

SOME NUMERICAL PROBLEMS

1. Periodically forced pendulum

We will integrate the equations of motion for the time dependent Hamiltonian

$$H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos \phi - \mu \cos(\phi - \nu t) \quad (1)$$

to make a map g^T from phase space ϕ, p to itself at a time T later

$$\phi(t), p(t) \xrightarrow{g^T} \phi(t + T), p(t + T)$$

We use

$$T = 2\pi/\nu$$

the period of the time dependent perturbation in the Hamiltonian, to create the map. We map at this period.

Starting with an initial condition $\mathbf{x}_0 = (p_0, \phi_0)$ we can iteratively apply this map. The orbit of an initial condition is the collection of points

$$g^T(\mathbf{x}_0), g^{2T}(\mathbf{x}_0), g^{3T}(\mathbf{x}_0), \dots$$

The orbits of the map are curves, points or area filling regions.

Our map is a function of parameters ν, ϵ, μ . The Hamiltonian has two resonances, one at $p = 0$ and the other at $p = \nu$. The two resonances are separated by a frequency ν . The peaks of the separatrices for one resonance are at $p = \pm 2\sqrt{|\epsilon|}$ and the other at $p = \pm 2\sqrt{|\mu|}$. The resonances are *overlapped* if

$$2\sqrt{|\epsilon|} + 2\sqrt{|\mu|} < \nu \quad (2)$$

If an orbit has $g^{jT}(\mathbf{x}_0) = \mathbf{x}_0$ and positive integer $j > 1$, then the orbit is periodic. If $j = 1$ then the orbit is a fixed point!

An orbit that gives a line in our area preserving map g^T is equivalent to a torus in the 3 dimensional space (p, ϕ, θ) with $\theta = \nu t$. An orbit of our map that fills an area fills a volume in this 3 dimensional space.

See <http://astro.pas.rochester.edu/~aquillen/ast233/lectures/perturbedpendulum.ipynb>

or

<http://astro.pas.rochester.edu/~aquillen/ast233/lectures/perturbedpendulum.html>

for sample python code.

By using different initial conditions (shown in different colors) a picture of the types of different orbits can be constructed; see Figure 1

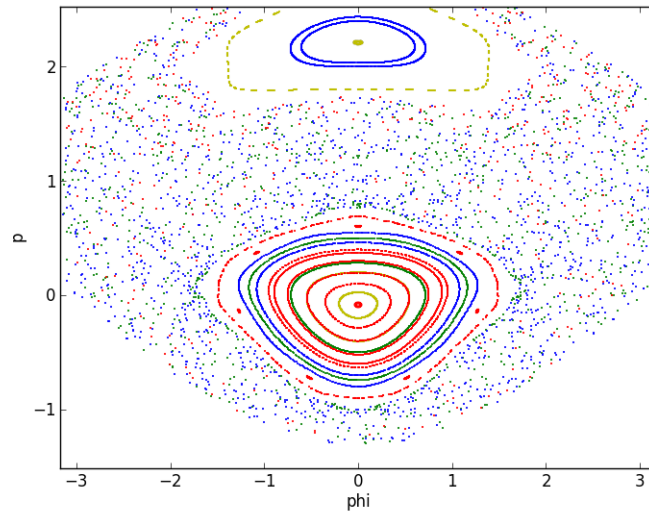


FIGURE 1. Mapping at a period $T = 2\pi/\nu$ for the Hamiltonian in equation 1 with $\nu = 2$, $\epsilon = 0.4$, $\mu = 0.15$.

- Compute a map for your choice of sets of initial conditions and parameters μ, ν, ϵ !
- We have three free parameters describing the map. Show that one of them can be removed by rescaling time and p .
- Create a map with only a single perturbation (set $\mu = 0$) and show that you see something like a pendulum.
- What do you expect for the map with $\epsilon = 0$? Verify numerically that the map is what you expect.

- e) Choose parameters that show different types of orbits (area filling and tori or lines). Show that resonant islands containing fixed points and chaotic regions depend on the parameters you choose. See if you can find a periodic orbit.
- f) Show that you only see a large chaotic zone if the two resonances are large and overlap.
- g) You can shift the phase of the second cosine term with a phase ϕ_0

$$H(p, \phi, t) = \frac{p^2}{2} - \epsilon \cos \phi - \mu \cos(\phi - \nu t + \phi_0) \quad (3)$$

but this is equivalent to starting your series of integrations with a different value for t_0 and still plotting every $T = 2\pi/\nu$. How does your map change and is this significant? (Think about slices in the 3D space).

2. On Area Preserving maps

Consider the area preserving map

$$\begin{aligned} x_{n+1} &= x_n + aI_n^\gamma \\ I_{n+1} &= I_n + \sin x_{n+1} \end{aligned} \quad (4)$$

with $x \in [0, 2\pi]$.

See <http://astro.pas.rochester.edu/~aquillen/ast233/lectures/Keplermap.ipynb>

or

<http://astro.pas.rochester.edu/~aquillen/ast233/lectures/Keplermap.html>

for sample python code.

- a) Show that the map is area preserving. (Hint: compute the Jacobian and show that its determinant is 1.)
- b) Show that with $W_n = a^{\frac{1}{\gamma}} I_n$ that the map can be written in the form

$$\begin{aligned} x_{n+1} &= x_n + W_n^\gamma \\ W_{n+1} &= W_n + K \sin x_{n+1} \end{aligned} \quad (5)$$

and find K .

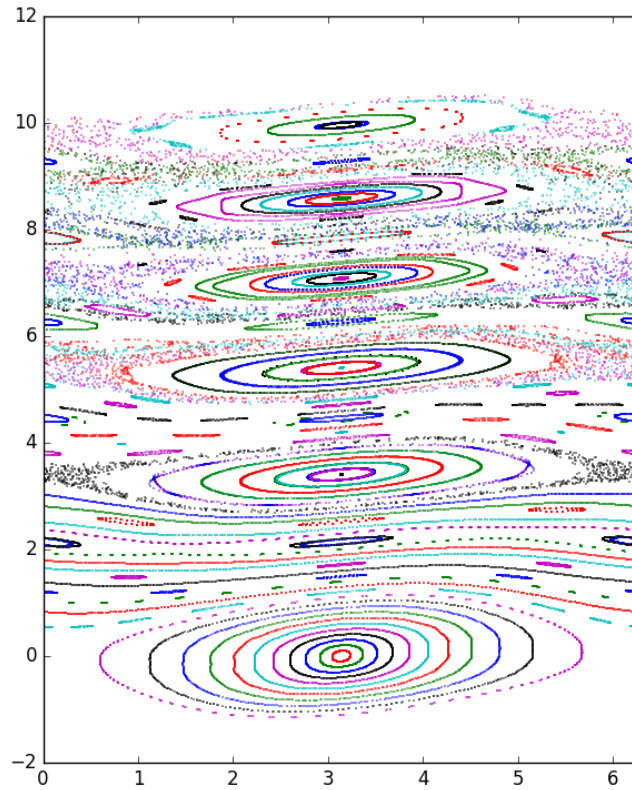


FIGURE 2. Orbits for the Kepler map (equations 8) with $K = 0.3$, $\gamma = 1.5$.

- c) For what value of γ is this map equivalent to the standard map (Chirikov's map), written like this

$$\begin{aligned}\theta_{n+1} &= \theta_n + p_n \\ p_{n+1} &= p_n + K \sin \theta_{n+1}\end{aligned}\tag{6}$$

for $p, \theta \in [0, 2\pi)$.

- d) The standard map is often written as this

$$\begin{aligned}p_{n+1} &= p_n + K \sin \theta_n \\ \theta_{n+1} &= \theta_n + p_{n+1}\end{aligned}\tag{7}$$

for $p, \theta \in [0, 2\pi)$. Is this essentially the same map as 6?

- e) Compute some orbits numerically. If $\gamma \neq 1$ then the above map might be ill defined for negative W . I find that this map gives nice results. See if you can identify different types of orbits.

$$\begin{aligned}x_{n+1} &= x_n + \text{sign}(y)|y_n|^\gamma \\y_{n+1} &= y_n + K \sin x_{n+1}\end{aligned}\tag{8}$$

and with $x \in [0, 2\pi)$ but y not constrained.

- f) The Kepler map with $\gamma = 1.5$ has lots of resonances (see Figure ??). Explain why the resonant width increases with larger p . For a given K value at what y do the resonances overlap?

Notes: Figures look better if you use a different backend than the notebook one. I found setting the dpi doesn't do anything in the notebook backend.