PROBLEMS

There is a sample python code included at the end of the assignment that shows how to plot streamlines or level curves of a function $E(x, y)$

1. The harmonic oscillator. The period is independent of amplitude

Consider the harmonic oscillator with Hamiltonian

$$H(p, q) = \frac{1}{2}\left(p^2 + q^2\right)$$

Show that the area $S(E)$ on the phase plane, (p, q), of an orbit with energy E, is proportional to E and so the period of oscillations for the harmonic oscillator is independent of energy or oscillation amplitude.

2. Dimensional analysis on the pendulum.

Consider a Hamiltonian for the pendulum with coefficients a, ϵ

$$H(p, \phi) = a\frac{p^2}{2} - \epsilon \cos\phi \tag{1}$$

Let us take H in units of energy per unit mass and ϕ as an angle. With this convention H has unit of velocity^2 and p has units of velocity times distance.

(a) Find all the fixed points of the Hamiltonian system

(b) Consider the level curves of the system. Which fixed points are stable and which ones lie on a separatrix?

(c) How do your previous answers change if ϵ is negative instead of positive?

(d) How do your previous answers change if a is negative instead of positive?

(e) In a physical system what are the units of a and ϵ?

(f) Construct a frequency using dimensional analysis and a, ϵ.

(g) What is the frequency of libration about a stable fixed point?

(h) Consider an initial condition that is very close to the separatrix. The orbit will move away from the separatrix exponentially fast. Compute the exponential timescale for the motion.

(i) What is the value of energy for an orbit in the separatrix? What is the value of energy for an orbit near a stable fixed point?

(j) Using a, ϵ construct a quantity with units of momentum, p.

(k) What is the maximum p value in the separatrix?

(l) What is the frequency of oscillation ($\dot{\phi}$) for $p \gg \sqrt{|\epsilon/a|}$?

(m) Construct a map rescaling both time and momentum. For τ a unit less time variable and P a unit less momentum that are related to t, p by two coefficients t_r and p_r:

$$\tau = t/t_r$$
$$P = p/p_r$$

find t_r, p_r such that the Hamiltonian in equation 1 becomes

$$H(P, \phi) = \frac{P^2}{2} + \cos\phi$$

3. Plotting level curves for an Andoyer Hamiltonian

A Hamiltonian that describes first order mean motion resonances in Celestial mechanics and Lindblad resonances in galactic dynamics is an Andoyer Hamiltonian

$$H(J, \phi) = J^2 + \delta J + J^{1/2}\cos\phi \tag{2}$$

Here J is an action momentum variable and ϕ an angle. It is customary to plot level curves for this Hamiltonian in a coordinate system (x, y)

$$x = \sqrt{2J}\cos\phi \qquad y = \sqrt{2J}\sin\phi$$

so that radius on the plot gives larger J values and angle on the plot corresponds to ϕ. The coordinate transformation is canonical so Hamilton's equations describe the equation of motion in the new coordinate system.

(a) Transfer the Hamiltonian into coordinates (x, y) showing that the Hamiltonian looks like

$$H(x, y) = \frac{1}{4}(x^2 + y^2)^2 + \frac{\delta}{2}(x^2 + y^2) + \frac{1}{\sqrt{2}}x \tag{3}$$

(b) Plot the level curves as a function of x, y for different values of δ including positive and negative ones. Illustrate that there are either 1 or 3 fixed points.
(c) Classify the fixed points as stable or unstable based on the phase curves.
(d) How do the Hamiltonian level curves change if you flip the sign of the cosine term in the Hamiltonian?
(e) Using Hamilton's equations for x, y find a cubic equation with roots giving the x values for the fixed points.
(f) Explain why this system either has 1 or 3 fixed points. Using the cubic equation, find what δ values give three fixed points rather than just one.
(g) Draw a bifurcation diagram for the fixed points.

4. Pendulum on a rotating axis

Consider a system with force $f(x) = -\sin(x) + M$;

$$\ddot{x} = -\sin x + M$$

(a) What is a Hamiltonian description for this system?
(b) Why is it called a pendulum on a rotating axis?
(c) Plot level curves
(d) Plot streamlines
(e) How do the streamlines, level curves and number of fixed points depend on the value of M?

Now add dissipation

$$\ddot{x} = \sin x + M - \alpha\dot{x}$$

$\alpha > 0$.
(f) Again, find the fixed points
(g) Taking into account energy loss with time, draw streamlines.
(h) Discuss the appearance of a periodic orbit in the case $M > 1$
(this system is much discussed and illustrated by Strogatz in his book on Non-linear Dynamics and Chaos).

5. Legendre Transform Problem

Find the Legendre Transform of

$$f(x) = x^3$$

Where is this function convex and allows a Legendre transform?

6. Time Crystals

Consider a separable Lagrangian

$$\mathcal{L}(q, \dot{q}) = T(\dot{q}) + V(q)$$

that does not depend on time. Because it does not depend on time, the associated Hamiltonian (energy) should be conserved.

A function $f(x)$, $f : \mathbb{R} \to \mathbb{R}$ is convex if for $\forall x_1, x_2 \in \mathbb{R}$ and $\forall t \in [0, 1]$

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$$

A twice differentiable function of one variable is convex on an interval if and only if its second derivative is non-negative in the interval.

Taking a Legendre transform ($\dot{q} \to p$) The associated Hamiltonian

$$H(p, q) = pq_*(p) - \mathcal{L}(q, \dot{q}_*(p))$$

where $q_*(p)$ inverts this condition

$$p = \frac{\partial \mathcal{L}(\dot{q}_*)}{\partial \dot{q}}$$

(a) Show that a minimum energy occurs at non-zero $\dot{q}$ only for non-convex kinetic energy functions.
(b) Show that the minimum energy occurs where p approaches a constant value (and stops being sensitive to $\dot{q}$). This is the same spot where $p(\dot{q})$ becomes multivalued.
(c) Using Lagrange's equations show that $\ddot{q}$ cannot be computed unless $\frac{\partial V}{\partial q} = 0$. There is a connection between the breakdown of the equations of motion and the break-down of time-translation symmetry.

Hint: show that

$$\frac{\partial H(p(q_*), q)}{\partial \dot{q}_*} = \frac{\partial}{\partial \dot{q}}\left[p\dot{q} - \mathcal{L}\right] = T''(q_*)\dot{q}_*$$

(even though it is a bad idea to write a dependence of H on $\dot{q}$ rather than $p(\dot{q}_*)$.

Based on recent works by Al Shapere and Frank Wilczek. *Classical Time Crystals*, Shapere, A. Wilczek F. 2012, Phys. Rev. Lett. 109, 160402

7. Distance to resonance

Consider a pendulum-like Hamiltonian

$$H(p, \phi) = \frac{1}{2}ap^2 + bp + \epsilon \cos \phi$$

(a) Assuming that ϕ is an angle, show that the coefficient b is a frequency. (This is a question of dimensional analysis).

(b) Find a canonical transformation and new coordinates that transform the Hamiltonian into the form

$$K(p', \phi') = \frac{1}{2}a'p^2 + \epsilon' \cos \phi'$$

Remarks: The coefficient b shifts the level curves vertically (in the p direction) on a plot of level curves of $H(p, \phi)$. Only with $b = 0$ are the fixed points located along $p = 0$. If you subtract a constant from a Hamiltonian, Hamilton's equations are unchanged.

We can consider $ap^2 + bp$ as the expansion of an integrable system around a particular value of p. The term $\epsilon \cos \phi$ can be considered a perturbation. We sometimes refer to b as a frequency that sets the distance to resonance.

8. First order canonical transformations

(a) Consider a Hamiltonian with a time dependent perturbation

$$H(I, \theta, t) = I\omega + \epsilon I^{1/2} \cos(\Omega_p t)$$

Find new variables J, ϕ such that the Hamiltonian becomes

$$K(J, \phi) = J\omega$$

and so is in action angle variables.

Hint: Try a generating function in the form

$$S_2(\theta, J) = \theta J + f(J)g(\Omega_p t)$$

with functions $f()$ and $g()$ to be determined.

(b) Show that the system (prior to canonical transformation) is equivalent to a Hamiltonian system in 4-dimensional phase space with angle α and $\dot{\alpha} = \Omega_p$.

9. The Shearing Sheet

The shearing sheet approximation gives equations of motion near a circular orbit in an axisymmetric potential. The shearing sheet has been been used to study structure and growth of instabilities in circumstellar, circumplanetary and galactic disks.

An approximate Hamiltonian (in units of angular rotation rate and radius equal 1).

$$H(x, y; p_x, p_y) \approx \frac{p_y^2}{2} + \frac{p_x^2}{2} - 2p_x y + \frac{\kappa^2 y^2}{2} \tag{4}$$

With epicyclic frequency $\kappa = 1$ (for the Keplerian setting) we can recover Hill's equations.

(a) Show that p_x is a conserved quantity.

As there is a fixed point at $x = 0, y = 0, p_x = 0, p_y = 0$ and the Hamiltonian is quadratic, this Hamiltonian can be written in the form

$$H = \frac{1}{2}\mathbf{x}M\mathbf{x}$$

with M the Hessian matrix and $\mathbf{x} = (x, y, p_x, p_y)$. Equations of motion are

$$\dot{\mathbf{x}} = \boldsymbol{\omega}M\mathbf{x}$$

where $\boldsymbol{\omega}$ is a symplectic type of identity matrix.

(b) Find M.

Eigenvalues of $(\boldsymbol{\omega}M)^2$ are the oscillation frequencies.

(c) Find a set of canonical coordinates that give action angle variables. One way to do this is to use a generating function that looks like

$$F(y, \phi, p_x, X) = \frac{\kappa}{2}(y - ap_x)^2 \tan\phi + p_x X$$

and to look for a suitable value of a.

10. Phase Lag in a drifting system captured into resonance

Consider the Hamiltonian

$$H(p, \phi) = ap^2 + bp + \epsilon p^{1/2} \cos\phi$$

that is often used to describe a first order mean motion resonance or Lindblad resonance. Consider an adiabatically drifting system with distance to resonance, $b(t)$, slowly variation. Above consider coefficients a, ϵ as constants.

Assume that the system captures into resonance. In resonance ϕ remains nearly constant, librating about either $\phi_o = 0$ or π depending upon the signs of a and ϵ.

(a) Using Hamilton's equation for $\frac{\partial H}{\partial p}$ and assuming that $\dot{\phi}$ averages to zero, show that as p grows that

$$p \sim -\frac{b}{2a} \qquad \text{and} \qquad \dot{p} \sim -\frac{\dot{b}}{2a}$$

This is relevant to how eccentricity can increase for a particle or object captured into resonance.

(b) Show that there is a phase lag

$$\delta\phi \sim \frac{-\dot{b}}{\epsilon\sqrt{-2ab}} \cos\phi_o$$

This is relevant to an asymmetry in the Earth's resonant dust ring that was detected by observations from infrared satellites such as COBE.

11. Periods for Conjunctions

(a) Consider a mean motion resonance $jn \sim (j+k)n_p$ where n and n_p are the mean motions of object and planet, respectively and j, k are positive integers and $k < j$. Assume both are on circular orbits. What is the time period between conjunctions?

Write your answer in terms of the planet's rotation period.

(b) Consider a Laplace resonance with argument $\phi = p\lambda_1 - (p+q)\lambda_2 + q\lambda_3$ where $\lambda_1, \lambda_2, \lambda_3$ are longitudes of three different bodies, $\dot{\phi} \sim 0$ and q, p are positive integers. The mean motions of the three bodies are n_1, n_2, n_3 with $n_1 > n_2 > n_3$ interior to exterior. What is the time period between identical configurations? Assume that p and q contain no common factors greater than unity.

(c) Find p, q for the 1:2:4 Laplace resonance between the Galilean moons Io, Europa and Ganymede. What is the time period between identical configurations for this system?

Remarks: Rather than consider a resonance overlap criterion for onset of chaos (Chirikov's approach, also see Arnold diffusion), Pierre Lochak and collaborators placed limits on divergence of orbits from an initial value using periodic orbits. In that setting it does not matter whether the system is chaotic or not just how far the system can vary in a period of time (see Nekhoroshev's Theorem).

Conjunction maps can be used to make a discrete mapping of planetary dynamics. I have heard that these can capture the dynamics remarkably well, though I have never tried making one myself.

12. Complex notation and polynomial expansions

It is sometimes convenient to define

$$\begin{aligned} z &= q + ip \\ \bar{z} &= q - ip \end{aligned}$$

Show that

$$\{z, \bar{z}\} = -2i$$

These are not canonical coordinates.

If $H(p, q)$ is written as $H(z, \bar{z})$ using Poisson brackets show that

$$\begin{aligned} \dot{z} &= -i\frac{\partial H}{\partial \bar{z}} \\ \dot{\bar{z}} &= i\frac{\partial H}{\partial z} \end{aligned}$$

Define an operator

$$D \equiv q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}$$

Show that

$$D z^a \bar{z}^b = i(a - b) z^a \bar{z}^b$$

for integers a, b. Define a pseudo inverse operator

$$D^{-1} z^a \bar{z}^b \equiv \frac{1}{i(a - b)} z^a \bar{z}^b$$

D and D^{-1} do not change the order of a polynomial.

Compute $Dz^2\bar{z}$ and $D^{-1}z^2\bar{z}$.

Let $I(p,q) = \frac{1}{2}(p^2 + q^2)$. Show that $D(I(p,q)^n) = 0$ for any integer $n > 0$. Here n is the exponent of I.

We look at a Hamiltonian that is a harmonic oscillator plus an additional higher order term

$$H(p,q) = I(p,q) + H_3(p,q) \tag{5}$$

where H_s is a polynomial in p, q that is degree three (contains terms like $q^2 p$ and p^3 so that when you sum the exponents of p, q you get 3).

Choose a function $w(p,q) = D^{-1}H_3$. What is the degree of w?

We make a canonical transformation with generating function

$$S(q,P) = qP + w(q,P)$$

Show that to second order in the polynomials of p, q,

$$\begin{aligned} p &= P + \frac{\partial w}{\partial q} \\ q &= Q - \frac{\partial w}{\partial P} \end{aligned}$$

Insert these into equation 5 and show that you can remove H_3 with the canonical transformation (to the next order).

Remarks. As long as there are no resonances this procedure can be used to transform a Hamiltonian expanded near a fixed point into action angle coordinates. The theory of Birkhoff normal forms is tersely introduced in Appendix 7 by Arnold in his book on Math Methods of Classical Mechanics.

13. On the small divisor problem with a single resonant perturbation

(a) Consider a Hamiltonian with a small perturbation term

$$H(I,\theta) = g(I) + \epsilon h(I)\cos\theta$$

where ϵ is small. Using a generating function in the form

$$S_2(\theta, J) = \theta J + \epsilon f(\theta, J)$$

show that the Hamiltonian can be put via canonical transformation into a form $K(J,\phi) = g(J) + O(\epsilon^2)$... that is to first order in action angle variables.

Hint: Assume that $f(\theta, J)$ is separable and either proportional to $\sin\theta$ or $\cos\theta$.

(b) Consider the multidimensional Hamiltonian

$$H(\mathbf{I}, \boldsymbol{\theta}) = g(\mathbf{I}) + \epsilon \cos(\mathbf{k} \cdot \boldsymbol{\theta})$$

where $\mathbf{k}$ is a vector of N integers. Momenta and angles $\mathbf{I}, \boldsymbol{\theta}$ are N dimensional. The frequencies

$$\boldsymbol{\omega}(\mathbf{I}) = \boldsymbol{\nabla} g(\mathbf{I})$$

What condition on $\mathbf{k}$ and $\boldsymbol{\omega}$ allows the Hamiltonian to be put via canonical transformation into the form $K(\mathbf{J}, \boldsymbol{\phi}) = f(\mathbf{J}) + O(\epsilon^2)$?

Remark: As long as there is no commensurability (the angle $\phi = \mathbf{k} \cdot \boldsymbol{\theta}$ is not slow), then it is possible to remove the perturbation term to first order from the Hamiltonian with a first order canonical transformation. Near the commensurability or resonance, to first order, the new coordinates diverge.

(c) Consider the possibility that ϕ is a slow angle. We can transfer to a new canonical coordinate system with new angle $\phi = \mathbf{k} \cdot \boldsymbol{\theta}$. Use the generating function of old angles ($\boldsymbol{\theta}$) and new momenta $(J_1, J_2, J_3 ... J_N)$

$$F_2 = J_1(\mathbf{k} \cdot \boldsymbol{\theta}) + \sum_{i=2..N} J_i \theta_i$$

to show that this Hamiltonian has N conserved quantities and so is integrable.

Hints: Energy can be a conserved quantity. If the Hamiltonian lacks a coordinate then the momentum conjugate to the coordinate is conserved.

Remark: In this coordinate system there are no divergences.

(d) Are there fixed points?

(e) Consider initial values for the momenta $\mathbf{I_0}$ and $\mathbf{k} = (2, 5)$ so that $\phi = 2\theta_1 + 5\theta_2$ and frequencies such that

$$\mathbf{k} \cdot \boldsymbol{\omega}(\mathbf{I_0}) = 0$$

The system has initial angles $\boldsymbol{\theta}_0$ where $\boldsymbol{\theta}$ is the vector of angles. Denote the frequency vector as $\boldsymbol{\omega}(\mathbf{I_0}) = (\omega_1, \omega_2)$. Assume that ϵ is extremely small (like zero).

In terms of period $P = \frac{2\pi}{\omega_1}$ how long does it take the system to return to the initial values of *all* its angles $\boldsymbol{\theta}_0$?

Remark: A fixed point can be called a periodic orbit.

14. Canonical Transformation to a frame in a Rotating Coordinate System

Consider the following Hamiltonian that has been used to represent a fourth order epicyclic approximation near a Lindblad resonance with a spiral or bar pattern with pattern speed Ω_p

$$H(I_1, \theta_1; I_2, \theta_2; t) = \Omega I_2 + \kappa I_1 + aI_1^2 + bI_2^2 + cI_1 I_2 + \epsilon I_1^{1/2} \cos(\theta_1 - m(\theta_2 - \Omega_p t))$$

where m is an integer. Here Ω and κ are the angular rotation rate and epicyclic frequency and both are approximately functions of orbital radius. Coefficients a, b, c are also approximately functions of radius in the galaxy. Here I_2 is related to the angular momentum of the orbit and I_1 the epicyclic amplitude. The parameter ϵ depends on the bar or spiral perturbation strength.

Consider the following generating function

$$F_2(\theta_1, \theta_2; J_1, J_2) = [\theta_1 - m(\theta_2 - \Omega_p t)] J_1 + \theta_2 J_2$$

that is a function of old coordinates and new momenta.

(a) Following a canonical transformation find the form of the Hamiltonian in new coordinates and show that $I_2 + mI_1$ is a conserved quantity. Is energy in the new coordinate system conserved?

(b) In what way is $[(c - 2bm)(I_2 + mI_1) + \kappa - m(\Omega - \Omega_p)]$ the distance to the resonance?

Hint: After transformation, the system is only a function of a single momentum and coordinate (as the other momentum is conserved). It may be possible to group terms so that the Hamiltonian resembles a lower dimensional one, in the form $H(p, \phi) = ap^2 + bp + f(p) \cos \phi$, with constants a, b.

(c) Is there a small divisor problem with this coordinate transformation?

Python script using *matplotlib* and *numpy* for plotting streamlines (or level contours)

```
# plot streamlines of the vector field for dynamics of a pendulum
import numpy as np
import matplotlib.pyplot as plt
xmax = np.pi*2.0             # setting range for grid
ymax = 2
fac=1.01    # so plotting area is slightly larger than grid
# make a grid of x and y values, Y = dot X
X,Y = np.meshgrid(np.arange(-xmax,xmax,.1),np.arange(-ymax,ymax,.1) )
epsilon=0.4
H = 0.5*Y*Y  - epsilon*np.cos(X)    #here is the Hamiltonian
# Hamilton’s equations define a vector field U,V.
# To compute U, V take gradient of H
U = Y
V = -epsilon*np.sin(X)
# set up the plot
plt.figure()
plt.xlabel(’x’)
plt.ylabel(’dx/dt’)
plt.axis([-xmax*fac, xmax*fac,-ymax*fac,ymax*fac])
# plot the vector field here with either of the two commands below
#Q = plt.quiver(X,Y,U,V)
Q = plt.streamplot(X,Y,U,V,density=2)

# if you only want level curves use the following commands
# cs=plt.contour(X,Y,H,20)      # plot 20 levels contour plot
# cc=plt.pcolormesh(X,Y,H)    # plot with color as an image
# plt.colorbar()
```