AST233 Lecture notes Part 2

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1 The Hyperbolic orbit

1.1 Two bodies, velocity changes

The center of mass $\mathbf{X}_{com}$

$$\mathbf{X}_{com} = \frac{\sum_i \mathbf{x}_i}{\sum_i m_i}$$

The center of velocity $\mathbf{V}_{com} = \dot{\mathbf{X}}_{com}$ is found by taking the time derivative of $\mathbf{X}_{com}$

$$\mathbf{V}_{com} = \dot{\mathbf{X}}_{com} = \frac{\sum_i \dot{\mathbf{x}}_i}{\sum_i m_i}$$

Two bodies, M, m with velocities $\mathbf{v}_m, \mathbf{v}_M$ and **relative velocity**

$$\mathbf{v} = \mathbf{v}_M - \mathbf{v}_m$$

$$\begin{aligned} m\mathbf{v}_m + M\mathbf{v}_M &= \text{constant} = (m+M)\mathbf{V}_{com} \\ m\mathbf{v}_m + M\mathbf{v}_M - M\mathbf{v}_m + M\mathbf{v}_m &= \text{constant} \\ (m+M)\mathbf{v}_m + M\mathbf{v} &= \text{constant} \qquad (1) \end{aligned}$$

$$\begin{aligned} \mathbf{v}_m &= -\frac{M}{m+M}\mathbf{v} + \text{constant} \qquad (2) \\ \mathbf{v}_M &= \frac{m}{m+M}\mathbf{v} + \text{constant} \qquad (3) \end{aligned}$$

Velocity changes

$$\begin{aligned} \Delta\mathbf{v}_m &= -\frac{M}{m+M}\Delta\mathbf{v} \qquad (4) \\ \Delta\mathbf{v}_M &= \frac{m}{m+M}\Delta\mathbf{v} \qquad (5) \end{aligned}$$

1.2 The two body problem

Two bodies M_1, M_2 with positions $\mathbf{r}_1, \mathbf{r}_2$ and velocities $\mathbf{v}_1, \mathbf{v}_2$.

$$\begin{aligned} E &= M_1\frac{v_1^2}{2} + M_2\frac{v_2^2}{2} - \frac{GM_1M_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad (6) \\ E &= (M_1+M_2)\frac{\mathbf{V}_{com}^2}{2} + \mu\frac{\mathbf{v}^2}{2} + \frac{G(M_1+M_2)\mu}{|\mathbf{r}|} \qquad (7) \end{aligned}$$

The first term is a coasting body of total mass $M = M_1 + M_2$ with a constant velocity $\mathbf{V}_{com}$ corresponding to the velocity of the center of mass. The reduced mass

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

The second two terms in 7 are a Keplerian system of reduced mass μ in orbit about a large mass $M = M_1 + M_2$. For the Keplerian system, the coordinate is the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ with relative velocity $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.

1.3 Angular momentum in polar coordinates

A single body at position $\mathbf{r}$ with velocity $\mathbf{v}$. Together tho vectors $\mathbf{r}, \mathbf{v}$ give us a plane for the orbit. Coordinate

$$\mathbf{r} = r\hat{\mathbf{r}}$$

Velocity

$$\begin{aligned} \mathbf{v} &= v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} && (8) \\ &= v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} && (9) \end{aligned}$$

where we take x, y to be coordinates spanning the plane containing both $\mathbf{r}$ and $\mathbf{v}$.

$$v_\theta = r\dot{\theta}$$

where θ is an angle on the xy plane.

Angular momentum per unit mass

$$\begin{aligned} \mathbf{L} &= \mathbf{r} \times \mathbf{v} && (10) \\ &= r v_\theta \hat{\mathbf{z}} \\ &= r^2 \dot{\theta} \hat{\mathbf{z}} && (11) \end{aligned}$$

The angular momentum is only sensitive to the tangential velocity component.

1.4 Conservation of Angular momentum

With a radial force law the force on a particle i associated with a particle j is $\mathbf{F}_{ij} \propto \mathbf{r}_i - \mathbf{r}_j$ is proportional to the vector between the two particles. Let us adopt $\mathbf{F}_{ij}(\mathbf{r}_i, \mathbf{r}_j) = a_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ with $a_{ii} = 0$. The force on particle i is opposite to that on particle j and this implies that a_{ij} is symmetric. The total angular momentum $\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$ where we are summing over particles.

The change in angular momentum

$$\begin{aligned}
\dot{L} &= \sum_i m_i \left(\dot{\mathbf{r}}_i \times \mathbf{v}_i + \mathbf{r}_i \times \dot{\mathbf{v}}_i\right) \\
&= \sum_i m_i \left(\mathbf{v}_i \times \mathbf{v}_i + \mathbf{r}_i \times \sum_j \mathbf{F}_{ij}/m_i\right) \\
&= \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij} \\
&= \sum_{i,j} \mathbf{r}_i \times a_{ij}(\mathbf{r}_i - \mathbf{r}_j) \\
&= \sum_{i,j} -a_{ij}\mathbf{r}_i \times \mathbf{r}_j \\
&= 0 \qquad (12)
\end{aligned}$$

Here a_{ij} is symmetric but $\mathbf{r}_i \times \mathbf{r}_j = -\mathbf{r}_j \times \mathbf{r}_i$ and is antisymmetric. For every pair i, j the coefficients a_{ij} and a_{ji} have the same sign, but the cross product factors have opposite signs and so the two terms cancel. As a consequence $\dot{L} = 0$ making the total angular momentum $\mathbf{L}$ a conserved quantity.

When forces are only applied along vectors connecting particles, angular momentum conservation is assured. Potentials that are two-body interactions of functions of interparticle distance fall into this category.

1.5 Keplerian orbit

Radial force with $\mathbf{r}$ the vector between two masses

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{G(M+m)}{r^2}\hat{\mathbf{r}} \qquad (13)$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{G(M+m)}{r^2} \qquad (14)$$

Angular momentum per unit mass

$$h \equiv r^2\dot{\theta} = L$$

It is useful to work with inverse radius

$$\begin{aligned}
u &\equiv \frac{1}{r} \\
\dot{u} &= -\frac{\dot{r}}{r^2} \qquad (15)
\end{aligned}$$

We cannot find $r(t)$ but we can find $r(\theta)$.

$$\begin{aligned}
\dot{u} &= \frac{du}{d\theta}\dot{\theta} && (16)\\
&= \frac{du}{d\theta}\frac{h}{r^2} && (17)
\end{aligned}$$

where I have used angular momentum per unit mass h which is conserved to get rid of $\dot{\theta}$. Putting these together

$$\begin{aligned}
\frac{du}{d\theta}\frac{h}{r^2} &= -\frac{\dot{r}}{r^2} \\
\frac{du}{d\theta}h &= -\dot{r} \\
\frac{d\dot{u}}{d\theta}h &= -\ddot{r} && (18)
\end{aligned}$$

where on the last step I took the time derivative and h is a constant. Now insert equation 17

$$\begin{aligned}
-\ddot{r} &= \frac{d}{d\theta}\left(\frac{du}{d\theta}\frac{h}{r^2}\right)h && (19)\\
\ddot{r} &= -\frac{d^2u}{d\theta^2}h^2u^2 && (20)
\end{aligned}$$

Now we go back to equation 14 and start replacing r with u.

$$r\dot{\theta}^2 = \frac{h^2}{r^3} = h^2u^3$$

$$\frac{G(M+m)}{r^2} = G(M+m)u^2$$

Inserting these two relations into equation 14 and using equation 20 we find

$$\frac{d^2u}{d\theta^2} + u = \frac{G(M+m)}{h^2} \quad (21)$$

This has a solution

$$\begin{aligned}
r &= \frac{p}{1+e\cos(\theta-\varpi)} && (22)\\
p &\equiv \frac{h^2}{G(M+m)} && (23)
\end{aligned}$$

where ϖ is the longitude of pericenter and sets the angle of mininum r. The parameter p is called the semi-lattice rectum and e is the orbital eccentricity.

The orbits are conic sections.

Ellipses: $0 < e < 1$, and $p = a(1-e^2)$. Pericenter radius is $q = a(1-e)$. Semi-major axis $a > 0$.

Hyperbolas $e > 1$, and $p = |a(e^2-1)|$. Pericenter radius is $q = |a(e-1)|$. Sometimes negative a is used so that energy per unit mass is positive with $E = -\frac{GM}{2a}$ (and that makes sense as the orbit is not bound).

Parabolas $e = 1$, and $p = 2q$ where q is pericenter.

$$p = |a(1-e^2)| \quad \text{for} \quad e \neq 1 \tag{24}$$
$$p = 2q \quad \text{for} \quad e = 1 \tag{25}$$

Our orbits are described by 3 parameters (see equation 23), a unitless eccentricity e, an orientation angle for the angle of pericenter ϖ, and the semi-lattice rectum p. The constant p is the only one that has units and it is in units of length. But note that it involves a ratio of the square of the angular momentum and $G(M+m)$. We should not be surprised that p is related to a, e and so can be written in terms of orbital energy and angular momentum.

In terms of a, e, the orbital energy per unit mass

$$E = -\frac{G(M+m)}{2a}. \tag{26}$$

Equations 23, 24 then gives the angular momentum per unit mass

$$h = \sqrt{G(M+m)|a(1-e^2)|} \tag{27}$$

1.6 True anomaly

$$r = \frac{p}{1 + e\cos(\theta - \varpi)} \tag{28}$$
$$= \frac{p}{1 + e\cos f} \tag{29}$$

where f is the true anomaly. Heliocentric coordinates

$$x = r\cos f \tag{30}$$
$$y = r\sin f \tag{31}$$

Angles from pericenter are **anomalies** (see Figure 1), whereas angles from a fixed reference direction are **longitudes**.

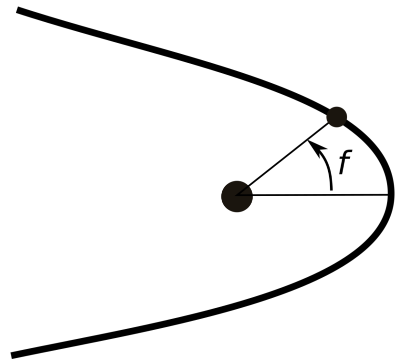

Figure 1: The true anomaly gives the angle of the test mass in the orbital plane with respect to percenter for a test particle in orbit about a larger mass.

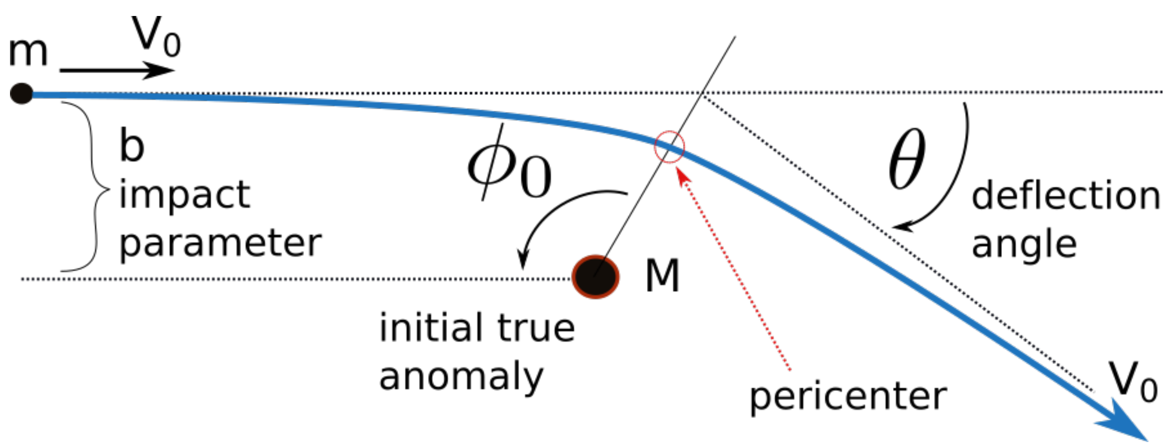

Figure 2: A gravitational encounter with impact parameter b and relative velocity V_0. The orbit is hyperbolic. The angle ϕ is also the true anomaly. Here the angular momentum per unit mass $h = bV_0$.

1.7 Energy and semi-major axis for a Hyperbolic encounter

Impact parameter b. Incoming velocity of mass m is V_0 coming toward initially fixed mass M. Hyperbolic orbit. See Figure 2. What is the velocity of the center of mass?

$$V_{com} = \frac{m}{m+M} V_0 \tag{32}$$

It is positive.

Initially the energy is kinetic only. The total energy is $E = \frac{mV_0^2}{2}$. This is equal to the sum of the kinetic energy of the center of mass and the total Keplerian energy of the two body system.

The total energy

$$E = \frac{mV_0^2}{2} = (m+M)\frac{V_{com}^2}{2} - \frac{GMm}{2a}$$

Note $G(M+m)\mu = GMm$. Insert the center of mass velocity (equation 32) and solve for semi-major axis a

$$a = -\frac{G(M+m)}{V_0^2} \tag{33}$$

Note, no factor of 2 here is correct. Here I am using the convention $E = -\frac{GMm}{2a} > 0$ and $a < 0$ for a hyperbolic (unbound) orbit.

1.8 Angular momentum and Eccentricity for a Hyperbolic encounter

Impact parameter b and mass m and velocity V_0 angular momentum per unit mass

$$h = bV_0 \tag{34}$$

We had two ways to write the semi-lattice rectum (equations 23, 24)

$$p = \frac{h^2}{G(m+M)} = |a(e^2-1)|$$

Insert the expression for h and solve for e^2

$$e^2 - 1 = \frac{b^2 V_0^4}{G^2(M+m)^2}$$

and convention $e > 1$ for a hyperbolic orbit.

Notice that we see $G(M+m)/b$ in the expression. Let us define a gravitational velocity scale

$$V_g \equiv \sqrt{\frac{G(M+m)}{b}}. \tag{35}$$

Then

$$e^2 = 1 + \frac{V_0^4}{V_g^4}$$

For $V_0 > V_g$ the eccentricity is large and the orbit strongly hyperbolic. For V_0 small the orbit approaches $e \to 1$ and the orbit is nearly parabolic.

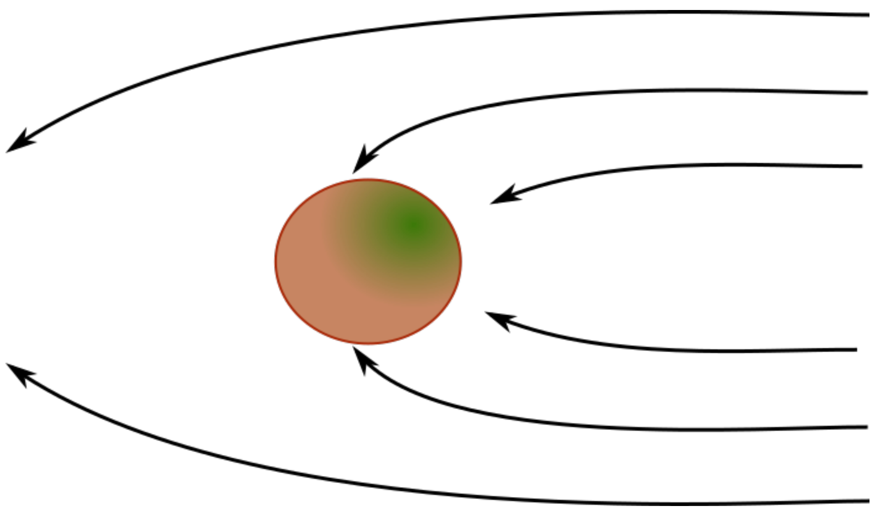

Figure 3: Gravitational focusing

1.9 Gravitational focusing

For a hyperbolic encounter the semi-major axis

$$|a| = \frac{G(M+m)}{V_0^2} \tag{36}$$

and eccentricity

$$e^2 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2} = 1 + A^{-2} \tag{37}$$

with

$$A = \frac{G(m+M)}{bV_0^2} = \left(\frac{V_g}{V_0}\right)^2, \tag{38}$$

and where the gravitational velocity scale is defined in equation 35. The pericenter radius $q = |a(e-1)|$. Inserting a, e into the equation for pericenter we find that

$$q = b\left(\sqrt{1+A^2} - A\right) \tag{39}$$

The pericenter is a minimum distance between the masses m, M during the encounter. For $V_0 > V_g$ the encounter has $q \sim b$ where as for $V_0 < V_g$ the pericenter distance q is much smaller than b.

When $V_0 > V_g$, the parameter $A < 1$ and pericenter $q \sim b$.

When $V_0 < V_g$, the parameter $A > 1$. In the limit of $V_0/V_g \to 0$, pericenter q approaches 0 (becomes smaller and smaller). Approximating this for small A^{-1},

$$\begin{aligned} \frac{q}{b} &= \sqrt{1+A^2} - A = A\left(\sqrt{A^{-2}+1} - 1\right) & (40) \\ &\sim A\left(1 + A^{-2}/2 - 1\right) \sim A^{-1}/2 & (41) \end{aligned}$$

or

$$q \sim \frac{b}{2}\left(\frac{V_0}{V_g}\right)^2 = \frac{b}{2}\frac{V_0^2 b}{G(M+m)}. \tag{42}$$

What does the pericenter distance have to do with *gravitational focusing*? The pericenter sets the cross section for collisions.

To discuss collisions we consider a mass M passing through a sea of smaller particles of mass m. I am flipping the picture (M vs m) because nothing we did above depends on which of the two particles was more massive. It makes more sense to use notation $M > m$ and have M be the moving particle. The mass M has velocity V_0 with respect to the fixed particles m. We ignore the radius of the m particles assuming that they are small.

A collision happens if M passes within a distance R of a smaller particle m where R would be the radius of M. A collision happens if the pericenter distance of the encounter $q < R$. We have introduced a new scale R to the problem. With a sea of particles the collision rate is set by all collisions with impact parameter b such that $q(b) < R$. Let us define b_R to be the impact parameter such that $q(b_R) = R$. If the encounters are slow then $b_R > R$ whereas if the encounters are fast then $b_R = R$. As $b_R > R$ in the slow setting, the collision rate is higher than in the fast setting. This effect is known as gravitational focusing because the encounters themselves pull trajectories toward M, increasing the collision rate. Gravity focuses in the sense that many more trajectories are encounters than estimated using the body's radius alone to estimate the cross section.

Given a velocity V_0, masses M, m and radius R what is the ratio b_R/R?

We need to solve the equation $q(b_R) = R$ for b_R. Taking equation 39 we can rewrite it as

$$q^2 + 2Abq = b^2$$

Now let $q = R$ and insert $A = \frac{G(m+M)}{bV_0^2}$ (equation 38)

$$R^2 + \frac{2G(m+M)R^2}{V_0} = b_R^2$$

We solve for b_R finding

$$b_R = R\left(1 + \frac{2G(M+m)}{V_0^2 R}\right)^{\frac{1}{2}} \tag{43}$$

It may be useful to define a new quantity

$$V_R \equiv \sqrt{\frac{G(M+m)}{R}}$$

$$\frac{b_R}{R} = \left(1 + \frac{2V_R^2}{V_0^2}\right)^{\frac{1}{2}} \tag{44}$$

By introducing a scale R we have also introduced a new velocity scale, V_R. If $V_R > V_0$ then gravitational focusing is a large effect, otherwise $b_R \sim R$.

In the slow V_0 velocity limit

$$b_R^2 \sim R^2 \frac{V_R^2}{V_0^2} \sim \frac{GMR}{V_0^2}$$

Collision probability can be estimated from the cross section

$$\int 2\pi b \; db = \pi b_R^2$$

Accretion rate depends on the number density of planetesimals n, their masses m, and relative velocity V_0

$$\dot{M} \sim nm\pi b_R^2 V_0 \sim nm\frac{GMR}{V_0}$$

Using $R \sim M^{\frac{1}{3}} \rho_{body}^{-\frac{1}{3}}$

$$\dot{M} \sim \frac{mn}{\rho_{body}^{\frac{1}{3}}} \frac{GM^{\frac{4}{3}}}{V_0}$$

A timescale for increasing mass is

$$t_M = \frac{M}{\dot{M}} \propto M^{-1/3}$$

and is very short for high mass objects. As higher mass objects double their mass faster than lower mass objects, accretion favors growth of a few high mass objects.

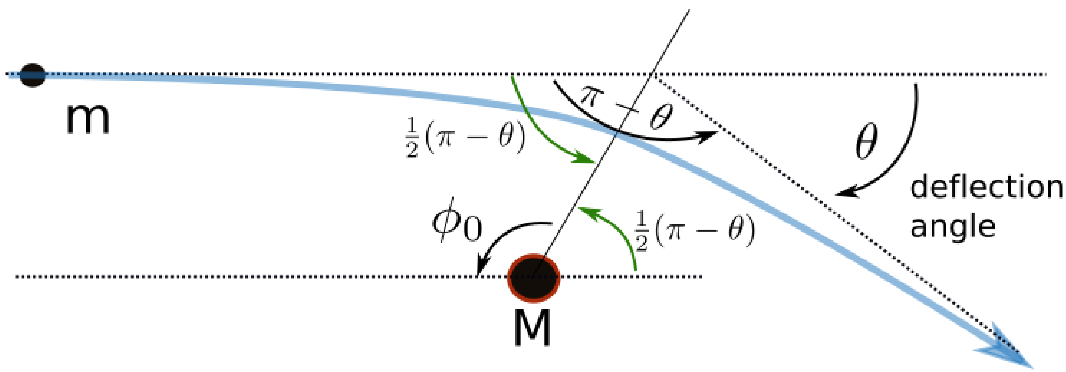

Figure 4: Deflection angle, θ, in terms of the initial true anomaly, ϕ_0, for a hyperbolic orbit. $\theta = 2\phi_0 - \pi$.

1.10 Deflection angle for the hyperbolic orbit

Looking at Figure 4 the deflection angle

$$\theta = 2\phi_0 - \pi$$

where ϕ_0 is the angle measured between initial velocity and pericenter. This angle is equivalent to the initial true anomali. Recall that an angle between the line connecting M to m and a reference direction aligned with M and m at pericenter is the true anomaly. Going back to our orbit equation

$$r = \frac{p}{1 + e\cos f}$$

When $f = 0$ we are at pericenter. So we can take $f = \phi_0$ equal to the initial true anomaly. The radius goes to infinity at an angle where the denominator vanishes or

$$1 + e\cos f = 1 + e\cos\phi_0 = 0$$

or

$$\sec\phi_0 = -e$$

Because $1 + \tan^2\phi = \sec^2\phi$ we find

$$e^2 = 1 + \tan^2\phi_0 \tag{45}$$

and this happens at angles $f = \pm\phi_0$. Using equation 37

$$e^2 = 1 + \tan^2\phi_0 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2}. \tag{46}$$

We should notice that this implies that

$$\tan\phi_0 = \frac{bV_0^2}{G(M+m)} \tag{47}$$

Inspection of Figure 5 helps us relate the changes in the relative velocity components to the deflection angle.

$$\begin{aligned}
\Delta V_\perp &= V_0 \sin\theta_d && (48)\\
\Delta V_\parallel &= -V_0(1-\cos\theta_d) && (49)
\end{aligned}$$

Here parallel is along the initial direction of the m and the perpendicular is perpendicular to this direction but in the plane containing the two masses and their trajectories. We can keep the signs straight if we remember that $\Delta V_\parallel$ must slow down the initially moving mass and $\Delta V_\perp$ is in the direction toward the other mass. With $\mathbf{V} = \mathbf{V}_m - \mathbf{V}_M$, and m the one initially moving with positive V_0 then $\Delta V_\parallel$ is negative and $\Delta V_\perp$ is m moving toward M.

Using some trig identities

$$\begin{aligned}
\sin\theta_d &= \sin(2\phi_0 - \pi)\\
&= -\sin(2\phi_0) = -2\sin\phi_0\cos\phi_0\\
&= -2\tan\phi_0\cos^2\phi_0\\
&= -\frac{2\tan\phi_0}{1+\tan^2\phi_0} && (50)
\end{aligned}$$

$$\begin{aligned}
1-\cos\theta_d &= 1+\cos 2\phi_0\\
&= 2\cos^2\phi_0\\
&= \frac{2}{1+\tan^2\phi_0} && (51)
\end{aligned}$$

1.11 Parallel and perpendicular velocity changes

Putting these trig functions (equations 47, 50, 51) together with equation 52 and equation 46,

$$\begin{aligned}
\Delta V_\perp &= -V_0\sin\theta_d = -\frac{2bV_0^3}{G(M+m)}e^{-2}\\
\Delta V_\parallel &= V_0(1-\cos\theta_d) = 2V_0 e^{-2} && (52)
\end{aligned}$$

with

$$e^2 = 1 + \tan^2\phi_0 = 1 + \frac{b^2V_0^4}{G^2(M+m)^2}. \tag{53}$$

Now we need to get out of the center of mass frame using equations 5. So far there is no dependence on which mass is the one initially moving. Taking M initially fixed and m the one that is initially moving if we want to know the change to M we need to multiply ΔV by $m/(m+M)$ giving

$$\Delta V_{M\perp} = \frac{2mbV_0^3}{G(M+m)^2}e^{-2} \quad (54)$$

$$\Delta V_{M\parallel} = \frac{2mV_0}{M+m}e^{-2} \quad (55)$$

With $\Delta V_{M\parallel}$ in the same direction as m's initial velocity (M is sped up) and $\Delta V_{M,\perp}$ in the direction toward m at pericenter.

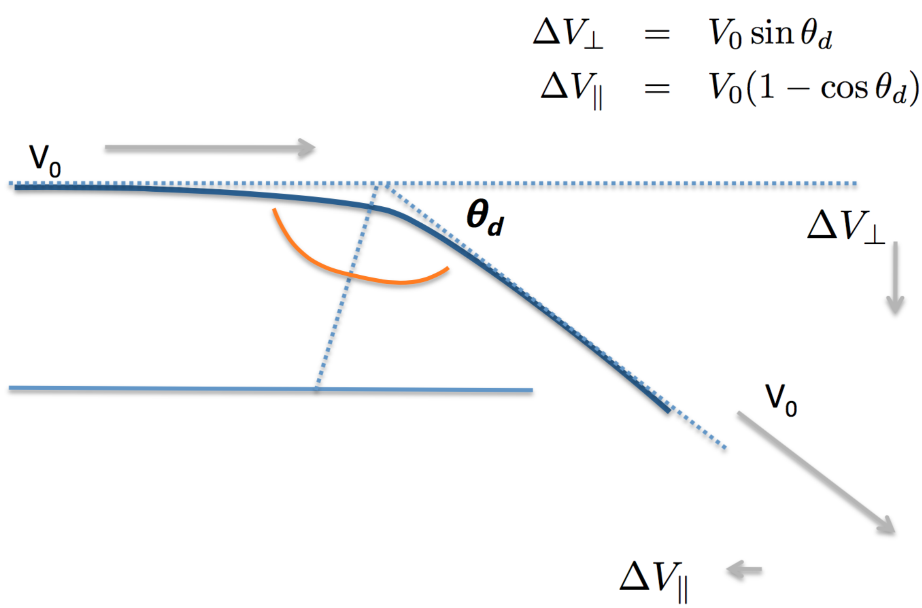

Figure 5: Components of the velocity change due to the encounter in terms of the deflection angle θ_d.

2 Applications

2.1 Dynamical friction

The number density of stars with mass m is f. The rate that a star with mass m impact parameter b and velocity v has an encounter with M

$$2\pi b\ db\ V_0 f.$$

Here V_0 is the relative velocity. Let us assume that M is moving at a velocity V_0 with respect to a sea of particles with mass m. To find the total rate of change in $\Delta V_{M\|}$ we integrate over all impact parameters

$$\frac{d}{dt}\Delta V_{M\|} = \int_0^\infty db\ 2\pi b V_0 f \Delta V_{M\|}(b) \tag{56}$$

$$= \int_0^\infty db\ 2\pi b V_0 f \frac{2mV_0}{(M+m)}\left[1+\frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1} \tag{57}$$

where I have used equation 55 for $V_{M\|}$. If the field of stars is uniform then we can neglect $\Delta V_\perp$ as it should cancel to zero when we integrate.

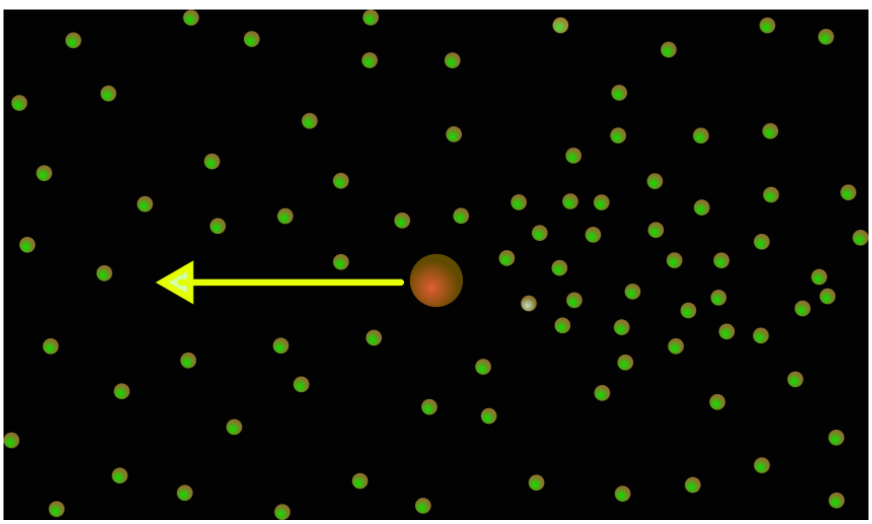

Figure 6: Dynamical Friction. Illustrating how a large mass, moving through a sea of particles accumulates a wake behind it due to gravitational scattering. The wake slows the mass down, causing the frictional force known as dynamical friction.

We notice that the integral in equation 57 is dominated by encounters at large impact parameter b. Let us simplify the integral taking this into account, and setting density $\rho = fm$, the mass density of our sea of particles with mass m;

$$\frac{d}{dt}\Delta V_{M\parallel} \sim \int_0^\infty db\ 2\pi b V_0 \rho \frac{2V_0}{M+m}\frac{G^2(M+m)^2}{b^2 V_0^4} \tag{58}$$

$$= \int_0^\infty db\ \frac{4\pi\rho G^2(M+m)}{bV_0^2} \tag{59}$$

This integral diverges so we can't let impact parameter $b \to \infty$. We can consider a maximum impact parameter b_{max} typical of our system. $G(M+m)/V_0^2$ is in units of length and this is the transition regime were the denominator in equation 57 is 1. Let $u = \frac{bV_0^2}{G(M+m)}$, with $db = du\frac{V_0^2}{G(M+m)}$.

$$\begin{aligned}\frac{d}{dt}\Delta V_{M\parallel} &\sim \frac{4\pi\rho G^2(M+m)}{V_0^2}\int_0^{b_{max}}\frac{db}{b} \\ &\sim \frac{4\pi\rho G^2(M+m)}{V_0^2}\int_1^{\frac{b_{max}V_0^2}{G(M+m)}}\frac{du}{u} \\ &= \frac{4\pi\rho G^2(M+m)}{V_0^2}\ln\left(\frac{b_{max}V_0^2}{G(M+m)}\right) \end{aligned} \tag{60}$$

It is customary to define a Coulomb log

$$\Lambda \equiv \frac{b_{max}V_0^2}{G(M+m)} \tag{61}$$

The change in velocity is in the same direction as M is moving so

$$\dot{\mathbf{V}}_M \sim -\frac{4\pi\rho G^2(M+m)\ln\Lambda}{V_M^3}\mathbf{V}_M \tag{62}$$

For $M > m$ the change in velocity is proportional to M. As the acceleration is proportional to M the actual force is proportional to M^2. The change in velocity depends on the velocity so this is a dissipative force and that is why it is called *dynamical friction*. The formula diverges for small V_M only because we did not correctly estimate the integral (because we took the limit of large impact parameter).

A similar formula taking into account the integral over relative velocities was derived by Chandrasekkar. In this case we would take $f(\mathbf{v})$ a distribution function and integrate over d^3v. The change in velocity depends on the relative velocity so we would integrate over $v - V_M$.

With an isotropic Maxwellian velocity distribution the integral over all encounter velocities

$$\frac{d\mathbf{V}_M}{dt} = -\frac{4\pi \ln\Lambda \; G^2 M\rho}{V_M^3}\left(\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}e^{-X^2}\right)\mathbf{V}_M \tag{63}$$

where σ is the velocity dispersion and $X \equiv \frac{v_M}{2\sigma}$.

As a mass M (globular cluster, black hole) passes through a sea of stars, it leaves a gravitational **wake** behind it of focused stars and this wake slowly pulls M backwards slowing it down.

Insert picture here!

2.2 Gravitational stirring and heating

With dynamical friction we primarily took into account the drag force from the component of the parallel component of the velocity change in a hyperbolic orbit.

We now think about the other components.

Each encounter gives a random change in velocity. So while perpendicular velocity changes do average to zero, they also cause random motions. The expectation of $\langle \Delta V \rangle = 0$ However $\langle \Delta V^2 \rangle$ is not zero. We can think of the problem with a random walk or diffuse like behavior.

The effect is **gravitational stirring** or **gravitational heating**.

Diffusion coefficients come from integrating components of ΔV over the velocity field. For mass M

$$\begin{aligned}
D(\Delta v_\parallel) &= \frac{4\pi G^2 \rho (M+m)\ln\Lambda}{\sigma^2} G(X) && (64)\\
D(\Delta v_\parallel^2) &= \frac{4\sqrt{2}\pi G^2 \rho m \ln\Lambda}{\sigma}\frac{G(X)}{X} && (65)\\
D(\Delta v_\perp^2) &= \frac{4\sqrt{2}\pi G^2 \rho m \ln\Lambda}{\sigma}\left(\frac{\mathrm{erf}(X) - G(X)}{X}\right) && (66)\\
G(X) &= \frac{1}{2X^2}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}e^{-X^2}\right] && (67)\\
X &\equiv \frac{v_M}{2\sigma} && (68)
\end{aligned}$$

Here $D(\Delta v_\parallel)$ gives a drift, whereas $D(\Delta v_\parallel^2), D(\Delta v_\perp^2)$ are diffusive, giving random motions. Statistics can be described in terms of an advective diffusion equation. A star moving through the galaxy is primarily heated by the diffusive terms, whereas a globular cluster moving through a sea of stars would be primarily slowed down by the advective or drift term. A star on a trajectory, is one of a bunch of possible randomly chosen trajectories. The distribution of trajectories widens due to heating and this is described by the diffusive terms. Gravitational heating is described by the diffusive terms whereas dynamical friction is described by the drift term. When the two balance we have *equipartition*.

Focusing on units only

$$
\begin{aligned}
D(\Delta v_\perp^2) &\sim G^2\rho m\sigma && (69)\\
\frac{\Delta v_\perp^2}{\Delta t} &\sim t_\rho^{-2}\frac{Gm}{\sigma^2}\sigma && (70)\\
&\sim t_\rho^{-2}R_m\sigma && (71)\\
\Delta v_\perp &\sim (\Delta_t R_m\sigma)^{\frac{1}{2}}t_\rho^{-1} && (72)
\end{aligned}
$$

where $t_\rho = (G\rho)^{-1/2}$ and $R_m = \frac{Gm}{\sigma^2}$.

2.3 Equipartition

The kinetic energy of a single particle of mass M

$$E = \sum_i \frac{Mv_i^2}{2}$$

where i is over x,y,z. Diffusion in energy

$$
\begin{aligned}
\frac{D(\Delta E)}{M} &= \sum_i v_i D(\Delta v_i) + \frac{1}{2}\sum_i D(\Delta v_i^2) && (73)\\
&= vD(\Delta v_\parallel) + \frac{1}{2}\left(D(\Delta v_\parallel^2) + D(\Delta v_\perp^2)\right) && (74)
\end{aligned}
$$

where convention is that perpendicular part takes into account both components of perpendicular part.

The parallel part is negative because this is dynamical friction. This is a cooling term. The other two terms are heating terms. Looking at equation 64, $D(\Delta v_\parallel) \propto M/\sigma^2$ with σ the velocity dispersion of the masses m. Looking at equation 65, $D(\Delta v_\parallel^2)$ and $D(\Delta v_\perp^2) \propto m/\sigma$. Setting a balance with $\frac{D(\Delta E)}{M} = 0$. The two terms in equation 74 are equivalent with

$$\frac{V_M M}{\sigma^2} \sim \frac{m}{\sigma}$$

or when

$$MV_M \sim m\sigma.$$

When two different masses are present the heating and cooling term balance giving what is called **equipartition**.

2.4 Eccentricity and Inclination evolution in a circumstellar disk

Gravitational heating of planetesimals or dust particles in a circumstellar disk is due to scattering from the larger masses in the disk.

$$\frac{d\langle e^2\rangle}{dt} = \frac{\Omega r^2 \sigma_* M_*^{-2}}{\sqrt{\pi}(\langle e_*^2\rangle - \langle e^2\rangle)^{\frac{1}{2}}(\langle i_*^2\rangle - \langle i^2\rangle)^{\frac{1}{2}}}\left[BJ_e m_* + 1.4AH_e\left(\frac{m_*\langle e_*^2\rangle - m\langle e^2\rangle}{\langle e^2\rangle + \langle e_*^2\rangle}\right)\right] \quad (75)$$

From Stewart and Ida 2000, Icarus 143, 28.

Ω is angular rotation rate about M_*. σ_* is surface mass density of particles m_* and σ is mass density of particles with mass m. Two heating terms and one damping term. Dynamical friction is negative so is a cooling term.

Inclination evolution is the same except one half the size and this is consistent with velocity isotropy. Here B, J_e, A, H_e are all terms of order unity. The ones subscripted with e depend on whether the system is in a dispersion or shear dominated regime. Observed debris disks are not in the shear dominated regime, whereas cold ring systems might be, but in that case there would also be cooling due to collisions and that is not taken into account here.

2.5 Relaxation timescale in a star cluster

Consider a star cluster of mass M with N stars so $M = Nm$ where m is the mass of each star in the cluster. The cluster has a radius R and a typical velocity scale

$$V = \sqrt{\frac{GM}{R}} = \sqrt{\frac{GNm}{R}} \quad (76)$$

which is also approximately the velocity dispersion in the cluster.

What time does it take for a star in the cluster to lose memory of its orbit? This is known as the **relaxation time**.

A gravitational encounter with impact parameter b gives a velocity kick in the perpendicular direction of order

$$\delta v \sim \frac{Gm}{bV}$$

Our star undergoes a random walk due to these velocity kicks. During a crossing time, our star experiences N kicks and the velocity kicks add in quadrature as they would on average all cancel when added. The number of stars per unit area is N/R^2 but each kick is due to an encounter with a different impact parameter. The total velocity change after passing through the cluster once (and having an encounter with every star in the cluster)

$$\begin{aligned}
\Delta v^2 &\sim \int_{b_{min}}^{b_{max}} 2\pi b \; db \frac{N}{R^2} (\delta v)^2 \\
&\sim \int_{b_{min}}^{b_{max}} 2\pi b \; db \frac{N}{R^2} \left(\frac{Gm}{bV} \right)^2 \\
&\sim \frac{N}{R^2} 2\pi \frac{G^2 m^2}{V^2} \int_{b_{min}}^{b_{max}} \frac{db}{b} \\
&\sim \frac{N}{R^2} 2\pi \frac{G^2 m^2}{V^2} \ln \left| \frac{b_{max}}{b_{min}} \right| \qquad (77)
\end{aligned}$$

Define

$$\Lambda \equiv \frac{b_{max}}{b_{min}} \qquad (78)$$

and use equation 76 to remove R

$$\frac{\Delta v^2}{V^2} = \frac{2\pi}{N} \ln \Lambda \qquad (79)$$

Losing memory of the initial velocity happens when $\Delta v^2 / V^2 \sim 1$. The number of crossing times required to lose all memory of initial conditions is

$$n_{relax} \sim \frac{N}{6 \ln \Lambda}. \qquad (80)$$

It makes sense that the maximum impact parameter $b_{max} = R$. The minimum impact parameter we can estimate from a gravitational sizescale $b_{min} \sim Gm/V^2$. Taking the ratio

$$\frac{b_{max}}{b_{min}} = \frac{RV^2}{Gm} \sim \frac{M}{m} \sim N$$

and giving

$$n_{relax} \sim \frac{N}{6 \ln N} \qquad (81)$$

To estimate a relaxation timescale

$$t_{relax} = n_{relax} t_{cross} \qquad (82)$$

with the crossing timescale

$$t_{cross} \sim \frac{R}{V}$$

2.6 Stochastic behavior, ergodicity and chaos

When we discuss gravitational heating in terms of diffusion or gravitational relaxation we assume that gravitational encounters are a *stochastic* phenomena. Stochastic here means involving random behavior. This contrasts with a Keplerian system which is analytically solvable. N-body systems are deterministic in the sense that trajectories are integrated and they are not chosen from a random distribution in any way. However for $N \geq 3$ an N-body system is likely to be chaotic. Our assumption of stochastic behavior rests in the way that N-body systems behave *ergodically*. With the word *ergodic* here meaning acting as if we can model the system as if it were random.

3 Problems

1. **Maximum deflection of a space craft near a planet**

   (M+D problem 2.3).

   A test particle approaches a planet of mass M and radius R from infinity with initial speed v_∞ and impact parameter b. The test particle's mass m
   M.

   a) Use the particle's energy, angular momentum and orbit to show that eccentricity

   $$e = 1 + 2\frac{v_\infty^2}{v_{esc}^2}$$

   where v_{esc} is the escape velocity at pericenter q.

   Hints: you need the following three equations Energy per unit mass

   $$E = \frac{1}{2}v_\infty^2 = -\frac{GM}{2a}$$

   Pericenter in terms of semi-major axis and eccentricity

   $$q = a(1-e)$$

   and an equation for the escape velocity at pericenter

   $$v_{esc} = \frac{2GM}{q}$$

   b) Show that the deflection angle of deflection ψ of the test particle satisfies

   $$\sin(\psi/2) = \frac{1}{e}$$

   c) Given that $q < R$ to avoid collision, calculate the maximum orbital deflection angle for a space craft skimming Jupiter with $v_\infty = 10$ km/s.

2. **Lagrangian and Jacobi coordinates for the 3 body system**

From Valtonen and Karttunnen Chapter 2.

In the three body system we have 3 coordinates $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and three masses m_1, m_2, m_3. In the Lagrangian formulation, the equations of motion can be written in terms of coordinate differences. Differences in positions $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ and similarly for velocities and accelerations and different pairs of masses.

The equations of motion using differences are:

$$\ddot{\mathbf{r}}_{12} = Gm_3\mathbf{W} - GM\frac{\mathbf{r}_{12}}{r_{12}^3} \tag{83}$$

$$\mathbf{W} \equiv \frac{\mathbf{r}_{12}}{r_{12}^3} + \frac{\mathbf{r}_{23}}{r_{23}^3} + \frac{\mathbf{r}_{31}}{r_{31}^3} \tag{84}$$

In the Jacobi coordinate system, the three body system is instead described hierarchically. The three body system is describe with two bodies in a compact binary and the third body orbiting the center of mass of the compact binary. The binary is described with distance between the two masses $\mathbf{r} = \mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$. The center of mass of the binary is $\mathbf{r}_B$. The coordinate of the third body from the binary center of mass is $\mathbf{R}_3 = \mathbf{r}_3 - \mathbf{r}_B$.

The total angular momentum for the three body system in the center of mass frame

$$\mathbf{L}_{cm} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 + m_3\mathbf{r}_3 \times \dot{\mathbf{r}}_3 - M\mathbf{r}_{cm} \times \dot{\mathbf{r}}_{cm}$$

Here

$$\begin{aligned} M &= m_1 + m_2 + m_3 \\ \mathbf{r}_{cm} &= \frac{1}{M}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3\right) \\ \dot{\mathbf{r}}_{cm} &= \frac{1}{M}\left(m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + m_3\dot{\mathbf{r}}_3\right) \end{aligned}$$

The angular momentum is a conserved quantity!

a. Show that the total angular momentum in Lagrangian coordinates (in the center of mass frame) is

$$\mathbf{L}_{cm} = \frac{m_1m_2m_3}{M}\left(\frac{\mathbf{r}_{12} \times \dot{\mathbf{r}}_{12}}{m_3} + \frac{\mathbf{r}_{23} \times \dot{\mathbf{r}}_{23}}{m_1} + \frac{\mathbf{r}_{31} \times \dot{\mathbf{r}}_{31}}{m_2}\right).$$

b. Show that the total angular momentum in Jacobi form (in the center of mass frame) is

$$\mathbf{L}_{cm} = \mu_{12}\mathbf{r} \times \dot{\mathbf{r}} + \mu_{3B}\mathbf{R}_3 \times \dot{\mathbf{R}}_3$$

where reduced masses are

$$\mu_{12} = \frac{m_1 m_2}{m_1 + m_2}$$
$$\mu_{3B} = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}$$

3. **On accretion rate and gravitational focusing**

   Consider a planetary embryo of mass M and radius R moving through a sea of planetesimals of mass m and number density n. Assume that M is moving with velocity V_0 with respect to the planetesimals and $M > m$. The velocity dispersion of the planetesimals is σ. Assume that anything that hits M is added to M so that M increases in mass. What is the accretion rate $\dot{M}$ and how does it depend on M? Consider the time-scale $M/\dot{M}$. How does this depend on M?

   There are three velocities in this problem, $\sqrt{GM/R}$, σ, V_0. Discuss the possible regimes.

   Here is the beginning list of regimes ....

   - $V_0 < \sqrt{GM/R} < \sigma$
   - $V_0 < \sigma < \sqrt{GM/R}$

4. **The impulse approximation as a limit**

   We computed the following velocity changes for M following an encounter by m with relative initial velocity V_0.

$$\Delta V_{M,\perp} = \frac{2mbV_0^3}{G(M+m)^2} e^{-2} \quad (85)$$
$$\Delta V_{M,\parallel} = \frac{2mV_0}{M+m} e^{-2} \quad (86)$$

   with

$$e^2 = 1 + \frac{b^2 V_0^4}{G^2(M+m)^2}. \quad (87)$$

   Show that the large V_0 limit gives the impulse approximation with

$$\Delta V_\perp \to \frac{2Gm}{bV_0} \quad (88)$$
$$\Delta V_\parallel \to 0 \quad (89)$$

   Note: The limit for the parallel component is actually to a higher negative power of V_0.

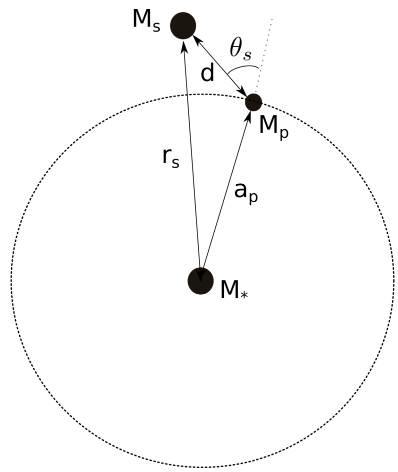

Figure 7: A fast encounter by a star, M_s passing through the ecliptic causes a change in eccentricity of the planet M_p in orbit about star M_*.

5. **Excitation of eccentricity from a gravitational encounter**

   Consider a planet of mass M_p at zero orbital inclination and in a circular orbit with semi-major axis a_p in orbit about its host star M_*.

   A star of mass M_s moving at a very high speed, V_0, passes through the ecliptic plane that contains the planet's orbit. The star's trajectory is perpendicular to the ecliptic orbital plane. It passes through the ecliptic at a radius r_s from the host star. At the moment that the star M_b passes through the ecliptic, it is a distance d from the planet. The point at which it passes through the ecliptic, is described by angle θ_s as shown in Figure 7.

   We assume that $d < a_p$ where a_p is the semi-major axis of the planet.

   a. Using the impulse approximation, estimate the size and direction of the velocity kick given to the planet?

   b. What is the planet's and semi-major axis after the encounter?

   c. What is the planet's eccentricity after the encounter?

   d. Is it possible to change the orbital energy (or semi-major axis) without affecting the eccentricity of the orbit?

6. **Impulse approximation for something other than a point mass. Cluster evaporation**

The impulse approximation assumes that the trajectory of a particle is not perturbed during a gravitational encounter. The velocity change $\Delta V \sim F\Delta t$ where F is a force and Δt is the timescale of the encounter. A somewhat more accurate approximation estimates

$$\Delta \mathbf{V} = \int_{-\infty}^{\infty} \mathbf{F}(t) dt$$

integrated over the linear trajectory during the encounter.

Consider a disk galaxy with mass surface density Σ and vertical scale height h and a globular cluster of mass M_c passing through the disk on a vertical trajectory with velocity V_0.

What is the tidal force from the disk on the cluster as it passes through the galactic disk? Estimate this during the encounter at a radius r from the center of the globular cluster.

What size velocity kick is given to a star at radius r as the cluster passes through the galactic disk?

At what radius does the star become unbound to the cluster?

Is this effect important for young clusters moving through spiral arms? Do we expect evaporating streams to be left behind every time a young cluster passes through a spiral arm?

7. **Stream broadening**

   Consider a star of mass m in a cold stream of stars in the Galaxy (such as Palomar 5). Each star in the stream has a similar velocity and they all lie along a single orbit but differing in position along the orbit.

   The stream is embedded in a sea of black holes of mass M with velocity dispersion σ and density ρ_{BH}.

   First consider the trajectory of the star in the absence of any black hole perturbers.

   How fast does the star diverge from its original trajectory due to gravitational encounters with black holes?

   Consider two nearby stars in the cold stream of stars. How fast do their trajectories diverge?

   Instead of black holes consider dark matter halos and a stream in the Galactic halo.

   Instead of a stream in the Galactic halo consider a stream from a cluster in the galactic disk and molecular clouds or spiral arms as perturbers.

8. **Debris Disk Thickening**

   The Beta Pictoris dusty disk has aspect ratio $h/r \sim 0.05$ and the star is about 10^8 years old. Here the disk thickness is h at radius r. The aspect ratio $h/r \sim \sqrt{\langle i^2 \rangle}$

is related to the inclination distribution of dust particles. Here i is the inclination. Assume that planetesimals embedded in the disk midplane have thickened the disk in 10^8 years. Place a constraint on the mass and surface density of these planetesimals.