AST233 Lecture notes

Alice Quillen

February 25, 2020

Contents

1 Eulerian and Lagrangian views

We view the system from a fixed coordinate system and describe each variable as a function of $(\mathbf{x}, t)$. The partial time derivative

$$\frac{\partial}{\partial t}$$

describes how variables change in time from the point of view of a fixed point in space attached to a coordinate system or an inertial frame. This is the Eulerian viewpoint.

We could also describe the system from the view point of particles moving with the fluid. Suppose we have a scalar quantity like T. We would like to predict what would cause a small change δT as our fluid element moves. Over a small change in time δt and with small changes in coordinates $\delta x, \delta y, \delta z$.

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

We now divide by δt.

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t} \tag{1}$$

If we chose $\delta x, \delta y, \delta t$ to be an element of the fluid that is moving along with the fluid then $\frac{\delta \mathbf{x}}{\delta t} = \mathbf{u}$ and we can write the above as

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T$$

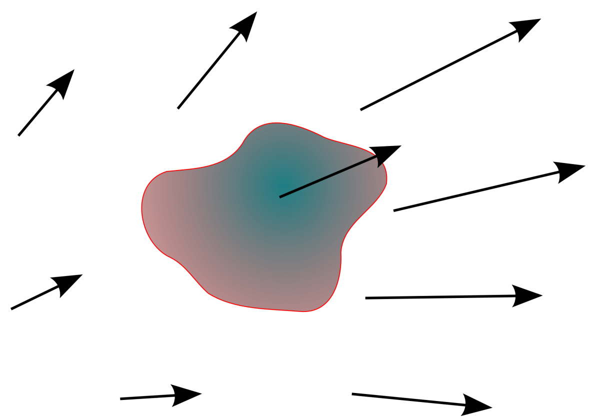

Figure 1: A fluid element moving within a larger flow.

If we consider derivatives from the point of view of particles moving with the fluid then we can describe changes with the Lagrangian time-derivative or

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}$$

Let us write this out in terms of components

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_i u_i \frac{\partial}{\partial x_i}$$

as we had done in equation (1). With summation notation it is understood that any repeated index is summed. With summation notation we would write

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}.$$

The index $i = 1$ gives x, $i = 2$ gives y and $i = 3$ gives the z coordinate.

Another way to think about this is to consider a fluid element at $\mathbf{x}$ that has moved by $\mathbf{u}\delta t$ in a time δt. If we consider T for that fluid element we can write T as

$$T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t)$$

so the change in T moving with the fluid element

$$\begin{aligned}\frac{DT}{Dt} &= \lim_{\delta t \to 0} \left(\frac{T(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) - T(\mathbf{x}, t)}{\delta t} \right) \\ &= \left[\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \right] T\end{aligned}$$

If we write equations from the view point of fluid elements that are moving we say we are using the Lagrangian view point.

Consider traffic flow. We can describe traffic flow in terms of density, ρ, (cars per unit length) and a velocity, u, the speed of cars on the road. If we describe ρ and u as a function of position on the road we are using the Eulerian view point. If we describe ρ and u in terms of those seen by individual drivers we say we are using the Lagrangian viewpoint.

Numerical methods that use fixed grids work in the Eulerian view point. Numerical methods that allow particles to move in the simulation and compute forces on these particles work in the Lagrangian viewpoint. Smooth Particle Hydrodynamics (SPH) codes use the Lagrangian viewpoint.

2 The collisionless Boltzman equation

We call $f(\mathbf{x}, \mathbf{v})$ the phase space distribution function. A volume element in real space

$$d\mathbf{x}^3 = dx\ dy\ dz$$

A volume element in velocity space

$$d\mathbf{v}^3 = dv_x dv_y dv_z$$

The distribution function $f()$ is the number of stars (or particles) per unit volume in space per unit volume in velocity space. For a specific phase space volume element the number of stars in it is

$$f(\mathbf{x}, \mathbf{v}, t) d\mathbf{x}^3 d\mathbf{v}^3$$

What is the number of stars per unit volume?

$$n(\mathbf{x}, t) = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z\ f(\mathbf{x}, \mathbf{v}, t) = \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)$$

If all the particles have the same mass m then the density at position $\mathbf{x}$ is

$$\rho(\mathbf{x}, t) = mn(\mathbf{x}, t)$$

What is the mean velocity at a position $\mathbf{x}$?

$$\langle \mathbf{v} \rangle (\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) = \frac{1}{n(\mathbf{x}, t)} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

This is similar to the expression for an expectation value where f gives a probability distribution.

Conservation of mass for a fluid gives

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

where density $\rho(\mathbf{x}, t)$.

If stars are not born and do not disappear then similarly

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n \mathbf{u}) = 0$$

This can be written in index form and using summation notation as

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_i}(n u_i) = 0$$

Stars can change velocity. If stars are not born and do not die then $Df/dt = 0$. We can take $f(\mathbf{x}, \mathbf{v}, t)$ and differentiate all variables w.r.t. to time

$$\begin{aligned} \frac{Df}{Dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial v_i}\frac{dv_i}{dt} = 0 \\ &= \frac{\partial f}{\partial t} + \boldsymbol{\nabla} f \cdot \mathbf{v} + \boldsymbol{\nabla}_v f \cdot \dot{\mathbf{v}} = 0 \end{aligned}$$

In the first line I used summation notation. I am using gradient operators

$$\begin{aligned} \boldsymbol{\nabla} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \\ \boldsymbol{\nabla}_v &= \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right) \end{aligned}$$

These are known as the collisionless Boltzmann equation. The acceleration is related to the gradient of the gravitational potential

$$\dot{\mathbf{v}} = -\boldsymbol{\nabla}\Phi$$

so the collisionless Boltzmann equation can also be written as

$$\frac{\partial f}{\partial t} + \nabla f \cdot \mathbf{v} - \nabla_v f \cdot \nabla \Phi = 0$$

Collisions and birth and death of stars would add terms to the collisionless Boltzmann equation.

We are only keeping track of the position and velocity of stars. We could also take into account more degrees of freedom, such as mass or age or metallicity.

2.1 The particle distribution function

To describe a distribution of particles we can consider a particle distribution function that depends on position, velocity and time, $f(\mathbf{x}, \mathbf{v}, t)$. Here $f(\mathbf{x}, \mathbf{v}, t)d^3\mathbf{x}d^3\mathbf{v}$ represents the number of particles found in a volume element of volume $d^3\mathbf{x}$ and in a velocity bin of size $d^3\mathbf{v}$ at time t. Here volume elements

$$d^3\mathbf{x} = dx\ dy\ dz \qquad d^3\mathbf{v} = dv_x\ dv_y\ dv_z$$

in Cartesian coordinates. The number density (number of particles per unit volume) at position $\mathbf{x}$ and at time t would be

$$n(x, t) = \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

where we perform the integral in 3 dimensions. If each particle has mass m then the density

$$\rho(\mathbf{x}, t) = mn(\mathbf{x}, \mathbf{t}).$$

We can consider the average of any function $Q(\mathbf{v})$ as

$$\langle Q \rangle(\mathbf{x}, t) = n^{-1} \int Q(\mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}.$$

For example the bulk or average velocity would be

$$\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{v} \rangle = n^{-1} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}$$

and

$$\int v_i v_j f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n \langle v_i v_j \rangle \qquad \text{for} \qquad i \neq j$$

For a single component (like v_x^2 or v_y^2) we can write

$$\int v_i^2 f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = n \langle v_i^2 \rangle$$

But this is not necessarily the same as $nu_i^2 = n(\langle v_i \rangle)^2$ which depends on the square of the average velocity. Usually

$$\langle v_i^2 \rangle \neq u_i^2 \qquad \langle v_i v_j \rangle \neq u_i u_j$$

We can define a total *velocity dispersion*, σ_a, averaged over all directions, as

$$\begin{aligned}
\sigma_a^2 &\equiv \frac{1}{3}\left(\langle (v_x - u_x)^2 \rangle + \langle (v_y - u_y)^2 \rangle + \langle (v_z - u_z)^2 \rangle\right) \\
&= \frac{1}{3n}\int |\mathbf{v} - \mathbf{u}|^2 f d^3\mathbf{v}
\end{aligned}$$

Evaluating σ_a^2

$$\begin{aligned}
\sigma_a^2 &= \frac{1}{3n}\int (v^2 + u^2 - 2\mathbf{u}\cdot\mathbf{v}) d^3\mathbf{v} \\
&= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3n}\mathbf{u}\cdot\int \mathbf{v} d^3\mathbf{v} \\
&= \frac{1}{3}(\langle v^2 \rangle + u^2) - \frac{2}{3}u^2 \\
&= \frac{1}{3}(\langle v^2 \rangle - u^2)
\end{aligned}$$

so we can write

$$n\langle v^2 \rangle = \int v^2 f d^3\mathbf{v} = n(u^2 + 3\sigma_a^2)$$

We can think about the velocity v_i as a sum of the mean velocity u_i plus a random component. Let us consider a velocity dispersion tensor

$$w_{ij} \equiv \langle (v_i - u_i)(v_j - u_j) \rangle = \langle v_i v_j \rangle - u_i u_j$$

Here w_{ij} is a symmetric dispersion tensor with two indexes where each index can assume one of three values (x, y, z). When w_{ij} contains off diagonal components or its diagonal components are not equal we say the dispersion tensor is "anisotropic." If the system is "isotropic" then the diagonal components would all be the same and the off diagonal components would be zero.

We can write the trace of w as w_{ii} in summation notation and

$$\sigma_a^2 = \frac{1}{3}(\langle v^2 \rangle - u^2) = \frac{w_{ii}}{3} = \frac{1}{3}\text{trace } w$$

If $w_{xx} = w_{yy} = w_{zz}$ then $\sigma_a^2 = w_{xx}$. The dispersion tensor is symmetric. We can decompose the dispersion tensor, w_{ij}, into the sum of a trace component that has zeros off the diagonal and a *symmetric traceless component*, y_{ij};

$$y_{ij} = \frac{w_{ij} + w_{ji}}{2} - \text{trace } w\frac{\delta_{ij}}{3} = \frac{w_{ij} + w_{ji}}{2} - \sigma_a^2 \delta_{ij}$$

Note that y_{ij} can contain components on the diagonal but their sum would be zero. If the system is isotropic then all components of y_{ij} would be zero.

We can associate pressure in a fluid or gas with the trace of w_{ij} or σ_a^2.

2.2 Collisionless Boltzmann equation

In the absence of collisions the collisionless Boltzmann equation describes the evolution of the density distribution.

$$\frac{Df}{Dt} = \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0.$$

The derivative here is done with respect to all degrees of freedom of the distribution function. As $\mathbf{v} = d\mathbf{x}/dt$ and $d\mathbf{v}/dt = -\boldsymbol{\nabla}\Phi$ for a force field with potential Φ we can write

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \boldsymbol{\nabla} f(\mathbf{x}, \mathbf{v}, t) \cdot \mathbf{v} - \boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}, t) \cdot \boldsymbol{\nabla}\Phi = 0. \tag{2}$$

Using summation notation this equation is

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial x_i} v_i - \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial v_i} \frac{\partial \Phi(\mathbf{x}, t)}{\partial x_i} = 0. \tag{3}$$

Equation 2 (or 3) is known as the collisionless Boltzmann equation. It is used to study the kinetic theory of gases, atomic nuclei and for stellar dynamical systems such as galaxies and globular clusters. The collisionless Boltzmann equation is sufficiently complex that it is usually difficult to solve. Equation 2 is sometimes written

$$\frac{Df}{Dt} = 0$$

where the Lagrangian derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} - \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}_v$$

Here the Lagrangian derivative describes a small element moving in *phase space* or $(\mathbf{x}, \mathbf{v})$. Previously we used a Lagrangian derivative for a small element moving only in Cartesian space.

When collisions are important we can use the full Boltzmann equation by adding a source term that is due to collisions

$$\frac{Df}{Dt} = \left(\frac{\partial f}{\partial t}\right)_C$$

where the term on the right hand side depends on the cross sections of particles and their velocity differences. In many situations collisions conserve mass, momentum and kinetic energy. When these are conserved

$$\int m \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$

$$\int m\mathbf{v} \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$

$$\int mv^2 \left(\frac{\partial f}{\partial t}\right)_C d^3\mathbf{v} = 0$$

2.3 Observables

The number density $n(\mathbf{x}, t)$ is an integrated quantity and so possibly an observable.

The mean velocity $\langle \mathbf{v} \rangle = \mathbf{u}$ can be considered an observable.

The velocity dispersion in a particular direction (here the z direction)

$$\sigma_z^2(\mathbf{x}, t) = \langle (v_z - \langle v_z \rangle)^2 \rangle = \frac{1}{n} \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)(v_z - u_z)^2$$

The velocity dispersion tensor

$$\begin{aligned} w_{ij}^2 &= \frac{1}{n} \int d^3\mathbf{v} f(\mathbf{x}, \mathbf{v}, t)(v_i - u_i)(v_j - u_j) \\ &= \langle v_i v_j \rangle - u_i u_j - u_i u_j + u_i u_j \\ &= \langle v_i v_j \rangle - u_i u_j \end{aligned}$$

We can integrate along the line of sight (here the z direction)

$$g(x, y, \mathbf{v}, t) = \int dz f(x, y, z, v_x, v_y, v_z, t)$$

A spectrum would be sensitive to Doppler shifts in one direction giving

$$h(x, y, v_z, t) = \int dz dv_x dv_y f(x, y, z, v_x, v_y, v_z, t)$$

This is what is measured from an integral field spectrograph at different positions x, y where z is the line of sight direction. In a galaxy absorption lines are broadened by the different Doppler shifts of stars. A velocity dispersion along the lines of sight direction at different positions on the sky

$$\sigma_z^2(x, y, t) = \frac{\int dz d^3\mathbf{v}(v_z - u_z)^2 f(x, y, z, v_x, v_y, v_z, t)}{\int dz d^3\mathbf{v} f(x, y, z, v_x, v_y, v_z, t)}.$$

2.4 Conservation of mass

The simplest continuum equation can be made by integrating the Boltzmann equation over all possible velocities. The first term in the collisionless Boltzmann equation ($\partial f / \partial t$) gives us the time derivative of the particle density. Integrating the first term in the collisionless Boltzmann equation over velocity space

$$\int_{-\infty}^{\infty} \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} d^3\mathbf{v} \approx \frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v} = \frac{\partial}{\partial t} n(\mathbf{x}, t)$$

The second term in the collisionless Boltzmann equation is $\mathbf{v} \cdot \boldsymbol{\nabla} f$. As derivatives with $\mathbf{x}$ and $\mathbf{v}$ commute we can integrate the second term in the following way

$$\int_{-\infty}^{\infty} \boldsymbol{\nabla} f(\mathbf{x}, \mathbf{v}, t) \cdot \mathbf{v}\, d^3\mathbf{v} = \boldsymbol{\nabla} \cdot \int f \mathbf{v} d^3\mathbf{v} = \boldsymbol{\nabla} \cdot (n\mathbf{u})$$

where we have rewritten the last term in terms of the average velocity $\mathbf{u}$. The last term in the collisionless Boltzmann equation is $-\boldsymbol{\nabla}_v f \cdot \boldsymbol{\nabla}\Phi(\mathbf{x})$. We integrate this over velocity space

$$-\boldsymbol{\nabla}\Phi(\mathbf{x}, t) \cdot \int d^3\mathbf{v} \boldsymbol{\nabla}_v f(\mathbf{x}, \mathbf{v}, t)$$

Consider one part of the sum

$$-\frac{\partial \Phi(\mathbf{x}, t)}{\partial x} \int dv_x dv_y dv_z \frac{\partial f}{\partial v_x} = -\frac{\partial \Phi(\mathbf{x}, t)}{\partial x} \int dv_y dv_z f(\mathbf{x}, \mathbf{v}, t) \Big]_{v_x=-\infty}^{v_x=\infty} = 0$$

This vanishes as long as we assume that the numbers of stars is small at large velocity, or $f \to 0$ as $v_i \to \pm\infty$.

Putting these together with the integral of the collision term (also zero) we find

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{u}) = 0 \tag{4}$$

To summarize: the integral over velocity space of the Boltzmann equation gives an equation that looks just like the equation for conservation of mass for a fluid.

2.5 Conservation of momentum and Jeans equations

To derive an equation similar to Euler's equation (which is a result of conservation of momentum) we multiply the Boltzmann equation by $\mathbf{v}$ and then again integrate over velocity space. Taking the i-the component of the velocity and using summation notation for the other indices

$$\int \left(\frac{\partial f}{\partial t} v_i + \frac{\partial f}{\partial x_j} v_j v_i - \frac{\partial f}{\partial v_j} \frac{\partial \Phi}{\partial x_j} v_i \right) d^3\mathbf{v} = \int \left(\frac{\partial f}{\partial t} \right)_C v_i d^3\mathbf{v} = 0 \tag{5}$$

Consider the first term

$$\int \frac{\partial f}{\partial t} v_i d^3\mathbf{v} = \frac{\partial}{\partial t} \int f v_i d^3\mathbf{v} = \frac{\partial}{\partial t}(n\langle v_i \rangle) = \frac{\partial (n u_i)}{\partial t}$$

Consider the second term of equation 5. This can be written

$$\int \frac{\partial f}{\partial x_j} v_j v_i d^3\mathbf{v} = \frac{\partial}{\partial x_j} [n\langle v_j v_i \rangle]$$

We can decompose this in terms of the dispersion tensor ($\mathbf{w}$) and then the traceless component of the dispersion tensor ($\mathbf{y}$) and the average dispersion (σ_a^2)

$$\begin{aligned}\frac{\partial}{\partial x_j}[n\langle v_j v_i\rangle] &= \frac{\partial}{\partial x_j}[n(u_i u_j + w_{ij})] \\ &= \frac{\partial}{\partial x_j}[n(u_i u_j + \sigma_a^2\delta_{ij} + y_{ij})] \\ &= \frac{\partial}{\partial x_j}[n(u_i u_j + y_{ij}) + P\delta_{ij}]\end{aligned} \tag{6}$$

where we define a pressure in terms of the trace of the dispersion tensor

$$P \equiv n\sigma_a^2 = \frac{nw_{ii}}{3}.$$

Altogether the second term in the momentum equation (5) becomes

$$\frac{\partial}{\partial x_j}(nu_i u_j + P\delta_{ij} + ny_{ij})$$

The first two terms inside the derivative, $nu_i u_j + P\delta_{ij}$ are known as the **stress tensor** in hydrodynamics. The last term ny_{ij} depends in the traceless component of the dispersion tensor and is only non-zero when the velocity distribution is *anisotropic*.

The third term in the momentum equation (5) can be integrated through integration by parts. The term is

$$\frac{\partial \Phi}{\partial x_j}\int \frac{\partial f}{\partial v_j} v_i \, d^3\mathbf{v}$$

First consider the case $i \neq j$ and let k be the third index

$$\frac{\partial \Phi}{\partial x_j}\int dv_k \int dv_i v_i \int dv_j \frac{\partial f}{\partial v_j} = \frac{\partial \Phi}{\partial x_j}\int dv_k \int dv_i v_i \; f(\mathbf{x},\mathbf{v},t)\Big|_{v_j=-\infty}^{v_j=\infty} = 0$$

Now consider the case $i = j$ and the denote k, l as the other two indices, and we integrate by parts

$$\begin{aligned}\frac{\partial \Phi}{\partial x_j}\int \frac{\partial f}{\partial v_j} v_i \, d^3\mathbf{v} &= \frac{\partial \Phi}{\partial x_j}\delta_{ij}\int dv_k \int dv_l \left(f(\mathbf{x},\mathbf{v},t) v_i\big|_{v_i=-\infty}^{v_j=\infty} - \int_{-\infty}^{\infty} f\mathbf{x},\mathbf{v},t) dv_i \right) \\ &= \frac{\partial \Phi}{\partial x_j}\delta_{ij}\,(0 - n) \\ &= -n\frac{\partial \Phi}{\partial x_i}\end{aligned}$$

This is the integrated third term of equation (5)

Altogether (5) becomes

$$\frac{\partial}{\partial t}(nu_i) + \frac{\partial}{\partial x_j}(nu_i u_j + P\delta_{ij} + ny_{ij}) + n\frac{\partial \Phi}{\partial x_i} = 0$$

This is an equation for momentum conservation. Except for the term associated with anisotropy this looks just like that derived in hydrodynamics but with n replaced by mass density ρ.

By making use of the equation of continuity we can manipulate this equation so that it becomes an equation for acceleration that resembles Euler's equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{n}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi - \frac{1}{n}\boldsymbol{\nabla}\cdot(n\mathbf{y})$$

where the last term is a divergence of the traceless component of the dispersion tensor. If the velocity dispersion is isotropic then $\mathbf{y} = 0$ and we recover Euler's equation. To summarize: by multiplying the Boltzmann equation by velocity and integrating over all velocities we recover an equation that looks remarkably like Euler's equation.

Here we have integrated over velocity. We have taken the first "moment" of the collisionless Boltzmann equation. If one also integrates over all space one can derive tensor "virial" equations. Integrating only over velocity and working in cylindrical or spherical coordinates the equations, and in the setting of stellar dynamics, the equations are called the *Jeans equations*.

Using equation 6 and not trying to use a pressure like term we can also write the momentum equation as

$$\frac{\partial}{\partial t}(nu_j) + \frac{\partial}{\partial x_i}(nu_i u_j + nw_{ij}) + n\frac{\partial \Phi}{\partial x_j} = 0 \tag{7}$$

and using summation notation.

Then combined with the equation of continuity (equation 4) this becomes

$$n\frac{\partial u_j}{\partial t} + nu_i\frac{\partial u_j}{\partial x_i} + n\frac{\partial \Phi}{\partial x_k} + \frac{\partial (nw_{ij})}{\partial x_i} = 0 \tag{8}$$

This is known as the Jeans equations.

3 Using moments of the Collisionless Boltzmann equation

3.1 The tensor virial equations

We will integrate the collisionless Boltzmann equation over all space.

We define something that is *like* a moment of inertia tensor

$$I_{ij} \equiv \int d^3\mathbf{x}\rho(\mathbf{x})x_i x_j = m\int d^3\mathbf{x}\ x_i x_j \int d^3\mathbf{v} f(\mathbf{x},\mathbf{v},t)$$

This is to be compared to the actual moment of inertia tensor for a rigid body about the origin which is the sum over mass elements inside the rigid body

$$I_{ij,actual} = \sum_k m_k (r^2 \delta_{ij} - x_i x_j) = \int d^3\mathbf{x} \rho(\mathbf{x})(r^2 \delta_{ij} - x_i x_j)$$

where r is the distance to the origin for each particle in the sum and x_i is x, y or z depending upon the index.

Kinetic energy per unit volume

$$\sum_i \frac{1}{2} \int d^3\mathbf{v} v_i^2 f(\mathbf{x}, \mathbf{v}, t) m = \sum_i \frac{1}{2} n(\mathbf{x}, t) m \langle v_i^2 \rangle = \sum_i \frac{1}{2} \rho(\mathbf{x}, t) \langle v_i^2 \rangle$$

The total kinetic energy

$$K = \sum_i \frac{1}{2} \int d^3\mathbf{x} n(\mathbf{x}, t) m \langle v_i^2 \rangle = \sum_i \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t) \langle v_i^2 \rangle$$

A more general total kinetic energy tensor we define as

$$K_{ij} \equiv \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t) \langle v_i v_j \rangle \tag{9}$$

The trace of this

$$\sum_i K_{ii} = K$$

is the total kinetic energy.

The total ordered velocity tensor

$$T_{ij} = \frac{1}{2} \int d^3\mathbf{x} \rho(\mathbf{x}, t) u_i u_j \tag{10}$$

The total random velocity tensor is an integral of the velocity dispersion

$$\begin{aligned} \Pi_{ij} &\equiv \int d^3\mathbf{x} \rho(\mathbf{x}) w_{ij}^2 \qquad (11) \\ &= \int d^3\mathbf{x} \rho(\mathbf{x}) (\langle v_i v_j \rangle - u_i u_j) \\ &= 2K_{ij} - 2T_{ij}. \end{aligned}$$

This gives a relation between the total kinetic energy tensor, the order velocity tensor and the the random velocity tensor

$$K_{ij} = T_{ij} + \frac{1}{2} \Pi_{ij} \tag{12}$$

Lastly we create a tensor for the gravitational energy. We tentatively define a gravitational potential energy tensor as

$$W_{jk} \equiv -\int d^3\mathbf{x}\rho(\mathbf{x})x_j \frac{\partial\Phi(\mathbf{x})}{\partial x_k} \tag{13}$$

This is also known as the Chandrasekkar potential energy tensor. The gravitational potential

$$\Phi(\mathbf{x}) = G\int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

The gradient of the gravitational potential

$$\frac{\partial\Phi(\mathbf{x})}{\partial x_k} = -G\int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^3}$$

This gives an alternative form for W_{jk}

$$\begin{aligned} W_{jk} &= G\int d^3\mathbf{x}\int d^3\mathbf{x}'\rho(\mathbf{x})\rho(\mathbf{x}')\frac{x_j(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^3} \\ &= -\frac{1}{2}G\int\int d^3\mathbf{x}d^3\mathbf{x}'\rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x_j - x'_j)(x_k - x'_k)}{|\mathbf{x}-\mathbf{x}'|^3} \end{aligned}$$

In the last step we infer that we can flip the indices to rewrite the integral in such a way that it is clear is symmetric. The trace of this

$$W = \sum_j W_{jj} = \frac{1}{2}\int d^3\mathbf{x}\rho(\mathbf{x})\Phi(\mathbf{x})$$

is equal to the total gravitational potential energy.

Now that we have a few definitions, we go back to the first moment of the Collisionless Boltzmann equation (equation 7) which I repeat here:

$$\frac{\partial}{\partial t}(nu_j) + \frac{\partial}{\partial x_i}(nu_iu_j + nw_{ij}) + n\frac{\partial\Phi}{\partial x_j} = 0 \tag{14}$$

We multiply this by mx_k and integrate this over all space

$$\int d^3\mathbf{x}x_k\frac{\partial}{\partial t}(\rho u_i) + \int d^3\mathbf{x}x_k\frac{\partial}{\partial x_i}(\rho u_iu_j + \rho w_{ij}) + \int d^3\mathbf{x}x_k\rho\frac{\partial\Phi}{\partial x_j} = 0 \tag{15}$$

The last term on the right is equal to the potential energy tensor $-W_{jk}$ from the definition in equation 13. The second term can be integrated by parts

$$\begin{aligned} \int d^3\mathbf{x}x_k\frac{\partial}{\partial x_i}(\rho u_iu_j + \rho w_{ij}) &= \delta_{ik}\left[x_k(\rho u_iu_j + \rho w_{ij})|_{x_k=-\infty}^{\infty} - \int d^3\mathbf{x}(\rho u_iu_j + \rho w_{ij})\right] \\ &= 0 - \delta_{ik}\left(2T_{ij} + \Pi_{ij}\right) \\ &= -(2T_{jk} + \Pi_{jk}) \end{aligned}$$

where we have neglected a term on the second line with the assumption that the density is zero at infinity, $\rho \to 0$ at $x \to \pm\infty$. This neglect means the outer boundaries might affect the results. The first term can be written as

$$\frac{d}{dt}\int d^3\mathbf{x} x_k \rho u_i$$

Equation 15 becomes

$$\frac{d}{dt}\int d^3\mathbf{x} x_k \rho u_i = 2T_{jk} + \Pi_{jk} + W_{jk} \tag{16}$$

With a bit more effort, the left hand side can be related to the moment of inertia tensor. The resulting **tensor virial equation** is

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2T_{jk} + \Pi_{jk} + W_{jk} \tag{17}$$

In steady state, there is a relationship between the gravitational potential energy which depends on shape, the velocity dispersion and the bulk motion or rotation. Elongated non rotating galaxies tend to have anisotropic velocity dispersions. Rotating galaxies tend to be flatter.

Taking the trace of the steady state equation, the tensor virial theorem becomes

$$2K + W = 0$$

which is the scalar version of the virial theorem.

3.2 Applications of Jean's equations

The velocity moments of the collisionless Boltzmann equation are called Jeans equations.

One application is known as *asymmetric drift.* Consider a disk of stars all in circular orbits about the center a galaxy and all confined to a single plane. The velocity dispersion is small. In a local region the average velocity is tangential and is equal to the circular velocity.

Now consider a similar disk of stars but the stars have some ellipticity to their orbits and undergo radial oscillations. The orbits have random phases so the stars do not move in and out together. The velocity dispersion arises from the radial oscillations of the orbits. What is the mean tangential velocity component? It must be slightly lower than the rotation velocity. This makes sense looking at the tensor virial equations. The difference between the mean tangential velocity and that of a star in a circular orbit is known as *asymmetric drift.*

Using Jeans equation in polar coordinates, it is possible to show that

$$v_a \equiv \langle v_\phi \rangle - v_c \approx \frac{\langle v_R^2 \rangle}{2v_c}\left[\frac{\sigma_\phi^2}{\langle v_R^2 \rangle} - 1 - \frac{\partial \ln(n\langle v_R^2 \rangle)}{\partial \ln R} - \frac{R}{\langle v_R^2 \rangle}\frac{\partial(\langle v_R v_z \rangle)}{\partial z}\right]$$

Another application of Jean's equations is similar to hydrostatic equilibrium giving a relation between the velocity dispersion and density in the z direction and the gradient of the potential. Repeating Jeans equations (equation 8)

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} + \frac{\partial \Phi}{\partial x_k} + \frac{1}{n}\frac{\partial (n w_{ij})}{\partial x_i} = 0 \tag{18}$$

We assume steady state and drop the first term. We assume symmetry about the galactic plane, no vertical bulk or average motion and take the z component. The result is this:

$$\frac{1}{n}\frac{\partial (n\langle v_z^2\rangle)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

Using Poisson's equation

$$\frac{\partial^2 \Phi}{dz^2} = 4\pi G\rho$$

Putting these two together we find

$$\frac{\partial}{\partial z}\left[\frac{1}{n}\frac{\partial (n\langle v_z^2\rangle)}{\partial z}\right] = -4\pi G\rho$$

The left hand side can be measured using vertical velocity measurements for stars as a function of distance above and below the Galactic plane and placing a constraint on the mass distribution. This has been used to estimate the fraction of dark matter in the vicinity of the Sun.

3.3 Tremaine-Weinberg method for measuring pattern speeds

The continuity equation in Cartesian coordinates

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\mathbf{u}) = 0$$

where $n(x,y,z,t)$ is the stellar number density. Assume that the density of a flat galaxy in 2D rotates at a fixed and steady pattern speed Ω, $n(r,\theta-\Omega t)$ in polar coordinates. We assume that the density distribution does not vary in a frame rotating with the pattern.

The continuity equation in 2D Cartesian coordinates becomes

$$-\Omega\left(x\frac{\partial n}{\partial y} - y\frac{\partial n}{\partial x}\right) + \frac{\partial (n u_x)}{\partial x} + \frac{\partial (n u_y)}{\partial y} = 0$$

Consider integrating the continuity equation along the y axis. This is as if we are integrating along a slit that is oriented along the y axis. The first term

$$\int dy\ \Omega x \frac{\partial n}{\partial y} = 0$$

because $n \to 0$ at large y. The second term is

$$\int dy\ \Omega y \frac{\partial n}{\partial x} = \Omega \frac{\partial}{\partial x} \int dy\ yn(x,y)$$

The third term

$$\int dy\ \frac{\partial (n u_x)}{\partial x} = \frac{\partial}{\partial x} \int dy\ n u_x(x,y)$$

The fourth term

$$\int dy\ \frac{\partial (n u_y)}{\partial y} = 0$$

because $n \to 0$ at large y. Putting this together

$$\frac{\partial}{\partial x} \left(\Omega \int dy\ yn(x,y) + \int dy\ n u_x(x,y) \right) = 0$$

Integrating this

$$\Omega \int dy\ yn(x,y) + \int dy\ n u_x(x,y) = C$$

where C is a constant. This relation must be true for any x value and C cannot depend on x. This means that it must be true at large x and we can let the constant C be zero. This gives the relation

$$\Omega = -\frac{\int dy\ n u_x(x,y)}{\int dy\ yn(x,y)}$$

The estimate for the pattern speed depends on the mean velocity component in the direction perpendicular to the slit, u_x. The denominator weights the stellar density by the distance along the slit. The estimate for the pattern speed is also valid if the number density is replaced by the light density. The light density would also be a conserved quantity, but again we assume that the density is fixed in a frame rotating with the pattern.

The galaxy is likely inclined with respect to the viewer. When measuring the mean velocity component u_x with a spectrum and using a Doppler shift, you would need to correct for galaxy inclination to get the full size of the in-plane velocity component.

This technique has been used to measure bar pattern speeds in some barred galaxies. We made a few assumptions. There is only a single pattern speed and the galaxy is nearly steady state. Both of these might be violated as galaxies can be changing shape and barred galaxies often also host spiral arms which may move at different or even varrying pattern speeds. Bars tend to have high surface brightness compared to spiral arms, making it easier to measure a mean velocity from a spectrum.

4 Jeans Theorem

It is possible to switch variables $f(L, E)$ for example, depending upon quantities that are conserved in a spherically symmetric gravitational potential, angular momentum L and energy E. Alternatively one can write or $f(\mathbf{I}, \boldsymbol{\theta}, t)$ where $\mathbf{I}, \boldsymbol{\theta}$ are pairs of action angle variables. The collisionless Boltzmann can be evaluated similarly with advective derivatives. If the potential is fixed and the system *relaxed*, the phase space distribution function only depends on the actions.

5 Problems

- **Problem 1**

  Show that in a frame that rotates with constant angular velocity $\boldsymbol{\Omega}$ the collisionless Boltzmann equation is

  $$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) f - \left[\boldsymbol{\nabla}\left(\Phi - \frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{r})^2\right) + 2\boldsymbol{\Omega} \times \mathbf{v}\right] \cdot \boldsymbol{\nabla}_v f = 0$$

  Note that acceleration $\mathbf{a}' = \dot{\mathbf{v}}$ in a rotating frame is

  $$\mathbf{a}' = \mathbf{a} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times \mathbf{v}$$

  It is helpful to use vector identities to evaluate the gradient operator.

- **Problem 2** (B+T 1-rst edition problem 4.9

  Consider a spherically symmetric system with an isotropic velocity distribution. The orbits are not circular. The rms speed in a local region is $\sqrt{\frac{3}{2}}v_c$ where v_c is the speed of a particle in a circular orbit about the center of the system. We can assume that v_c is independent of radius.

  Now consider a different spherically symmetric system. Here all stars are on circular orbits (about the origin) with velocity v_c, however all the orbits are randomly orientated w.r.t to each other.

  Assume that the two systems have identical density distributions.

  How are these systems consistent with the virial theorem?

  The rms speed is $\langle v^2 \rangle$ and is equal to the velocity dispersion as by symmetry there is no bulk motion.

- **Problem 3**

Consider a phase space density distribution that depends on time in the following way $f(x-ut, y, z, v_x, v_y, v_z)$ where u is the velocity of a wave that passes through the distribution of stars.

a) Show that the density distribution (that independent of velocities) depends on $x - ut$ and so exhibits a traveling wave.

b) Show that the collisionless Boltzman equation resembles

$$-u\frac{\partial f}{\partial x} + \mathbf{v}\cdot\boldsymbol{\nabla} f + \boldsymbol{\nabla}_v f\cdot\boldsymbol{\nabla}\Phi = 0$$

c) Show that at a peak in the velocity distribution (where $\boldsymbol{\nabla}_v f = 0$) that

$$u = \frac{\mathbf{v}\cdot\boldsymbol{\nabla} f}{\frac{\partial f}{\partial x}}$$

If there is a peak in the velocity distribution function, it is possible to estimate the pattern speed from the spatial gradients of the distribution function.

This is related to the Weinberg-Tremaine method for measuring pattern speeds of bar or spiral wave like patterns in disk galaxies.

- **Problem 4:** Averaging over z

  The collisionless Boltzmann equation in cylindrical coordinates R, ϕ, z is

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + v_z\frac{\partial f}{\partial z} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \left(\frac{v_R v_\phi}{R} + \frac{1}{R}\frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \tag{19}$$

  a. Consider integrating the collisionless Boltzmann equation over v_z. Why would this be true?

$$\int dv_z \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

  b. Consider integrating the collisionless Boltzmann equation over z. Why would this be true?

$$\int dz v_z \frac{\partial f}{\partial z} = 0$$

  c. In two dimensions we can describe the problem in terms of a distribution function $f(x, y, v_x, v_y, t)$ or in polar coordinates $f(R, \phi, v_R, v_\phi, t)$. The collisionless Boltzmann equation in 2D polar coordinates is the same as equation 19 except lacking those terms that depend on z, v_z or their gradients.

$$\frac{\partial f}{\partial t} + v_R\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \left(\frac{v_R v_\phi}{R} + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} = 0 \tag{20}$$

Using parts a, b, argue that by integrating in z and v_z we derive the same equation. In other words if $f_3(R, \phi, z, v_R, v_\phi, v_z, t)$ satisfies equation 19 then

$$f(R, \phi, v_R, v_\phi, t) = \int dz\ dv_z f_3(R, \phi, z, v_R, v_\phi, v_z, t)$$

satisfies equation 20.