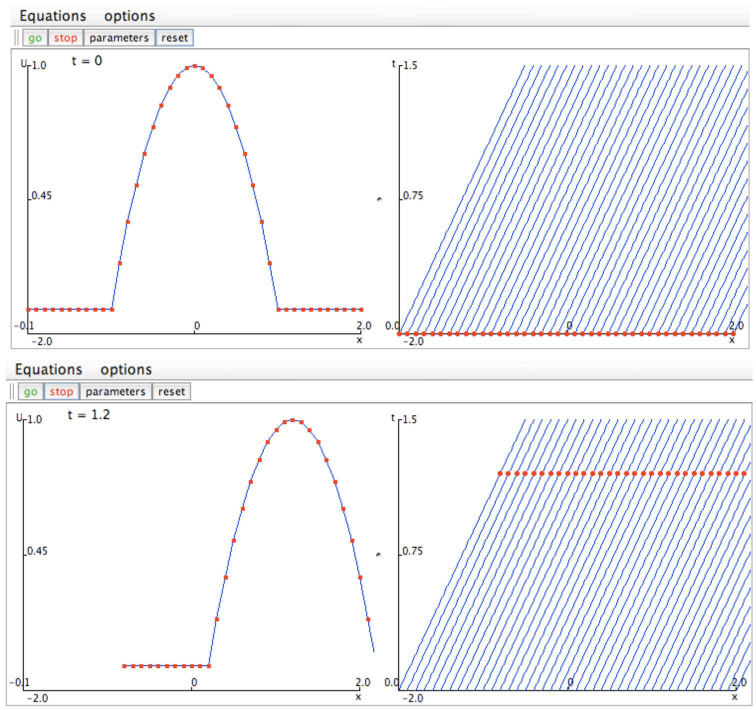

FIGURE 9. On top are initial conditions and on the bottom showing evolution after a period of time. The left most panels show u vs x the rightmost panels show t vs x. The differential equation is $u_{,t}+au_{,x}=0$ so that characteristics are parallel. Figure made with `http://www.scottsarra.org/shock/shockApplet.html` Here u is just a quantity advected at velocity a.

2.1.2. *A more difficult example.* Consider differential equation and initial condition

$$2xtu_{,x} + u_{,t} = u \tag{54}$$

$$u(x,0) = x. \tag{55}$$

The characteristic equations are

$$\frac{dt}{ds} = 1 \quad \rightarrow \quad t = s + \text{constant}(x_0) \tag{56}$$

$$\frac{dx}{ds} = 2x(s)t(s) \quad \rightarrow \quad d\ln x = 2t(s)ds \tag{57}$$

Setting $s = 0$ at $t = 0$ we find that $t = s$. The second characteristic equation gives us

$$dx/x = 2sds \tag{58}$$

with the solution $\ln x = s^2 + c(x_0)$ and constant function $c(x_0)$. This gives us

$$x = c'(x_0) \exp s^2 \tag{59}$$

with another function $c'(x_0)$. At $t = 0$ or $s = 0$ we would like $x = x_0$. This lets us specify the function of x_0;

$$x(x_0, s) = x_0 \exp s^2 \tag{60}$$

or

$$x_0(x, t) = x \exp(-t^2) \tag{61}$$

Our characteristic equations are more complex than our last example. Here they are exponentially increasing lines. Let us write variables in terms of one another

$$\begin{array}{ll} x(x_0, s) = x_0 \exp(s^2) & x_0(x, t) = x \exp(-s^2) \\ t(x_0, s) = s & s(x, t) = t \end{array}$$

Our first order differential equation is

$$\frac{du}{ds} = u \tag{62}$$

with solution $u = c(x_0)e^s$. Note u is not conserved along our characteristic equations but increases exponentially along it. We have been given the initial condition $u(t = 0) = x$ so $c(x_0) = x_0$ and

$$u(x_0, s) = x_0 e^s \tag{63}$$

But $x_0 = x \exp(-t^2)$ so we have a solution at later times

$$u(x, t) = x \exp(-t^2 + t). \tag{64}$$

What do the characteristics look like? On a x, t plot, $s = t$ and x_0 sets the root of each characteristic on the x axis. If we set t on the y axis then we want a plot of t as a function of x and x_0. Inverting equation 60

$$\sqrt{\ln(x/x_0)} = t = s \tag{65}$$

See Figure 10. And along each characteristic, the u value will increase exponentially according to equation 63, however the characteristics turn over so quickly that the solution for $u(x, t)$ drops at large t. The u stuff is advocated really quickly to large x along the characteristics, so quickly that this overcomes the fact that u increases along each characteristic.

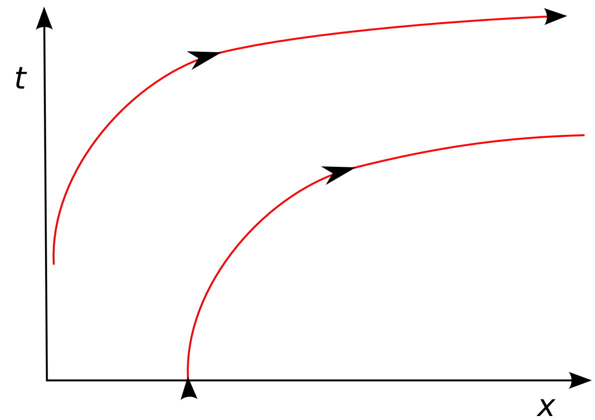

FIGURE 10. Characteristics where $x(x_0, s) = x_0 \exp s^2$ and $t = s$. The lines drawn are $t = \sqrt{\ln(x/x_0)}$. Each x_0 gives a different curve.

2.2. **Vector Linear case.** Consider vectors $\mathbf{y}$, a linear equation and initial condition

$$\mathbf{y}_{,t} + \mathbf{A}\mathbf{y}_{,x} = 0 \tag{66}$$

$$\mathbf{y}(\mathbf{x}, t = 0) = \mathbf{F}(\mathbf{x}) \tag{67}$$

with $\mathbf{A}$ a matrix and $\mathbf{F}$ a vector of functions. We diagonalize the matrix with $\mathbf{U}$ so that

$$\mathbf{\Lambda} = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} \tag{68}$$

is diagonal and set

$$\mathbf{z} = \mathbf{U}\mathbf{y}. \tag{69}$$

The matrix $\mathbf{U}$ gives a coordinate transformation so that $\mathbf{A}$ becomes diagonal ($\mathbf{\Lambda}$). In the new coordinate system, the basis vectors are eigenvectors of $\mathbf{A}$.

Insert $\mathbf{y} = \mathbf{U}^{-1}\mathbf{z}$ into our differential equation and we find

$$\mathbf{U}^{-1}\mathbf{z}_{,t} + \mathbf{A}\mathbf{U}^{-1}\mathbf{z}_{,x} = 0 \tag{70}$$

Multiply by $\mathbf{U}$ and we find

$$\mathbf{z}_{,t} + \mathbf{\Lambda}\mathbf{z}_{,x} = 0. \tag{71}$$

Our equation becomes a series of equations (one for each component)

$$z_{i,t} + \lambda_i z_{i,x} = 0 \tag{72}$$

where λ_i are the eigenvalues of $\mathbf{A}$. Our initial condition for the eigenvectors $\mathbf{z}(\mathbf{x}, t = 0) = \mathbf{U}\mathbf{F}(\mathbf{x})$. We define a new vector function $\mathbf{G}(\mathbf{x})$ so that

$$\mathbf{z}(\mathbf{x}, t = 0) = \mathbf{G}(\mathbf{x}) = \mathbf{U}\mathbf{F}(\mathbf{x}) \tag{73}$$

The eigenvalues are the velocities and determine the characteristics for the solutions of $\mathbf{z}(\mathbf{x},t)$ and so for $\mathbf{y}(\mathbf{x},t)$.

$$z_i(\mathbf{x},t) = G_i(\mathbf{x} - \lambda_i t) \tag{74}$$

Now that we know $\mathbf{z}(\mathbf{x},t)$ we also know

$$\mathbf{y}(\mathbf{x},t) = \mathbf{UG} \qquad \Longleftrightarrow \qquad y_i(\mathbf{x},t) = U_{ij} G_j(\mathbf{x} - \lambda_j t) \tag{75}$$

where on the right I have written the solution out in terms of components. This is the solution at all times.

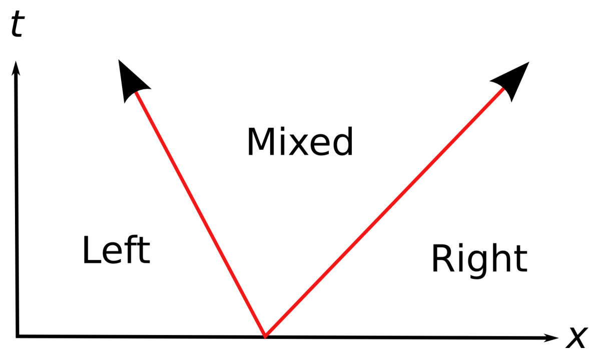

Figure 11. For the two dimensional case, there can be two characteristic velocities. If the initial condition is a constant vector on the left and a different constant on the right, the solution in the middle is a mixed state at later times.

For every eigenvector we can have a characteristic velocity λ_i. For every dimensional degree of freedom we can have a different characteristic velocity. While in one dimension we only had a single characteristic velocity, in 3-dimensions we can have three. These turn out to be the different velocities that information can propagate. For Euler's equation in three dimension, we will find characteristic velocities like $u, u + c, u - c$.

The vector form is useful to think about because our fluid equations can be written in conservation law form like

$$\rho_{,t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{76}$$

$$(\rho \mathbf{u})_{,t} + \nabla \cdot (P\mathbf{g} + \rho \mathbf{u} \otimes \mathbf{u}) = 0 \tag{77}$$

So we can think of ρ and $\rho\mathbf{u}$ as the components of our vector $\mathbf{y}$. The analogy is that we will have eigenvalues, $v \pm c$, for information propagating at the sound speed and that our solution will be a vector combination of information propagating at these two velocities. However the above fluid equations are non-linear so we will explore this case in one dimension before considering characteristics for the fluid equations in two or three dimensions.

2.3. **Non-linear 1-dimensional case.** Consider an equation in conservation law form

$$u_{,t} + \frac{\partial F(u)}{\partial x} = 0 \tag{78}$$

We can write this as

$$u_{,t} + \frac{dF(u)}{du} u_{,x} = 0 \tag{79}$$

so that we can think of

$$c = F'(u) \tag{80}$$

as our velocity and we would expect our solutions to depend on $x - ct$. However c is no longer a constant and depends on $u(x,t)$.

Let's try following our method of characteristics. We find $s = t$ again from our first characteristic equation. Our second one

$$\frac{dx}{ds} = F'(u(x,t)) \tag{81}$$

$$\frac{d}{ds}u(x,t) = u_{,x}\frac{dx}{dt} + u_{,t} = u_{,x}F'(u(x,t)) + u_{,t} = 0 \tag{82}$$

$$\frac{du}{ds} = 0 \tag{83}$$

This means that the solution $u(x(t),t)$ will not change with time along the characteristic curve. As u is fixed along a characteristic so is $F'(u)$ along each characteristic. This means the characteristic velocity (and slope in a x,t plot) is stays fixed along each characteristic.

So we start with an initial condition $u(x_0, t = 0)$ and then figure out the velocity $F'(u)$ at x_0 and $t = 0$. Each position x_0 has a characteristic that is a straight line but each position will have a characteristic that is a different slope. On an t,x with t along the vertical axis the slope of each characteristic is $1/F'(u(x_0, t = 0))$. Each characteristic crosses the $t = 0$ axis at initial x_0. Each characteristic satisfies

$$x = x_0 + F'(u(x_0, t = 0))t \tag{84}$$

or

$$t = \frac{x - x_0}{F'(u(x_0, t = 0))t}. \tag{85}$$

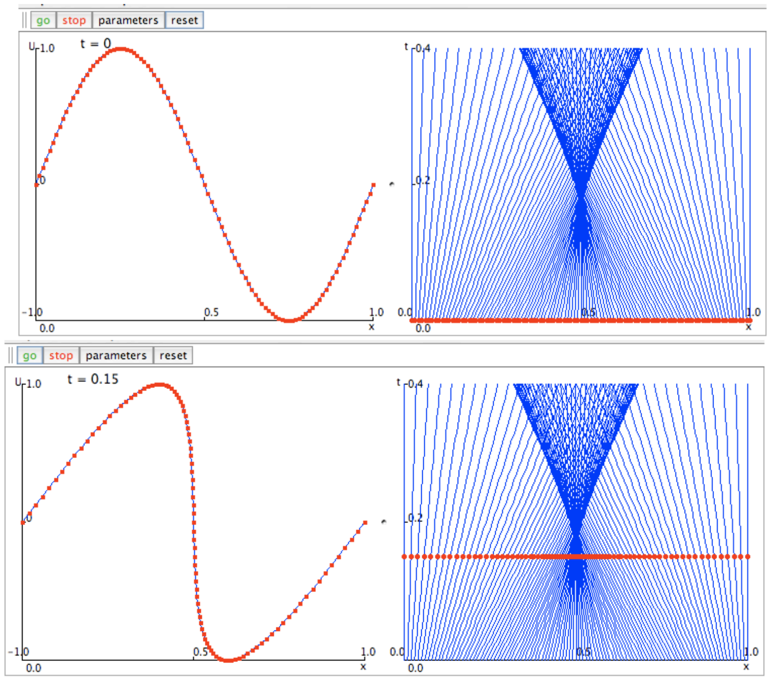

FIGURE 12. On top are initial conditions and on the bottom showing evolution after a period of time. The left most panels show u vs x the rightmost panels show characteristics on a plot of t vs x. The differential equation is $u_{,t} + uu_{,x} = 0$. Characteristics are not parallel. Note the steepening to a shock at later times. The peaks of the sine wave travel faster than the troughs. Figure made with `http://www.scottsarra.org/shock/shockApplet.html`

Previously we described things moving with the fluid with a derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \tag{86}$$

where c is the velocity. Now the effective velocity depends on u but u itself still remains constant along its own streamlines or characteristics. See figure 12 for an example using Burger's equation.

2.4. **Steepening into a shock.** The example of the inviscid Burger's equation shows that the velocity remains constant on characteristics but that the characteristics do not have the same slope. An initial condition of a sine wave steepens till there is an infinite slope (the characteristics cross). At this time a shock develops and the fluid properties change rapidly over a small distance in distance that is of order a few mean free path lengths.

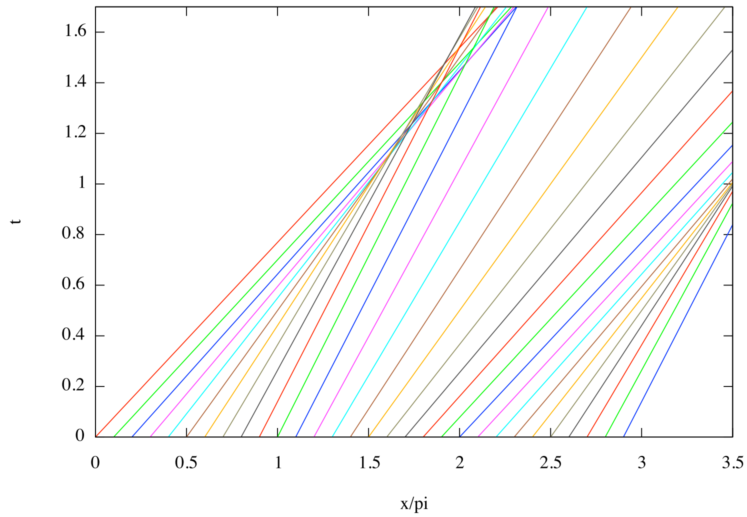

FIGURE 13. Characteristics are shown on a t vs x plot for Burger's equation with initial conditions that are a sine wave. The time to development of a discontinuity can be estimated by computing the time it takes characteristics to cross.

Using characteristics we can estimate the time it takes for a small discontinuity to develop into a shock. Consider the following non-linear equation

$$u_{,t} + cu^{1.5}u_{,x} = 0 \tag{87}$$

with initial condition

$$u(x, t = 0) = u_0 + A\cos kx \tag{88}$$

with $A \ll u_0$. At what time does a shock develop? Here the coefficient c has units of velocity times $u^{-1.5}$ and A has units of u.

Solutions have constant u along lines in x, t space with slope that is determined from the initial u value at each initial position or x_0. They satisfy

$$x = x_0 + u(x_0, t = 0)^{1.5}t \tag{89}$$

or

$$t = \frac{(x - x_0)}{u(x_0, t = 0)^{1.5}} \tag{90}$$

On a t vs x plot like shown in Figure 13 the slopes of the characteristics are given by the initial value of $1/(cu^{1.5})$. On a plot like that shown in Figure 13 we need to invert equations 92, 93 to give lines for the characteristics at $x_0 = 0$ and $x_0 = \pi/k$ or

$$
\begin{aligned}
t &= \frac{x}{cu_0^{1.5}\left(1 + A/u_0\right)^{1.5}} \\
t &= \frac{x - \pi/k}{cu_0^{1.5}\left(1 - A/u_0\right)^{1.5}}
\end{aligned} \tag{91}
$$

The characteristic with $x_0 = \pi/k$ is steeper than that for $x_0 = 0$.

The characteristic going through $x_0 = 0$ is

$$
x = cu_0^{1.5}\left(1 + \frac{A}{u_0}\right)^{1.5} t \sim cu_0^{1.5}\left(1 + 1.5\frac{A}{u_0}\right) t \tag{92}
$$

At $x_0 = \pi/k$ the characteristic has

$$
x = cu_0^{1.5}\left(1 - \frac{A}{u_0}\right)^{1.5} t - \frac{\pi}{k} \sim cu_0^{1.5}\left(1 - 1.5\frac{A}{u_0}\right) t + \frac{\pi}{k} \tag{93}
$$

By subtracting one equation from the other, we solve for the time when the two characteristics intersect, finding

$$
t \approx \frac{\pi}{k}\frac{u_0}{A}\frac{1}{3cu_0^{1.5}}. \tag{94}
$$

A quick check of units is comforting.

If A is very small then it takes a long time for a discontinuing to develop. If the differential equation is non-linear, then even very smooth initial conditions can eventually give discontinuities.

2.5. **Characteristics and discontinuities.** Consider Burger's equation

$$
u_{,t} + uu_{,x} = 0 \tag{95}
$$

with an initial discontinuity

$$
u(x, t = 0) = \begin{cases} u_L & \text{for} \quad x < 0 \\ u_R & \text{for} \quad x \geq 0 \end{cases} \tag{96}
$$

Figure 15 shows a situation with $u_L < u_R$. No characteristics propagate into the green region so the solution is not unique. However not all solutions are physically meaningful or stable. Techniques to find physically good solutions include using an entropy condition or specifying a vanishing viscosity at discontinuities.

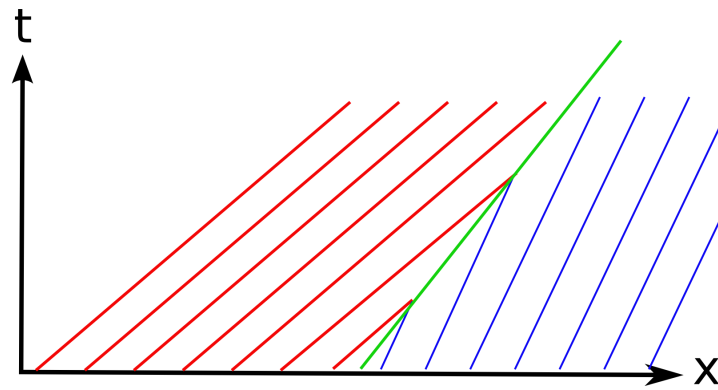

FIGURE 14. Here we show converging characteristics for Burger's equation with $u_L > u_R$. The shock appears at the intersections of the characteristics.

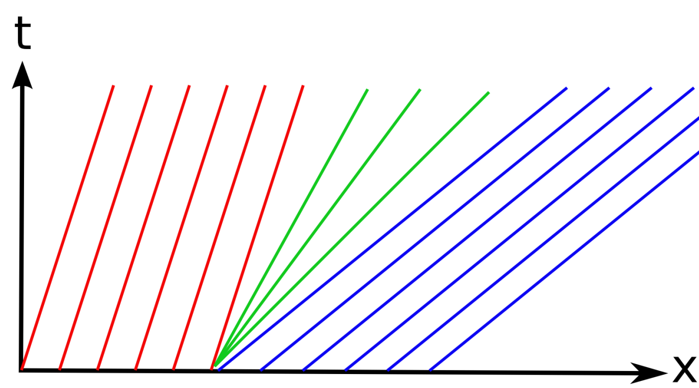

FIGURE 15. Here we show diverging characteristics with $u_L < u_R$. This is a **rarefaction** wave. No information can propagate into the green region so the solution in this region is not unique. However the fan-like solution is preferred as it is stable and would arise in the presence of viscous processes.

2.6. **Characteristics for isentropic fluid flow in one dimension.** Starting in conservation law form, conservation of mass and Euler's equation can be written for fluid flow in 1 dimension

$$\rho_{,t} + (\rho u)_{,x} = 0 \tag{97}$$

$$(\rho u)_{,t} + (p + \rho u^2)_{,x} = 0 \tag{98}$$

Define a new variable $j = \rho u$. The above two equations become

$$\rho_{,j} + j_{,x} = 0 \tag{99}$$

$$j_{,t} + \left(p + \frac{j^2}{\rho}\right)_{,x} = 0 \tag{100}$$

which we can write in terms of vectors as

$$\mathbf{y}_t + \mathbf{F}(\mathbf{y})_{,x} = 0 \tag{101}$$

with conserved quantities

$$\mathbf{y} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \tag{102}$$

and flux vector $\mathbf{F}$

$$\mathbf{F}(\mathbf{y}) = \begin{bmatrix} j \\ p(\rho) + \frac{j^2}{\rho} \end{bmatrix}. \tag{103}$$

We need not add a third conservation law if we adopt an equation of state relating p, ρ.

Our flux vector $\mathbf{F}(\mathbf{y})$ has Jacobian matrix $\mathbf{A}$ (with components $A_{ij} = \frac{\partial F_i}{\partial y_j}$)

$$\mathbf{A}(\mathbf{y}) = \begin{pmatrix} 0 & 1 \\ c_s^2 - u^2 & 2u \end{pmatrix} \tag{104}$$

with sound speed c_s such that

$$\frac{\partial p}{\partial \rho} = c_s^2 \tag{105}$$

Using this Jacobian we can write our original equation as

$$\mathbf{y}_{,t} + \mathbf{A}(\mathbf{y})\mathbf{y}_{,x} = 0. \tag{106}$$

Eigenvalue and eigenvector pairs for $\mathbf{A}$ are

$$\begin{pmatrix} 1 \\ u \pm c_s \end{pmatrix} \qquad\qquad u \pm c_s \tag{107}$$

The **eigenvalues** define our characteristic velocities. We can rewrite the Jacobian as a product

$$\mathbf{A} = \mathbf{R\Lambda L} \tag{108}$$

where $\mathbf{R}$ and $\mathbf{L}$ are composed of right and left eigenvectors and the diagonal matrix $\mathbf{\Lambda}$ has the eigenvalues in it.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ u + c_s & u - c_s \end{pmatrix} \begin{pmatrix} u + c_s & 0 \\ 0 & u - c_s \end{pmatrix} \begin{pmatrix} -(u - c_s)/2c_s & 1/2c_s \\ (u + c_s)/2c_s & -1/2c_s \end{pmatrix} \tag{109}$$

Using $\mathbf{A}$ we can write our original differential equation as

$$\mathbf{y}_{,t} + \mathbf{R\Lambda L y}_{,x} = 0. \tag{110}$$

It would be nice to write our differential equation as two equations that look like

$$\frac{\partial J_\pm}{\partial t} + (u \pm c_s)\frac{\partial J_\pm}{\partial x} = 0 \tag{111}$$

and then we would have conserved quantities $J_\pm$ (known as Riemann invariants) along the characteristic curves defined by $u \pm c_s$. Our eigenvectors (which are proportional to $(u \pm c_s, 1)$) would satisfy this if we could find an integral form for them.

Let us write

$$\mathbf{A} = \mathbf{U\Lambda U}^{-1} \tag{112}$$

associating $\mathbf{U}^{-1} \propto \mathbf{L}$. Multiplying both sides of equation 110 by $\mathbf{U}^{-1}$ then our differential equation can be written

$$\mathbf{U}^{-1}\mathbf{y}_{,t} + \mathbf{\Lambda U}^{-1}\mathbf{y}_{,x} = 0 \tag{113}$$

If we can find a new variable

$$d\mathbf{v} = \mathbf{U}^{-1}d\mathbf{y} \tag{114}$$

then we can write our above differential equation as

$$\mathbf{v}_{,t} + \mathbf{\Lambda v}_{,x} = 0 \tag{115}$$

and our new vector $\mathbf{v}$ is conserved along characteristics. Note that the equation will still be non-linear as $\mathbf{\Lambda}$ depends on $\mathbf{v}$.

Going back to our form for $\mathbf{A}$ let us try

$$\mathbf{U}^{-1} = \frac{k}{2c_s}\begin{pmatrix} -(u - c_s) & 1 \\ (u + c_s) & -1 \end{pmatrix} \tag{116}$$

with constant k. It is convenient to compute

$$dj = \rho du + u d\rho \tag{117}$$

Multiplying equation 116 by the vector

$$d\mathbf{y} = \begin{pmatrix} d\rho \\ dj \end{pmatrix}, \tag{118}$$

using our expression for dj and taking the first component

$$\frac{(c_s - u)d\rho + dj}{\rho} = \frac{c_s d\rho}{\rho} + du \tag{119}$$

and a similar equation for the second component. Above we have used $k = 2c_s/\rho$. Using an equation of state and $c_s^2 = K\gamma\rho^{\gamma-1}$ we can show that

$$c_s \frac{d\rho}{\rho} = (K\gamma)^{1/2}\rho^{(\gamma-3)/2} d\rho = \frac{2}{\gamma-1} dc_s \tag{120}$$

This means that

$$\frac{(c_s - u)d\rho + dj}{\rho} = du + \frac{2dc_s}{\gamma - 1} \tag{121}$$

We find that with

$$J_\pm = u \pm \frac{2c_s}{\gamma-1} \tag{122}$$

we can write

$$\mathbf{U}^{-1}\begin{pmatrix} d\rho \\ dj \end{pmatrix} = \begin{pmatrix} dJ_+ \\ dJ_- \end{pmatrix} \tag{123}$$

we can write our matrix equation in the form of equation (111) or

$$\frac{\partial}{\partial t}\begin{pmatrix} J_+ \\ J_- \end{pmatrix} + \begin{pmatrix} u + c_s & 0 \\ 0 & u - c_s \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} J_+ \\ J_- \end{pmatrix} = 0 \tag{124}$$

Thus our 1D fluid equations are consistent with 2 characteristics and 2 conserved quantities (the Riemann invariants) that are conserved along these 2 characteristics.

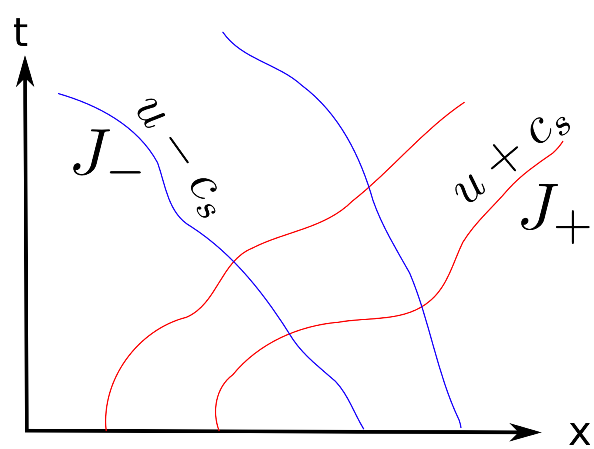

FIGURE 16. Solutions at a later time must conserve both Riemann invariants but each one is conserved along a different characteristic velocity line.

The Riemann invariants

$$
\begin{aligned}
u &= \frac{1}{2}(J_+ + J_-) \\
c_s &= \frac{\gamma - 1}{4}(J_+ - J_-)
\end{aligned}
$$

When an energy equation is added it is not always possible to find a simple form for conserved quantities along the characteristics and there is a third characteristic velocity that is u.

2.7. **Riemann invariants.** We illustrate another way to show that our functions $J_\pm$ (known as Riemann invariants) are conserved along trajectories of speed $u \pm c_s$. Conservation of mass in one dimension can be written

$$
\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) + \frac{\partial u}{\partial x} = 0 \tag{125}
$$

(where I have divided the continuity equation by ρ). Euler's equation in one dimension

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} \tag{126}
$$

Notice the $d\rho/\rho$ terms in both equations. We will replace these with expression that depend on dc_s.

We manipulate derivatives of the sound speed for a barytropic gas

$$
\begin{aligned}
c_s^2 &= K\gamma\rho^{\gamma-1} \\
2c_s dc_s &= (\gamma - 1)K\gamma\rho^{\gamma-2}d\rho \\
\frac{2}{\gamma - 1}\frac{dc_s}{c_s} &= \frac{d\rho}{\rho} \qquad (127)
\end{aligned}
$$

Inserting the expression for $d\rho/\rho$ into our 1 dimensional equations for conservation of mass and Euler's equation (and multiplying by c_s)

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{2}{\gamma - 1}c_s\right) + u\frac{\partial}{\partial x}\left(\frac{2}{\gamma - 1}c_s\right) + c_s\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c_s\frac{\partial}{\partial x}\left(\frac{2}{\gamma - 1}c_s\right) &= 0 \qquad (128)
\end{aligned}
$$

Adding and subtracting these two equations we obtain

$$
\begin{aligned}
\left[\frac{\partial}{\partial t} + (u + c_s)\frac{\partial}{\partial x}\right]\left(u + \frac{2}{\gamma - 1}c_s\right) &= 0 \\
\left[\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right]\left(u - \frac{2}{\gamma - 1}c_s\right) &= 0 \qquad (129)
\end{aligned}
$$

Equivalent to what we had shown before by diagonalizing the linearized equation and searching for integral forms for the eigenvectors. For a general problem, it may not be possible to find integral forms for the eigenvectors, however eigenvalues (characteristic velocities) and eigenvectors may still be used to approximate the solution. The matrix decomposition used here is sometimes used in numerical methods.

3. Shocks

3.1. **Jump Conditions.**

Consider the conservation law form for conservation of mass, momentum and energy for a gas can be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = 0 \tag{130}$$

with conserved variables

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ \rho(u^2/2 + e) \end{bmatrix} \tag{131}$$

and flux

$$\mathbf{F} = \begin{bmatrix} \rho \mathbf{u} \\ p\mathbf{g} + \rho \mathbf{u} \otimes \mathbf{u} \\ \rho \mathbf{u}(e + u^2/2 + p/\rho) \end{bmatrix} \tag{132}$$

with $\mathbf{g} = \delta_{\mathbf{ij}}$ a metric tensor that allows us to write the momentum flux density as a tensor.

Consider applying Gaus's theorem to a short cylindrical volume at a stationary discontinuity with normal $\mathbf{n} = (0, 0, 1)$ in the z direction. This is equivalent to working in the frame moving with the discontinuity. Gaus's law for any divergence

$$\int \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{A} \tag{133}$$

Adjusting our volume so that it is a narrow slab and oriented with normal $\mathbf{n}$

$$\mathbf{F} \cdot \mathbf{n} \mid_1^2 = (\mathbf{F_2} - \mathbf{F_1}) \cdot \mathbf{n} = 0 \tag{134}$$

We find that conservation of mass, with $\mathbf{F} = \rho \mathbf{u}$,

$$[\rho \mathbf{u} \cdot \mathbf{n}]_1^2 = 0 \tag{135}$$

across a discontinuity (where the subscripts refer to quantities on one side subtracted by those on the other side). If we consider density on each side as ρ_1, ρ_2 and take velocity components only in the direction perpendicular to the discontinuity in the frame of the shock, conservation of mass implies

$$\rho_1 u_{z1} = \rho_2 u_{z2} \tag{136}$$

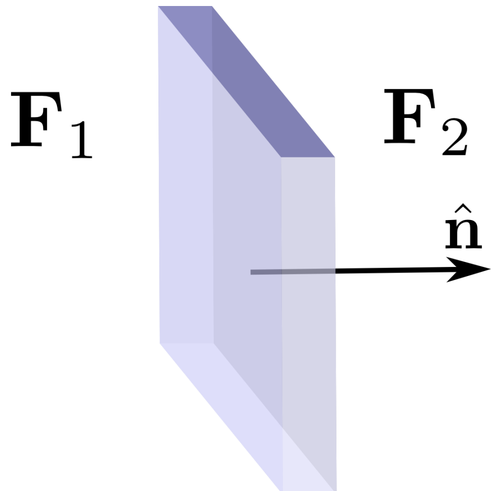

FIGURE 17. Apply Gaus's law to a conservation law to give a relation ship between fluxes on either side of a discontinuity, in a frame moving with the discontinuity; $\mathbf{F_1} \cdot \hat{\mathbf{n}} = \mathbf{F_2} \cdot \hat{\mathbf{n}}$.

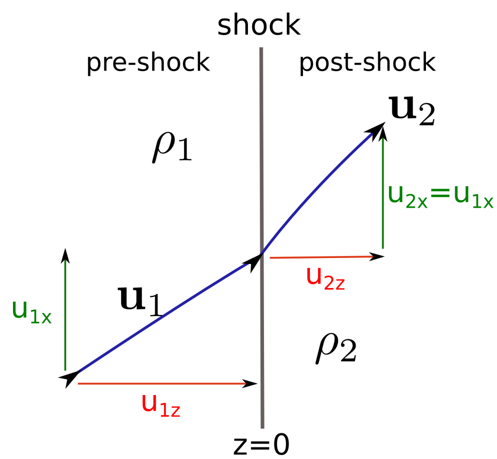

FIGURE 18. Jump conditions on velocity and density across a shock, in the frame moving with the shock. The u_x, u_y velocity components remain unchanged but the u_z velocity component and the density both can differ on either side of the discontinuity.

We now consider momentum density and momentum flux. Our momentum flux is given by the stress energy tensor, $\boldsymbol{\pi}_{ij} = p\delta_{ij} + \rho v_i v_j$. Applying Gaus's theorem to each component we have three equations (each one through surfaces with direction

n).

$$\boldsymbol{\pi} \cdot \mathbf{n} \,|_1^2 = 0 \tag{137}$$

or in summation notation

$$\pi_{ij} n_j \,|_1^2 = 0 \tag{138}$$

or orienting the shock normal along z (and in the shock frame)

$$[\pi_{xz}]_1^2 = [\pi_{yz}]_1^2 = [\pi_{zz}]_1^2 = 0 \tag{139}$$

For each one of these equations conservation of momentum implies

$$\left[p + \rho u_z^2\right]_1^2 = 0 \tag{140}$$

$$[\rho u_x u_z] = 0 \tag{141}$$

$$[\rho u_y u_z] = 0 \tag{142}$$

The second two equations (along with that for conservation of mass; $[\rho u_z] = 0$) imply that velocity components parallel to the discontinuity don't change. The first of the equations can be written

$$p_1 + \rho_1 u_{z1}^2 = p_2 + \rho_2 u_{z2}^2 \tag{143}$$

Lastly we look at the energy conservation. Conservation of energy leads to the following shock condition

$$\left[\rho \mathbf{u} \cdot \mathbf{n} \left(\frac{u^2}{2} + e + \frac{p}{\rho} \right)\right]_1^2 = 0 \tag{144}$$

or using conservation of mass again

$$\left[\frac{u^2}{2} + e + \frac{p}{\rho}\right]_1^2 = 0 \tag{145}$$

These jump conditions are known as the **Rankine-Hugoniot conditions**.

In terms of the *upstream Mach number*, $M_1 = u_1/c_1$, and using an equation of state with adiabatic index γ (relating $p(\rho)$) it is possible to show that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \tag{146}$$

and

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \tag{147}$$

Also the downstream Mach number

$$M_2^2 = \frac{2 + (\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)} \tag{148}$$

There are maximum density and velocity changes allowed across a shock jump that depend on the adiabatic index. By considering the entropy change in a shock one can show that only compressive shocks (downstream density greater than upstream density) occur in nature (have entropy increase). We have problems on our problem set illustrating this.

3.2. **Strong shocks.** Strong shocks are those in the limit of $M_1 \to \infty$ giving

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \to \frac{(\gamma+1)}{(\gamma-1)} \qquad \frac{p_2}{p_1} \to \frac{2\gamma M_1^2}{\gamma+1} \tag{149}$$

For $\gamma = 5/3$ the density and velocity ratio is 4. This is the maximum value. A strong shock for a gas with $\gamma = 5/3$ can have a maximum density and velocity contrast across the shock of 4.

3.3. **Comment on Entropy.** We note that we could look at $p/\rho^\gamma = K$ where K is the coefficient in $P = K\rho^\gamma$. The ratio

$$\frac{p_1 \rho_1^{-\gamma}}{p_2 \rho_2^{-\gamma}} \tag{150}$$

is not in general equal to 1. This means that K is not the same on either side of the shock and that entropy is not conserved across shocks. The gas jumps from one adiabat to another one of higher entropy. It may be puzzling to consider that we have specified an equation of state on either side of the shock that implies that variations are adiabatic. While we have assumed that $P \propto \rho^\gamma$ on either side of the shock and with the same γ, we have not specified that the constant K is the same on either side of the discontinuity. There is a direction to the problem in that shocks cannot decrease the entropy, only increase it. Because there is an entropy change at the shock discontinuity, energy must be dissipated in the shock interface itself. Dissipation such as from viscosity and thermal condition set the shape of the actual shock interface on small scales and do dissipate energy.

3.4. **Isothermal shocks.** After a shock front the gas is heated but this gas may cool in some cases back to its original temperature. The temperature may increase at the shock but then cool over a cooling length back to its original temperature. If the cooling length is short then we call the shock *isothermal*. We can consider the change in velocity, pressure and density between the initial values and those following the cooling phase assuming that $T_2 = T_1$. The Rankine-Hugoniot conditions for conservational of mass and momentum still hold so

$$\rho_1 u_1 = \rho_2 u_2 \tag{151}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{152}$$

Since the temperature is the same before and after the shock $p_1 = \rho_1 c_s^2$ and $p_2 = \rho_2 c_s^2$. By replacing ρ_2 in the second equation with $\rho_1 u_1/u_2$ we find

$$\rho_1(u_1^2 + c_s^2) = \rho_1 \frac{u_1}{u_2}(u_2^2 + c_s^2) \tag{153}$$

We can write this as

$$u_1^2 [u_1 - u_2] = c_s^2 \left[\frac{u_1}{u_2} - 1\right] \tag{154}$$

as long as $u_1 \neq u_2$ This becomes

$$c_s^2 = u_1 u_2 \qquad \text{or} \qquad M_1 M_2 = 1 \tag{155}$$

We can take equation 152 and write it as

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2 + 1}{M_2^2 + 1} \tag{156}$$

Inserting $M_2 = 1/M_1$ we find that

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = M_1^2 \tag{157}$$

This means that the compression factor can be arbitrarily large. This is consistent with the limit of $\gamma \to 1$ and the ratio in equation 149.

3.5. **Going back into the inertial frame from the shock frame.** Note that the above was done in the frame of the shock. After calculating velocity changes in the shock frame one must transfer back into a coordinate frame. If the shock velocity is U_s in the lab frame and the gas velocities in the lab frame are v_{1z}, v_{2z} and the shock normal is in the z direction, then

$$\begin{aligned} u_{1z} &= v_{1z} - U_s \\ u_{2z} &= v_{2z} - U_s \end{aligned} \tag{158}$$

where u_{1z} and u_{2z} are gas velocities in the shock frame.

3.6. **Example of transferring from lab to shock frame.** In the lab frame we can detect line of sight components of velocities and we can place constraints on temperatures and densities based on line diagnostics. Either temperature or density diagnostics on both sides of the shock (and an estimate for γ) are enough to estimate the Mach speed of the shock, M_1 and so u_1 and u_2. Suppose you use temperature and density line diagnostics and estimate a Mach number $M_1 = 10$ and a preshock temperature of 10^4K. The sound speed of 10^4K gas is about 10 km/s so that in the shock frame (and using our value for M_1 and sound speed we find $u_1 \sim 100$km/s. The shock is strong (high Mach number) so we expect a density ratio of 4 (for $\gamma = 5/3$; see equations 149) . The density ratio tells us that $u_2 = 100/4 = 25$ km/s.

$$u_1 = 100\text{km/s} \qquad u_2 = 25\text{km/s} \tag{159}$$

Suppose the shock normal is oriented along the line of sight and we measure a post-shock velocity of $v_2 = 20$ km/s. What is the pre-shock velocity observed, v_1, and what is the speed of the shock in the observer's frame? The difference between pre and post shock velocity in lab frame is equivalent to that in shock frame

$$v_1 - v_2 = u_1 - u_2 = 100 - 25 = 75\text{km/s}$$
$$v_1 = v_2 + 75 = 90\text{km/s}$$

Using equation 158, the shock front velocity in the observer's or lab frame

$$U_s = v_1 - u_1 = 90 - 100 = -10\text{km/s}$$

Supposing using Doppler shifts of spectral lines that we measure in the lab frame a line of sight pre-shock velocity of 20 km/s and a line of sigh post-shock velocity of 10 km/s. What is the shock propagation speed and at what angle does it propagate with respect to the line of sight? We can subtract the two lab frame velocities and find a difference of $v_1 - v_2 = 10$ km/s. However this is much lower than $u_1 - u_2 = 100 - 25 = 75$ km/s. This means that the shock is not oriented along the line of sight. If it were we would have measured a full Doppler difference of 75 km/s. To correct for this we must multiply by the cosine of the angle between the shock normal and the line of sight.

$$(v_1 - v_2)_{los} = (u_1 - u_2)\cos\theta \tag{160}$$

We solve this finding that

$$\theta = \text{acos}(10/75) = 1.437\text{radians} = 82.3^\circ.$$

We correct v_1 by the angle finding that the preshock velocity (all components) has length of

$$v_1 = 20 \times 75/10 = 150\text{km/s}.$$

Likewise the postshock velocity lab frame has

$$v_2 = 10 \times 75/10 = 75\text{km/s}.$$

The difference between the pre-shock velocity in lab and shock frame is $u_1 - v_1 = 100 - 150 = -50$km/s. That on the other side is $u_2 - v_2 = 25 - 75 = -50$km/s so the shock velocity is 50 km/s in the observer's frame, but only if the gas we measured did not contain a velocity component parallel to the shock normal. This illustrates how one could begin to relate observational diagnostics to an underlying shock model.

Note I did not add in a velocity component for motion parallel to the shock direction but radial velocity measurements should have an additional velocity component from motion parallel to the shock surface. This would affect our the estimate of the shock front propagation speed. Sometimes transverse velocity components can be measured with proper motions or estimated by other means – using the geometry of the system or shock itself.

Interpretation of observations of shocks can also be complicated by the presence of multiple unresolved shocks, radiative precursors, cooling and additional sources of pressure such as that from a magnetic field. Because high temperatures reached in shocks are often transient, models often consider ionization and recombination processes to predict line diagnostics. As recombination timescale may not be short, emission line ratios seen in shocks may differ from those emitted in a plasma that is in equilibrium. These exotic line ratios then provide evidence of shock excitation and are used to differentiate shock excitation from other excitation mechanisms such as photo-ionization.

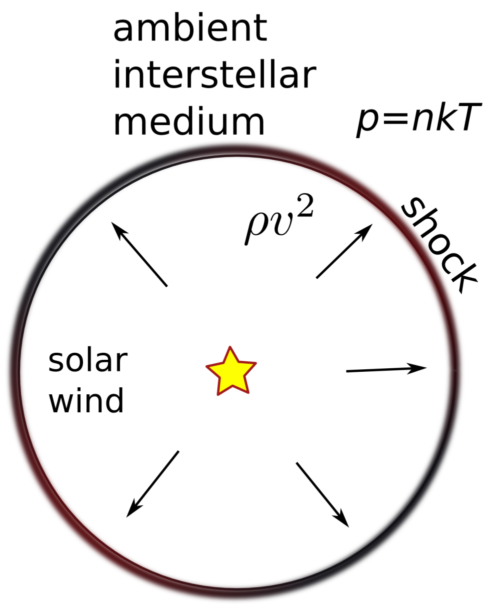

FIGURE 19. A balance between pressure of the ambient interstellar medium and solar wind ram pressure approximately determines the location of a shock that is being encountered by the Voyager probes. The structure of the shock is complex.

3.7. **Estimating bow shock location.** Let's go back to our momentum condition across the shock

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{161}$$

where I am only listing the velocity component along the shock front. This equation looks like a balance between ram pressure and pressure so sometimes we can use it to estimate the position of a bow shock. Supposing on one side of the shock the ram pressure is high and the other side the velocity is zero. This gives us a relation between ram pressure on one side and the pressure on the other side. For example let's consider the solar wind with a mean velocity of about 300 km/s and mass loss rate about $10^{-14} M_\odot$/yr. At about 100 AU from the Sun the wind encounters a shock. Before the Voyager probes encountered the shock, its location was estimated based on estimates of the physical properties of the interstellar medium in what is known as our local bubble. Now that the Voyager probes have encountered the shock we can use the shock location to estimate the properties of the ISM beyond the shock.

The wind velocity is well above the escape velocity in the inner solar system so we can assume that it is nearly constant with radius. However the density then drops as a function of r^{-2}. We are interested in how the ram pressure ρv^2 scales with radius. Here $\rho v^2 \sim \frac{\dot{M} v}{4\pi r^2}$ where I have assumed v is independent of radius and $\dot{M} = 4\pi r^2 \rho v$. Using our shock condition we can relate the pressure outside and in the ambient ISM to the ram pressure in the wind at the radius of the shock

$$p_{outside} = \rho v^2 \sim \frac{\dot{M} v}{4\pi r^2} \tag{162}$$

The interstellar medium just external to the Sun is probably ionized. Suppose its temperature is 10^4 K. We can use $p = nkT$ to determine n the density of gas in the ambient ISM outside the solar sytem.

In this case we can estimate the density of the gas outside the solar system with

$$n \sim (k_B T)^{-1} \frac{\dot{M} v}{4\pi r^2} \tag{163}$$

Inserting $r = 100$ AU and $T \sim 10^4$K we can estimate the gas density n.

3.8. **Shock velocity for conservation laws.** For a general set of conservation laws

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \tag{164}$$

the jump condition for a single discontinuity can be written

$$s(\mathbf{U_2} - \mathbf{U_1}) = [\mathbf{F}(\mathbf{U}_2) - \mathbf{F}(\mathbf{U_1})] \cdot \hat{\mathbf{n}} \tag{165}$$

for a shock velocity s and a shock face with normal $\hat{\mathbf{n}}$. Here $\mathbf{U}_1, \mathbf{U}_2$ are values of $\mathbf{U}$ on either side of the discontinuity. The shock velocity times the difference in $\mathbf{U}$ must be equal to the differences in the fluxes (components perpendicular to the shock face). The above relation is also sometimes called the **Rankine-Hugoniot condition**. It means that s must be an eigenvalue of the operator $\mathbf{F}$ on the vector

$\mathbf{U}_2 - \mathbf{U}_1$. This is the same thing as saying that speeds s are characteristic velocities for small $\mathbf{U}_2 - \mathbf{U}_1$.

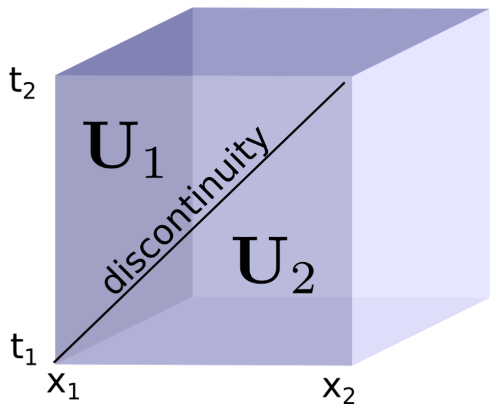

FIGURE 20. Integrating conservation laws in x, t space with a traveling discontinuity.

The above jump condition comes from integrating the conservation law. We place a box around our discontinuity and integrate in both volume and time (see Figure 20). Let the surface normal

$$d\mathbf{A} = (y_2 - y_1)(z_2 - z_1)\hat{\mathbf{x}} \tag{166}$$

$$\begin{aligned} \int_{x_1,y_1,z_1}^{x_2,y_2,z_2} \int_{t_1}^{t_2} d^3x \; dt \; \partial_t \mathbf{U}(x,t) &= -\int_{t_1}^{t_2} \int_{x_1,y_1,y_3}^{x_2,y_2,z_2} d^3x \; dt \nabla \cdot \mathbf{F}(\mathbf{U}(x,t)) \qquad (167) \\ \int_{x_1}^{x_2} dx \; dA[\mathbf{U}(x,t_2) - \mathbf{U}(x,t_1)] &= -\int_{t_1}^{t_2} \int_{box \; surface} dt \; d\mathbf{A} \cdot \mathbf{F}(\mathbf{U}) \\ \int_{x_1}^{x_2} dx[\mathbf{U}(x,t_2) - \mathbf{U}(x,t_1)] &= -\int_{t_1}^{t_2} dt \; \hat{\mathbf{n}} \cdot [\mathbf{F}(\mathbf{U}(x_2,t)) - \mathbf{F}(\mathbf{U}(x_1,t))] \qquad (168) \end{aligned}$$

with $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ the discontinuity normal. The above is an integral form for a conservation law.

We define our discontinuity as passing from x_1 at time t_1 to x_2 at time t_2 (see Figure 20). Let

$$dX = x_2 - x_1 \qquad dT = t_2 - t_1$$

At time t_2 the shock has passed and the first part of integral on the left of equation 168 gives us $\mathbf{U}_1 dX$. At time t_1 the shock has just come into our box and the second part of the integral on the right gives us $\mathbf{U}_2 dX$. At position x_2 the shock arrives only at t_2 the first term of the integral on the right hand side gives us $\mathbf{F}(\mathbf{U}_1)dT$. At position x_1 the the shock arrives at t_1 and so the second term on the integral on the right hand side gives us $\mathbf{F}(\mathbf{U}_2)dT$.

$$dX\ (\mathbf{U}_1 - \mathbf{U}_2) = -dT\ \hat{\mathbf{n}} \cdot [\mathbf{F}(\mathbf{U}_1) - \mathbf{F}(\mathbf{U}_2)] \tag{169}$$

We set

$$s \equiv \frac{dX}{dT} \tag{170}$$

as the shock or discontinuity travel speed. Altogether we get our jump condition (the Rankine-Hugoniot condition) in equation (165) which I repeat below.

$$s(\mathbf{U}_2 - \mathbf{U}_1) = \hat{\mathbf{n}} \cdot [\mathbf{F}(\mathbf{U}_2) - \mathbf{F}(\mathbf{U_1})] \tag{171}$$

It is also possible to get the sign correct by considering solutions of the form $\mathbf{U}(x-st)$ and integrating them over both t and x.

3.9. **Computing the velocity of a discontinuity for a one-dimensional non-linear conservation law.** Consider the inviscid Burger's equation

$$u_{,t} + uu_{,x} = 0$$
$$u_{,t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

with initial condition

$$u(x,t) = \begin{cases} u_1 = 1 & \text{for} \quad x < 0 \\ u_2 = \frac{1}{2} & \phantom{\text{for}} \quad x > 0 \end{cases} \tag{172}$$

The flux

$$f(u) = \frac{u^2}{2}. \tag{173}$$

We check that the characteristics *do* converge so we do not get a rarefaction wave. The fluxes

$$\begin{aligned} f(u_1) &= \frac{u_1^2}{2} = \frac{1}{2} \\ f(u_2) &= \frac{u_2^2}{2} = \frac{1}{8} \end{aligned}$$

Using our fluxes, we use equation 171 and solve for s

$$s = \frac{f(u_2) - f(u_1)}{u_2 - u_1} = \frac{1/8 - 1/2}{1/2 - 1} = \frac{3/8}{1/2} = \frac{3}{4} \tag{174}$$

3.10. **The Hugoniot Locus.** Supposing one side of a jump we have $\mathbf{U}_1$ and flux $\mathbf{F}(\mathbf{U}_1)$. We can ask what values of $\mathbf{U}_2$ and velocity s are allowed. The Rankine-Hugoniot jump condition relates s and $\mathbf{U}_2$ for a specific $\mathbf{U}_1$. The jump condition gives curves for $\mathbf{U}_2$, where each value corresponds to a particular velocity, s. The set of points on these curves is often called the **Hugoniot locus**. There may be more than one curve. If $\mathbf{U}_2$ lies along the p-th Hugoniot curve then we say that $\mathbf{U}_2$ and $\mathbf{U}_1$ are connected by a p-shock. We can parametrize each curve with a variable ξ where $s_p(\xi)$. At $\xi = 0$, we assert that $\mathbf{U}_{2,p}(\xi = 0) = \mathbf{U}_1$. corresponding to no discontinuity.

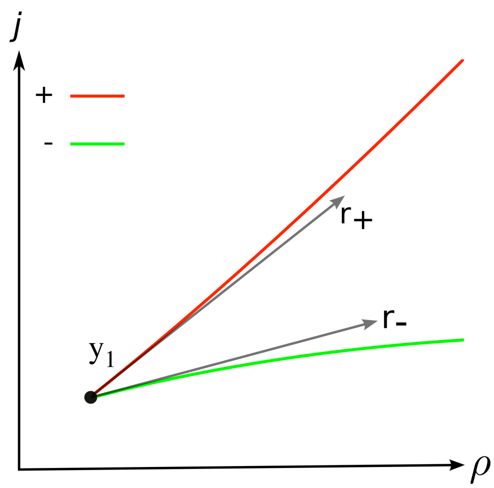

Figure 21. For a one dimensional isentropic or isothermal fluid flow the state vector is $\mathbf{y} = (\rho, j)$ with $j = \rho u$ the mass flux. There are two different characteristic velocities. From an initial condition $\mathbf{y}_1$, after a discontinuity, there can be a new condition $\mathbf{y}_2$ that lies on one of two different lines. Along each line or locus, the discontinuity velocity s varies. These curves are known as the Hugoniot locus. Near $\mathbf{y}_1$ the two directions are the two eigenvectors of the Jacobian matrix $\mathbf{F}'$ at $\mathbf{y}_1$ and the discontinuity velocities are the eigenvalues of the Jacobian matrix (the characteristic velocities) at $\mathbf{y}_1$.

The jump condition gives for each curve

$$\mathbf{F}(\mathbf{U}_{2,p}(\xi)) - \mathbf{F}(\mathbf{U}_1) = s_p(\xi)(\mathbf{U}_{2,p}(\xi) - \mathbf{U}_1). \tag{175}$$

Differentiating this expression with respect to ξ and setting $\xi = 0$ gives

$$\mathbf{F}'(\mathbf{U}'_{2,p}(0))\mathbf{U}'_{2,p}(0) = s'_p(0)(\mathbf{U}_{2,p}(0) - \mathbf{U}_1) + s_p(0)\mathbf{U}'_{2,p}(0), \tag{176}$$

and using the condition for $\xi = 0$,

$$\mathbf{F}'(\mathbf{U}_1)\mathbf{U}'_{2,p}(0) = s_p(0)\mathbf{U}'_{2,p}(0). \tag{177}$$

The above relation implies that $\mathbf{U}'_{2,p}(0)$ is a right eigenvector of $\mathbf{F}'(\mathbf{U}_1)$ and that $s_p(0)$ is an eigenvalue of this matrix.

For example consider the one dimensional gas dynamic equations for an **isothermal** fluid.

$$\rho_t + j_x = 0 \tag{178}$$

$$j_t + \left(\frac{j^2}{\rho} + a^2\rho\right)_x = 0 \tag{179}$$

where j is the mass flux and a is the sound speed and a constant. This can be written

$$\mathbf{y}_t + \mathbf{F}(\mathbf{y})_x = 0 \tag{180}$$

where

$$\mathbf{y} = \begin{pmatrix} \rho \\ j \end{pmatrix} \tag{181}$$

and

$$\mathbf{F}(\mathbf{y}) = \begin{pmatrix} j \\ \frac{j^2}{\rho} + a^2\rho \end{pmatrix} \tag{182}$$

The Jacobian of the matrix is

$$\mathbf{F}'(\mathbf{y}) = \begin{bmatrix} 0 & 1 \\ a^2 - \frac{j^2}{\rho^2} & 2j/\rho \end{bmatrix} \tag{183}$$

and eigenvalues are

$$\lambda_\pm = \frac{j}{\rho} \pm a \tag{184}$$

and eigenvectors

$$\mathbf{r}_\pm = \begin{pmatrix} 1 \\ j/\rho \pm a \end{pmatrix}. \tag{185}$$

The Rankine-Hugoniot condition becomes

$$j_2 - j_1 = s(\rho_2 - \rho_1) \tag{186}$$

$$\left(\frac{j_2^2}{\rho_2} + a^2\rho_2\right) - \left(\frac{j_1^2}{\rho_1} + a^2\rho_1\right) = s(j_2 - j_1). \tag{187}$$

Solving for j_2 and s in terms of ρ_2

$$j_2 = \frac{\rho_2 j_1}{\rho_1} \pm a\sqrt{\frac{\rho_2}{\rho_1}}(\rho_2 - \rho_1) \tag{188}$$

$$s = \frac{j_1}{\rho_1} \pm a\sqrt{\frac{\rho_2}{\rho_1}}. \tag{189}$$

We can parametrize the curves with ξ using

$$\rho_{2,p} = \rho_1(1+\xi) \tag{190}$$

Rewriting our solutions

$$\mathbf{y}_{2,-} = \mathbf{y}_1 + \xi \begin{pmatrix} \rho_1 \\ j_1 - a\rho_1\sqrt{1+\xi} \end{pmatrix}, \qquad s_- = \frac{j_1}{\rho_1} - a\sqrt{1+\xi} \tag{191}$$

$$\mathbf{y}_{2,+} = \mathbf{y}_1 + \xi \begin{pmatrix} \rho_1 \\ j_1 + a\rho_1\sqrt{1+\xi} \end{pmatrix}, \qquad s_+ = \frac{j_1}{\rho_1} + a\sqrt{1+\xi}. \tag{192}$$

Note that equation 177 related the eigenvalues and eigenvectors of the Jacobian matrix at u_1 to the Hugoniot locus. We can verify that the derivative

$$\lim_{\xi\to 0} \frac{\partial \mathbf{y}_{2,+}}{\partial \xi}(\xi) \propto \mathbf{r}_+ \tag{193}$$

is proportional to the positive right eigenvector and that

$$\lim_{\xi\to 0} \frac{\partial \mathbf{y}_{2,-}}{\partial \xi}(\xi) \propto \mathbf{r}_- \tag{194}$$

is proportional to the left eigenvector. Likewise the velocities approach the eigenvalues,

$$\lim_{\xi\to 0} s_\pm(\xi) = \lambda_\pm \tag{195}$$

Not all solutions of the Rankine-Hugoniot condition may be physically relevant (this problem is related to entropy conditions and limits of equations with finite viscosity). Also the existence of solutions with a single discontinuity is not necessarily assured.

With some feeling for the Hugoniot locus and shock conditions it is interesting to see how scientists at or visiting the LLE attempt to achieve certain high pressure and density conditions. A single shock cannot achieve post-shock conditions in any spot in P vs ρ. One solution is to use a laser in such a way as to drive two shocks, one after another (for example hole in plate and then plate ablates). Another idea is to use the laser to accelerate a plate that continually pushes on the gas.

4. Self-similar flows and blast waves

4.1. **Dimensional analysis.**

Consider an explosion of energy E into an ambient gas of density ρ. For example we can consider a supernova into the ISM or an atomic explosion into the atmosphere. Energy has units of g cm^2/s^2 and density g cm^{-3}. If we divide energy by density we find a unit of $E/\rho \sim \mathrm{cm}^5/\mathrm{s}^2$ so we expect a size scale for our blast wave of

$$R(t) \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5} \tag{196}$$

as a function of time.

This estimate is remarkably simple and likely to be accurate to an order of magnitude as long as energy is conserved. The size of a shell and its age or velocity can be used to estimate the total energy of the blast.

We can ask, how is it that energy is conserved but not momentum? As long as the energy cannot be radiated away, then energy will be conserved. However the blast wave can sweep up mass and so the total momentum may change with time. The time when energy is conserved but material is swept up is sometimes called the *snow plow* or *adiabatic expansion* phase and in this phase the above self-similar solution is good.

As a supernova shock expands radiation can be important. If energy is efficiently radiated the cooling time will be short. In this case it may be a good approximation to consider the shell momentum as the variable setting the blast wave solution and a similar self-similar solution can be derived but depending on the total momentum imparted initially. This approximation is sometimes used for stellar outflows to estimate their affect on the ambient ISM and can be used to describe a supernova remnant once it starts to radiate efficiently. Because energy can be radiated efficiently during this phase of expansion much of the total energy from a supernova is put into radiation and only a percent or so is put into kinetic motion into the nearby interstellar medium.

At earliest times the ambient density may be irrelevant and the explosion can be "coasting" at constant velocity. This earliest phase is sometimes called the free expansion phase.

Sizes of blast waves can be roughly estimated with dimensional or self-similar estimates. However these estimates do not predict the density or temperature as a function of radius, nor do they capture the complexity of the real objects. Supernova remnants can show small scale structure implying that there are instabilities in the flow.

4.2. **Similarity variables.** We can define a similarity variable

$$\xi = r \left(\frac{\rho_0}{E}\right)^{1/5} t^{-2/5} = \frac{r}{R(t)} \tag{197}$$

where ρ_0 is the density outside the shock and where $R(t)$ is from equation 196. At time t the value of ξ is just proportional to r. Thus the shock location for an explosion is at a fixed value of $\xi = \xi_s$. The above variable is a dimensionless distance parameter. We assume that all variables can be described as a function of ξ and so will have similar functional forms at all times.

Any variable $X(r,t)$ we now write as

$$X(\xi,t) = X_1(t)\tilde{X}(\xi) \tag{198}$$

At any time our variable X has the same shape (w.r.t ξ) but scaled up and down by a factor $X_1(t)$.

For strong shocks (equation 149 repeated here)

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{u_1}{u_2} \to \frac{(\gamma+1)}{(\gamma-1)} \\
\frac{p_2}{p_1} &\to \frac{2\gamma M_1^2}{\gamma+1}
\end{aligned}$$

Which side is pre-shock and which side is post-shock? The ambient outside medium is pre-shock. So the density should be higher just inside the shock than outside. We chose a function for density

$$\rho(r,t) = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0\tilde{\rho}(\xi) \tag{199}$$

where the factor is chosen based on the strong shock jump condition, allowing a normalization of $\tilde{\rho}(\xi_s) = 1$. To be consistent with a shock we expect that outside the shock $\xi > \xi_s$ we have ρ_0 but just inside the shock we have $\rho(\xi_s) = (\gamma+1)/(\gamma-1)\rho_0$. We look for solutions for $\tilde{\rho}(\xi)$ inside the shock for $\xi < \xi_s$ that have

$$\lim_{\xi\to\xi_s} \tilde{\rho}(\xi) \to 1 \tag{200}$$

At $\xi = \xi_s$ we have a jump condition and for $\xi > \xi_s$ the density is equal to the ambient value $\rho(\xi > \xi_s) = \rho_0$. With our choice for $R(t)$ we expect that the discontinuity occurs at $\xi_s = 1$.

Now let us consider the velocity. In the lab frame, the shock has velocity $s = \dot{R}$ and the pre-shock velocity, outside the blast wave, $v_1 = 0$. The ratio of the shock frame velocities

$$\frac{u_1}{u_2} = \frac{v_1 - \dot{R}}{v_2 - \dot{R}} = \frac{\gamma+1}{\gamma-1} \tag{201}$$

where I have used the strong shock condition in the last step. We can solve for v_2, the velocity inside the shock in the lab frame

$$v_2 = \dot{R}\frac{2}{\gamma+1} \tag{202}$$

Using this we can also find

$$\begin{aligned} u_2 &= \dot{R}\frac{1-\gamma}{\gamma+1} \\ u_1 &= -\dot{R} \end{aligned}$$

The jump condition on pressure we can write

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} = \frac{2}{\gamma+1}\frac{p_1}{\rho_1}u_1^2 = \frac{2}{\gamma+1}\frac{p_1}{\rho_1}u_1^2\dot{R}^2 \tag{203}$$

For $\rho(x,t)$ above (equation 199) the factor $X_1(t)$ is a constant but for velocity and pressure we expect it to depend on time. We can choose similar functions for velocity and pressure

$$u(r,t) = \frac{2}{\gamma+1}\dot{R}\tilde{u}(\xi) \tag{204}$$

$$p(r,t) = \frac{2}{\gamma+1}\rho_0\dot{R}^2\tilde{p}(\xi) \tag{205}$$

and these should resemble equation 202 and equation 203.

To convert our fluid equations into self-similar variables (those that depend on t and ξ) we need to compute derivatives,

$$\frac{\partial X}{\partial r} = X_1(t)\frac{d\tilde{X}}{d\xi}\left.\frac{\partial \xi}{\partial r}\right|_t \tag{206}$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi)\frac{dX_1}{dt} + X_1(t)\frac{d\tilde{X}}{d\xi}\left.\frac{\partial \xi}{\partial t}\right|_{r(\xi,t)} \tag{207}$$

with

$$\frac{\partial \xi}{\partial r} = \frac{1}{R(t)} \qquad \frac{\partial \xi}{\partial t} = -\frac{r\dot{R}}{R^2} \tag{208}$$

These relations for derivatives are inserted into Euler's equation, the continuity equation and the energy equation in spherical coordinates. These equations are simplified by assuming there are only radial motions and spherical symmetry. The result is a series of equations that depend only on ξ which when solved give the shapes of solutions during all times. Integrating these equations is called "laborious" by Landau and Lifshitz. Clarke and Carswell say that "the important thing here is not the details of the resultant equations...." However, these equations are carefully described and solved by Gordon Ogilvie in his lecture notes, illustrating that by using self-similar variables a set of complex non-linear differential equations can sometimes be reduced to a solvable set of ordinary differential equations.

The equations with the new self-similar variables give what is known as the Sedov-Taylor blast wave solution and is shown in Figure 22. The cusp in density near the shock is material swept up by the blast wave. Post shock flow (that interior) is subsonic so the pressure inside does not change rapidly with radius. The pressure inside is higher than outside allowing the acceleration of freshly swept up fluid. The energy in the blast can also be estimated by considering the mass and velocity in the swept up shell.

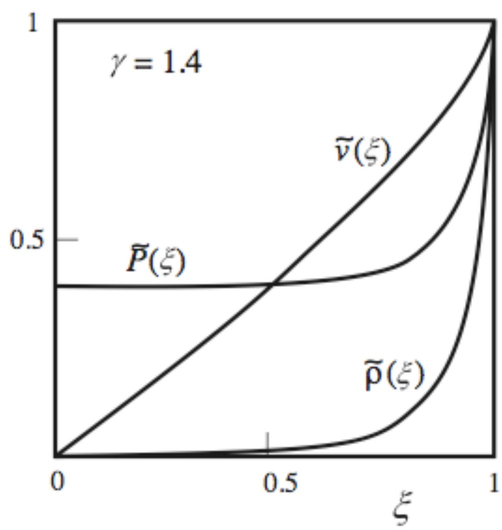

FIGURE 22. Scaled pressure, density and velocity as a function of scaled radius behind a Sedov-Taylor blast wave in air with $\gamma = 1.4$. Figure from Kip's lecture notes.

4.3. **Example of using dimensional analysis in interpretation of shells and cavities.** A famous example is the estimate of the energy of an atomic bomb only knowing the size of the blast wave at a particular time. The estimate relies only on the density of the ambient air and a photograph labelled with the time since ignition.

Here we will consider the momentum conserving phase rather than the energy conserving phase. Supposing that the blast wave has expanded sufficiently that the density in the shell has dropped. In this case radiation can easily escape the blast shell and so the shock face can be isothermal. Because radiation escapes the total energy is no longer conserved, however the total outward radial momentum in the shell might be conserved. Momentum p has units g cm/s. If we divide this by density we get units of cm^4/s. We expect a scaling

$$R \sim \left(\frac{p}{\rho}\right)^{1/4} t^{1/4} \tag{209}$$

Late stages of supernova blast shells are expected to enter a momentum conserving phase.

If there is gas in a shell like structure then its velocity could be measured from its Doppler shift. We can predict the velocity of the shell as a function of time by differentiating the above equation

$$V \sim \left(\frac{p}{\rho}\right)^{1/4} t^{-3/4} \tag{210}$$

We could also search for a scaling relation that depends on V and R instead of R and t. Note the age of a shell can be estimated by dividing the scaling relation for radius by that for the velocity and this is pretty much equivalent to simply dividing the observed radius by its observed velocity. Here I have neglected factors of 1/4. An observed shell radius, velocity and an ambient density estimate can be used to estimate the age of a blast as well as its total momentum (or energy if that is what is conserved).

Dimensional analysis and self-similar estimates can be used to estimate energies for supernova blast waves but they are also sometimes useful when any shell like feature is detected. For example groups of supernovas or a star burst can evacuate a region in a galaxy and create an expansion shell. Cavities evacuated by stellar outflows or jets could follow nearly self-similar solutions even if they are not spherically symmetric.

Young stellar objects can have an outflow of total mass $M_w \sim 0.1 M_\odot$ going into a molecular cloud at at velocity of $v_w \sim 100$ km/s over a time period of about 10^5 years. The total momentum imparted would be

$$p \sim 0.1 \times 2 \times 10^{33}\text{g } 10^7\text{cm/s} \sim 2 \times 10^{39}\text{g cm/s} \tag{211}$$

Using an estimate for the density in a molecular cloud, one can use scaling laws to estimate the sizes and velocities of cavities that could have been opened by the previous epochs of outflows. Supposing the momentum is sent into a molecular cloud of density $n \sim 10^3$ cm^{-3}. If the molecular cloud is molecular hydrogen this corresponds to a density of $\rho \sim 3 \times 10^{-21}$g/cm^3. So $\left(\frac{p}{\rho}\right)^{1/4} \sim 9 \times 10^{14}$ cm/s$^{-1/4}$. How would the cavity size depend on time?

$$R \sim 0.7\text{pc} \left(\frac{M_w}{0.1 M_\odot}\right)^{1/4} \left(\frac{v_w}{100\text{km/s}}\right)^{1/4} \left(\frac{n}{10^3\text{cm}^{-3}}\right)^{-1/4} \left(\frac{t}{10^6\text{yr}}\right)^{1/4} \tag{212}$$

The above illustrates that stellar outflows are expected to leave observable cavities in molecular clouds and that the properties of these cavities might be used to estimate the properties of previously active outflows.

5. Acknowledgments

Roughly following Pringle & King, Shu, Clarke & Carswell books for Bondi flow and self-similar flow. For Riemann invariants and characteristics Alan Hood and Scott Sarra's on-line reviews and Blandford and Thorne's chapter. Figures on characteristics from Alan Hood's applet. The blast wave figure from Blandford and Thorne's lecture notes. The Bondi flow figure from Clarke and Carswell. Following Randall Leveque's book on Numerical Methods for Conservation Laws with the Hugoniot Locus.