Notes on Hamiltonians Part 1

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March 21, 2017

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1 Introduction

1.1 Hamiltonian Mechanics

In this formalism the dynamics is described with a physically motivated function that depends on two types of coordinates, coordinates $\mathbf{q}$, and momenta $\mathbf{p}$ that are vectors of the same dimension.

$$H(\mathbf{p}, \mathbf{q})$$

The Hamiltonian could also depend on time t. A Hamiltonian obeys Hamilton's equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

Here the difference in q, p is evident because the time derivative for one contains a minus sign. Hamilton's equations treat coordinates and momenta differently.

Let's look for the moment at the lowest dimensional Hamiltonian system We can consider the dynamics of the system with a 2dimensional vector $\mathbf{x} = (q, p)$. This 2-dimensional space is called *phase space*. Our Hamiltonian is $H(\mathbf{x})$. Hamilton's equations involve partial derivatives of H or the gradient operator

$$\boldsymbol{\nabla} H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right)$$

Hamilton's equation in vector form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

can also be written as

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \boldsymbol{\nabla} H$$

where the matrix

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A symplectic transformation, g, is one such that

$$g^t \boldsymbol{\omega} g = \boldsymbol{\omega} \tag{1}$$

where g^t is the transpose of g. Hamiltonian systems are special dynamical systems in that the equations of motion generate symplectic maps of coordinates and momenta and as a consequence preserve volume in phase space.

One can attempt to describe the dynamics in a coordinate independent or geometrical manner. Or using a canonical formalism, search for nice coordinate systems that make the dynamics appear simpler.

If the Hamiltonian is only dependent on momenta

$$H(\mathbf{p})$$

then Hamiltonian flow preserves these momenta and we can call them conserved quantities. The system is said to be *integrable*. Integrable systems are interesting particularly when the dimension is high. There are examples of nonlinear system systems that are completely integrable. The dynamics of an integrable system, is easy to predict. All the coordinates advance at velocities or frequencies $\dot{q} = \boldsymbol{\nabla} H$ and these velocities (if $q \in \mathbb{R}$) or frequencies (if q is an angle) are constant. One way to determine whether a system is integrable is to do Fourier analysis on its trajectories. If the Fourier transform only contains a few frequencies and their harmonics then the system is behaving as if it were integrable.

But there are low dimensional Hamiltonian systems that cannot be written in this form (*chaotic ones!*). Even systems with small perturbations

$$H = H_0(\mathbf{p}) + \epsilon H_1(\mathbf{q})$$

where ϵ is a small parameter display chaotic behavior, though if ϵ is *small enough* then integrability is recovered. This class of systems is the focus of KAM theory.

Delineating where a complex system behaves chaotically and where it behaves like an integrable system is a challenge. There are complex looking nonlinear systems that are completely integrable. And sometimes we find through numerical integration that a system behaves chaotically and would like to understand why, and predict physically relevant quantities like the rate of diffusion. In celestial mechanics settings there can be a boundary between long lived and unstable systems that can be predicted by estimating a boundary for chaotic behavior.

1.2 Lagrangian mechanics

In this context, the physics is dependent on a choice of Lagrangian

$$\mathcal{L}(q, \dot{q}, t)$$

A trajectory

$$\int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt$$

is minimized in the sense that it must be on a path where the above integral is a minimum. Using calculus of variations, the system obeys Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

A Hamiltonian system may be transformed into a Lagrangian one and vice versa so these two approaches can be equivalent. The Lagrangian viewpoint is particularly powerful when studying constrained systems.

Exploitation of symmetries in the two formalisms is somewhat different. In Lagrangian systems we use Noether's theorem to find conserved quantities. In Hamiltonian systems we look at operators that commute with the Hamiltonian.

One can think of the Lagrangian as a way to measure *distance* between two points on a manifold or a configuration space. In this sense trajectories are *geodesics* as they minimize the distance between two points.

1.3 Dynamical systems

Instead of focusing on physically motivated systems, one can focus on simpler mathematical descriptions that are capable of exhibiting complex phenomena, for example,

$$\dot{x} = f(x)$$

A Hamiltonian system be written in the above way with vector $\mathbf{x} = (\mathbf{q}, \mathbf{p})$.

These systems can exhibit behavior that is exhibited by Hamiltonian systems, such as fixed points, bifurcations of fixed points, periodic orbits, ergodic behavior. While time independent Hamiltonian systems preserve energy, here we can also study dissipative systems. While Hamiltonian systems are area preserving, dissipative ones are not. *Strange attractors* are fractal-like features that arise in dissipative chaotic maps.

1.4 Go down a dimension

Starting with a Hamiltonian system we can look at the dynamics on a smaller dimensional surface, for example orbits in 4D phase space as they cross a 2D surface. This makes a

surfaces of section. Or we can study the dynamics every orbital period or a system with a periodic perturbation. In this setting *Floquet Theory* can be helpful. In both settings maps are created from a space back to itself. The dynamical system can be viewed by looking at iteratively applied maps. To understand their properties we can study *discrete dynamical systems* or iteratively applied maps from a space on to itself. Some nice examples with complex behavior include the logistic map (a map on a line segment), circle maps, the standard map (this is a 2 d map where one dimension is the circle), and Julia sets (of rational maps on the complex plane).

Billiards are even simpler dynamical systems. Given a boundary shape, an initial condition is a location on the boundary s and a trajectory angle θ. The billiard dynamics is a map

$$\mathbf{x}_{i+1} = \mathbf{f}(\mathbf{x}_i)$$

with $\mathbf{x}_i = (s_i, \theta_i)$. It's a 2 dimensional map but based on a dynamical system, just that the trajectories are all lines between bounced on the boundary.

1.5 Even more abstractly - Symbolic dynamics

We can do down a further level of abstraction and explore what is called *symbolic dynamics* where rules are made for going between different states. We can still get things like periodic orbits and fixed points.

Our goal here is to explore the connections between these formalisms/relations for physical systems. Visualization is very helpful in this field so we will spend some time integrating trajectories and making maps on a computer. I have created some python examples using *matplotlib* and *numpy* that can be modified for use in problem sets. One approach to making progress in understanding complex phenomena is to formulate the dynamics with toy model dynamical descriptions. Checking or understanding the behavior of these models computationally can be helpful when trying to understand or predict the behavior of a complex system.

2 Equations of motion with one degree of freedom

We consider Newton's equations

$$\ddot{x} = f(x)$$

The acceleration depends on the force $f(x)$ that is a function of position x. The kinetic energy

$$T = \frac{1}{2}\dot{x}^2$$

and potential energy

$$U(x) = -\int_0^x f(h)dh$$

If the acceleration is downward and constant, $f = -g$, then $U(x) = gx$ and the sign is such that the potential energy is positive for a particle with higher x that is higher off the ground.

In one dimension a function $U(x)$ can be found with $U' = -f(x)$. In more than one dimension, forces are vectors but not all vectors are gradients of a function. Not every vector force is *conservative*. Vector forces that are conservative are a special case.

2.1 Energy is conserved

Energy is the sum of the kinetic and potential energies, $E = T + U$ with kinetic energy $T = \dot{x}^2/2$. Taking the time derivative

$$\frac{d}{dt}(T+U) = \dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0$$

The energy $E = T + U$ is conserved.

The above conservation law can be generalized for a vector force with $\boldsymbol{\nabla} U = -\mathbf{F}$

$$\frac{d}{dt}(T+U) = \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \boldsymbol{\nabla} U \cdot \dot{\mathbf{x}} = \dot{\mathbf{x}} \cdot (\ddot{\mathbf{x}} - \mathbf{F}) = 0$$

In more than one dimension, energy is conserved if the force is conservative. We have assumed that force $\mathbf{F}$ and potential energy U are independent of time.

2.2 Phase flow

Energy can be written

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + U(x)$$

We can considering plotting the motion as a trajectory on a plane defined by (x, y) where

$$y = \dot{x}.$$

$$E(x, y) = \frac{1}{2}y^2 + U(x)$$

Because energy is conserved, orbits on this plane correspond to level curves of E. The equations of motion are now

$$\dot{x} = y \qquad \dot{y} = f(x)$$

A phase curve consists of functions of time such that

$$x = \phi_x(t) \qquad y = \phi_y(t)$$

describing a trajectory. These are functions that depend on the initial conditions so we can write $\mathbf{x} = \boldsymbol{\phi}(t, \mathbf{x}_0)$ where $\mathbf{x} = (x, y)$ and the initial condition is $\mathbf{x}_0$. The equations of motion determine these functions. The trajectory gives a vector field on the phase plane. Level curves of $E(x, y)$ are trajectories. We can also refer to trajectories as *orbits*.

2.2.1 Example: The Harmonic oscillator

Working with a dimensionless system (spring constant and mass are 1) the force

$$f(x) = -x \tag{2}$$

and energy for a harmonic oscillator

$$E(x, y) = \frac{1}{2}\left(x^2 + y^2\right) \tag{3}$$

with $y = \dot{x}$. The phase curves for the harmonic oscillator are shown in Figure 1. There is one fixed point, at the origin. The period of oscillations is independent of the amplitude. Orbits are circles.

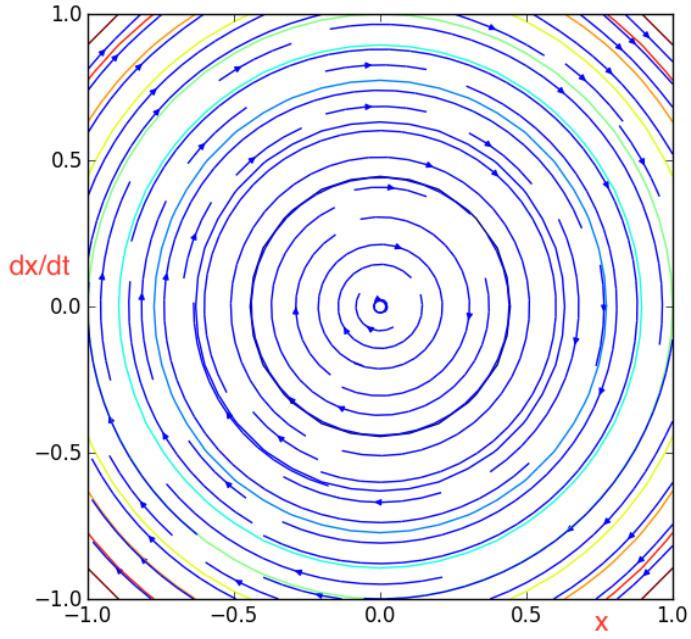

Figure 1: Level curves and phase flow of the Harmonic oscillator.

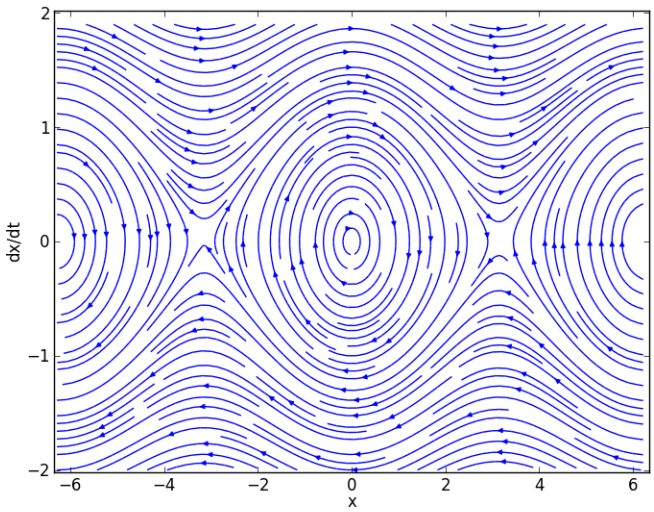

Figure 2: Level curves and phase flow of the Pendulum.

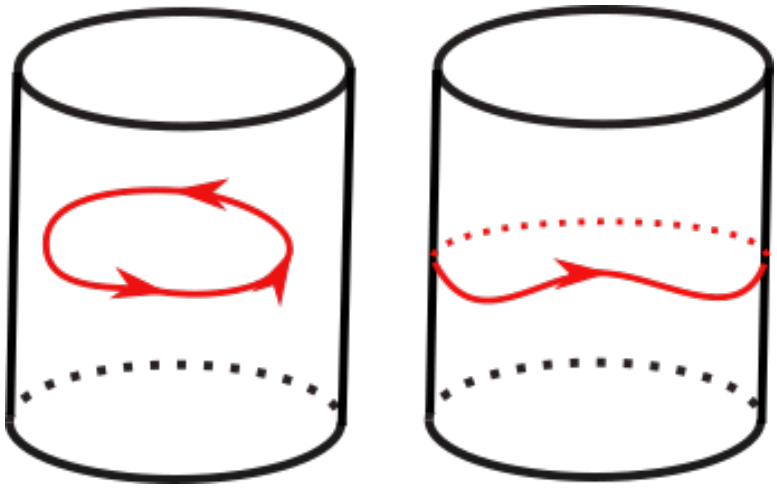

Figure 3: The difference between librating and oscillating orbits for the pendulum (equation 5). Here the vertical axis is y and the x axis corresponds to an angle around each cylinder.

2.2.2 Example: Pendulum

Here x is an angle and the force depends on the sin of the angle.

$$f(x) = -\sin x \tag{4}$$

The energy is

$$E(x, y) = -\cos x + \frac{y^2}{2} \tag{5}$$

with $\dot{x} = y$. Whereas y can take any value, the system is periodic with respect to x, so we can thing of x as an angle. The dynamical system on the plane can be written

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x \end{aligned}$$

At teach time the vector $\dot{x}, \dot{y}$ gives a *tangent* velocity vector. The level curves for this system are shown in Figure 2. Near the origin the phase curves resemble that of the harmonic oscillator. In fact we can expand the cosine for small x finding an energy

$$E(x, y) \sim -1 + \frac{x^2}{2} + \frac{y^2}{2} \tag{6}$$

which is a harmonic oscillator. However trajectories approach and leave $(x, y) = (\pi, 0)$. This is an unstable fixed point. The orbit containing the unstable fixed point is known as the separatrix and separates librating from oscillating trajectories. Librating trajectories correspond to small oscillations about the origin. Oscillating trajectories corresponds to those at large $\dot{x}$ (see Figure 3. A pendulum going around and around in a big circle has large $\dot{x}$ with orbit on the top or bottom of Figure 2. A pendulum has three types of orbits and two types of fixed points (unstable and stable). The types of orbits are the librating orbits (ellipses), hyperbolic orbits connecting unstable fixed points that have infinite period (the separatrix), and circulating orbits.

2.2.3 Example: Cubic Potential

For a Hamiltonian

$$H(p, q) = \frac{1}{2}p^2 + U(q)$$

fixed points (those with $\dot{p} = \dot{q} = 0$) occur where $p = 0$ and $dU/dq = 0$. If $U(q)$ is a cubic polynomial (with a non-zero q^3 coefficient) then fixed points are solutions of a quadratic equation in q. This means there are either zero, one or two real roots. If the potential is continuously rising or falling then there are no fixed points. If there is an inflection point and a single root then the fixed point is unstable. If there are two real roots then one fixed

point is unstable and the other stable (see Figure 4). We can draw a bifurcation diagram (see Figure 5). If use a cubic potential

$$U(q) = -aq + q^3/3$$

that is a function only of the parameter a, then fixed points occur were $U'(q) = -a+q^2 = 0$. We have two real roots for $a > 0$ and no real roots for $a < 0$. The roots are at $\pm\sqrt{a}$ and so their separation increases with a. The bifurcation diagram can be drawn as a function of a.

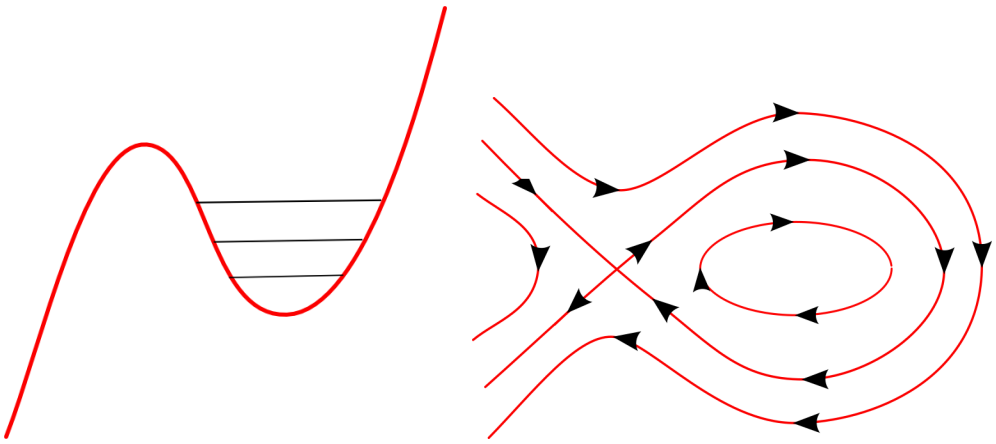

Figure 4: A cubic potential shown on the left. Phase flow and level curves shown on the right.

For positive a, we can expand the Hamiltonian around one of the fixed points. Near $q = \sqrt{a}$

$$x = q - \sqrt{a}$$

giving

$$\begin{aligned} H(p,x) &= \frac{p^2}{2} - a(x+\sqrt{a}) + \frac{1}{3}(x+\sqrt{a})^3 \\ &= \frac{p^2}{2} - a(x+\sqrt{a}) + \frac{x^3}{3} + \frac{a^{3/2}}{3} + x^2\sqrt{a} + xa \\ &\approx \frac{p^2}{2} + \sqrt{a}x^2 + \text{constant} \end{aligned} \tag{7}$$

and we have assumed that x is small. The system looks like a harmonic oscillator. Near $q = -\sqrt{a}$ and using

$$x = q + \sqrt{a}$$

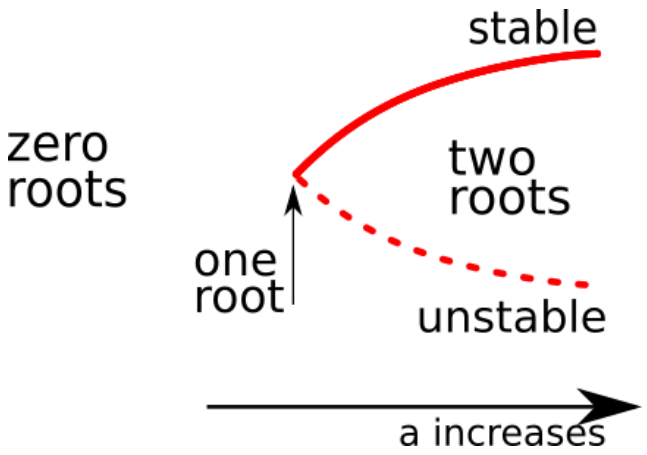

Figure 5: An example of a bifurcation diagram for fixed points of a cubic potential. For a cubic potential $U(q) = -aq + q^3/3$, two fixed points exist at $\pm\sqrt{a}$ when $a > 0$. The solid line corresponds to the stable fixed point and the dotted one to the unstable fixed point.

giving

$$H(p, x) \approx \frac{p^2}{2} - \sqrt{a}x^2 + \text{constant}$$

Here the level curves are hyperbolic.

Note: these are canonical transformations and Hamilton's equations are valid in the new coordinate system.

2.2.4 Example: The radial degree of freedom in Kepler's problem

We consider the problem of a massless particle in orbit around a massive body of mass M. Restricting motion to a single plane. The potential energy is a function of radius alone and is $-GM/r$ for mass M with G the gravitational constant. The kinetic energy

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$$

As the potential does not depend on ϕ there is a conserved quantity, the angular momentum, $L = r^2\dot{\phi}$. The tangential velocity component of the kinetic energy

$$\frac{1}{2}r^2\dot{\phi}^2 = \frac{L^2}{2r^2}$$

The radial motion can be described with an effective potential

$$U(r) = \frac{L^2}{2r^2} - \frac{GM}{r} \tag{8}$$

and equations of motion

$$\ddot{r} = -\frac{\partial U}{\partial r}$$

with energy

$$E(\dot{r}, r) = \frac{\dot{r}^2}{2} + \frac{L^2}{2r^2} - \frac{GM}{r}$$

This system resembles a harmonic oscillator in terms of its level curves. Interestingly we know that the period of radial oscillation in this system only depends on energy E and is independent of L. However it is not obvious from the shape of the potential that this is true.

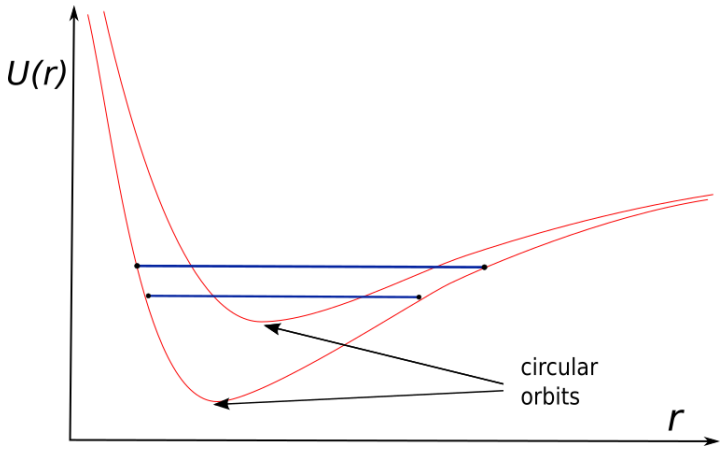

Figure 6: The effective potential for a Keplerian system $U(r) = \frac{L^2}{2r^2} - \frac{GM}{r}$ drawn for two different values of L. Circular orbits are located at the potential minimum. Orbital period depends only on energy not angular momentum, though it is not obvious from the shape of the effective potential.

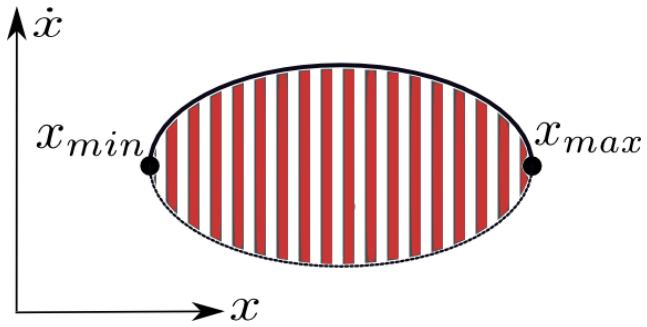

Figure 7: Integrating the area of an orbit. The area enclosed by an orbit with energy E is $S(E) = 2\int_{x_{min}}^{x_{max}} \dot{x}dx$. There is symmetry about $\dot{x} = 0$ for an orbit with $\dot{x} = \pm\sqrt{2(E - U(x))}$.

2.3 Period as a function of area

Taking a particle in a potential

$$E = \frac{\dot{x}^2}{2} + U(x)$$

We can solve for $\dot{x}$

$$\dot{x} = \frac{dx}{dt} = \pm\sqrt{2(E - U(x))}$$

giving (for the positive root)

$$dt = \frac{dx}{\sqrt{2(E - U(x))}}$$

and the time between two points x_1, x_2 at t_1, t_2

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}} \tag{9}$$

For a periodic orbit (one that comes back to itself) we can compute the area of the orbit $S(E)$ as a function of energy.

$$S(E) = 2\int_{x_{min}}^{x_{max}} \dot{x}dx = 2\int_{x_{min}}^{x_{max}} \sqrt{2(E - U(x))}dx$$

where by symmetry the area for positive $\dot{x}$ is the same as that for negative $\dot{x}$. See Figure 7. If we take the derivative of this with respect to energy

$$\begin{aligned} \frac{\partial S(E)}{\partial E} &= 2\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{2(E - U(x))}} \\ &= 2[t(x_{max}) - t(x_{min})] = P \end{aligned}$$

where P is the orbital period. So the orbital period

$$P(E) = \frac{\partial S(E)}{\partial E} \tag{10}$$

Again look at the Kepler system to reflect that it is not obvious that P is independent of L.

2.4 The separatrix of a pendulum

The time it takes to get from one unstable fixed point to another one is infinite. This can be argued from uniqueness of solutions to our second order differential equation. Also we can compute the energy at the separatrix and show that it is zero at $x = 0$ and so the derivative of the area with respect to energy (equation 10) diverges.

Particles located near a stable fixed point, undergo small oscillations about the fixed point. Particles located near the unstable fixed point exponentially diverge from it or exponentially slowly approach it. For example with

$$H(p, \phi) = \frac{p^2}{2} - \epsilon \cos \phi$$

we can expand near $\phi = \pi$ with $\phi = \pi + x$,

$$\cos(\pi + x) \sim -1 + \frac{x^2}{2},$$

Hamilton's equations give

$$\begin{aligned} \dot{\phi} &= \dot{x} = \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial \phi} = -\frac{\partial H}{\partial x} = \epsilon x \end{aligned}$$

Taking the time derivative of $\dot{x}$ and combining equations we find

$$\ddot{x} = \epsilon x$$

with solution

$$x \propto e^{\pm\sqrt{\epsilon}t}$$

near the unstable fixed point.

2.5 On Hamiltonian systems with one degree of freedom

The phase space is two dimensional. There is a single conserved quantity, the energy or Hamiltonian. All orbits fill at most a 1-dimensional space. Fixed points are points. A single quantity (energy) defines a trajectory. The system is said to be *integrable*. All smooth forces are *conservative*.

When we consider two degrees of freedom, many of these statements are no longer true. Not all forces are conservative. Phase space is 4 dimensional and so one conserved quantity is not enough to define a trajectory. An interesting bridge between the one-dimensional and two dimensional systems are periodically forced systems such as the periodically forced pendulum or the kicked rotor that is the physical analog for the standard map.

3 Relation between Lagrangian and Hamiltonian Descriptions

3.1 Extremum of a path

A Lagrangian is a function $\mathcal{L}(q, \dot{q}, t)$. We consider a path γ that gives $q(t)$ for $t_1 \leq t \leq t_2$. The equations of motion are found by minimizing the path length

$$\Phi(\gamma) = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t) dt$$

Here $q(t)$, $\dot{q}(t)$. By using calculus of variations we can show that Lagrange's equation is true on the extremum curve.

We differentiate this function by changing the path by an amount $q(t) \to q(t) + h(t)$ (see Figure 8). We chose $q_1(t_1)$ and $q_2(t_2)$ and assert that the ends of the path remain fixed; $h(t_1) = h(t_2) = 0$.

$$\begin{aligned} \Phi(\gamma + h) - \Phi(\gamma) &= \int_{t_1}^{t_2} \left[\mathcal{L}(q + h, \dot{q} + \dot{h}, t) - \mathcal{L}(q, \dot{q}, t) \right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} h + \frac{\partial \mathcal{L}}{\partial \dot{q}} \dot{h} \right] dt + O(h^2) \end{aligned} \tag{11}$$

Integrating by parts

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{q}} \dot{h} dt = -\int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} h dt + \left(h \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \Big|_{t_1}^{t_2}$$

We insert this back into equation 11

$$\Phi(\gamma + h) - \Phi(\gamma) = \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] h dt + \left(h \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \Big|_{t_1}^{t_2}$$

Because we have asserted that the ends of the path remain fixed, the rightmost term must be zero. Hence the extremum of the integral $\Phi(\gamma)$ is a path on which Lagrange's equations are satisfied or

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0$$

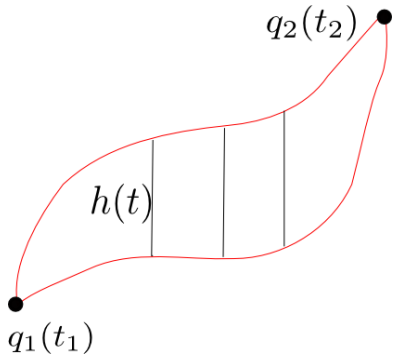

Figure 8: Paths between points $q_1(t_1)$ and $q_2(t_2)$

3.2 Lagrangian's equations for a Newtonian system

For a mechanical system $L = T(\dot{q}) - U(q)$ and Lagrange's equations are equivalent to Newton's equations.

$$L(q, \dot{q}, t) = \frac{1}{2}\dot{q}^2 - U(q)$$

$$\frac{\partial L}{\partial \dot{q}} = \dot{q} \tag{12}$$

$$\frac{\partial L}{\partial q} = -\frac{dU}{dq} = f(q) \tag{13}$$

Hence Lagrange's equations give

$$\ddot{q} = f(q)$$

which is Newton's equation.

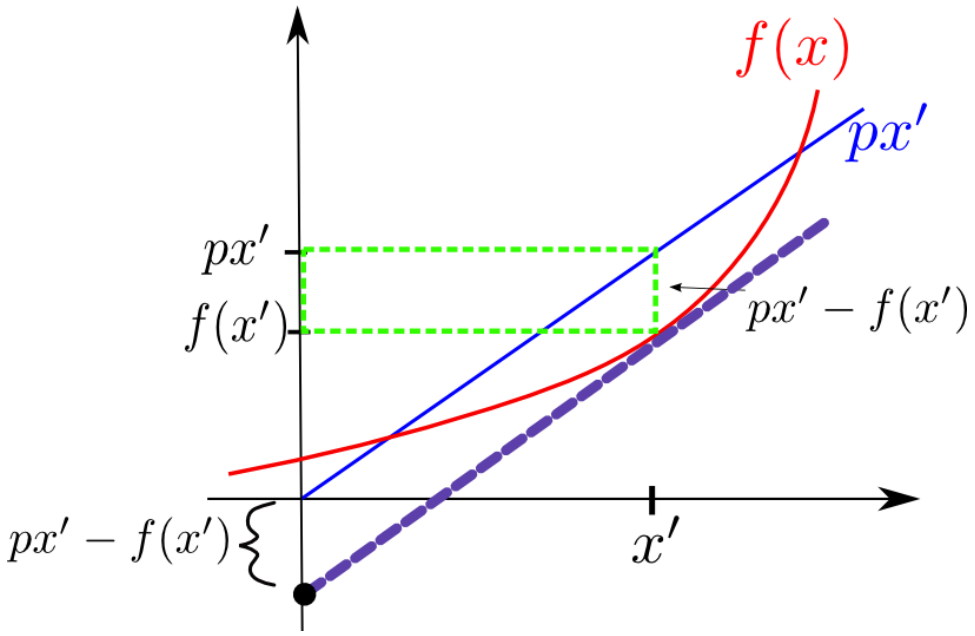

Figure 9: A figure that helps visualize the Legendre transform. The transform $g(p)$ of a convex function, $f(x)$ is $g(p) = \sup_x(xp - f(x))$. The function $F(x, p) = xp - f(x)$ is maximized at x' where $f'(x) = p$. The transform takes $(x, f(x))$ and gives (slope, intercept(slope)) using lines tangent to f. We define a new function $g(p)$ that is the negative of the y intercept of the tangent line to f that has slope p.

3.3 Hamilton's equations for a Newtonian system

The Hamiltonian for a Newtonian system is

$$H(p,q) = \frac{p^2}{2} + U(q) \tag{14}$$

Hamilton's equations give

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -U' = f(q)$$

where $f(q)$ is the force associated with the potential $U(q)$. Taking the time derivative of $\dot{q}$ we find

$$\ddot{q} = f(q)$$

which is Newton's equation.

Both Hamiltonian and Lagrangian viewpoints give Newton's equations. In the next section we will show how a Lagrangian can be turned into a Hamiltonian and how Lagrange's equations are consistent with Hamilton's equations.

3.4 Legendre Transforms

A Lagrangian system can be converted into a Hamiltonian system using a Legendre transform.

A Legendre transform transforms functions on a vector space to functions on a dual space. It transforms functions of a coordinate x to functions that are dependent on the slope of $f(x)$. Let $y = f(x)$ and look at Figure 9. We specify that $f(x)$ is convex so that $f''(x) > 0$ everywhere. We chose a value p for a slope and draw the line px. We look for the point where $px - f(x)$ is a maximum. Let us define a function $F(p,x) = px - f(x)$ that gives this distance. The point where $F(p,x)$ is a maximum is the point where the vertical distance between $f(x)$ and px is the largest. The condition that $F(p,x)$ is an extremum at x' means that $\frac{\partial F}{\partial x}\big|_{x'} = 0$. And this implies that $f'(x') = p$. The point x' is special because $f'(x') = p$ so p is the slope of $f(x)$ at x'. On the figure the special point is labelled as x'. For any p this point, x' is unique as $f(x)$ is convex. We define a new function $g(p) = F(p,x') = px' - f(x')$. This function gives the negative of the y intercept of the tangent line to f that has slope p. The Legendre transform is $f(x) \to g(p)$. Starting with $(x, f(x))$ the Legendre transform gives a new function $(p, g(p))$ where p is a slope and $g(p)$ is the negative of the y intercept of the tangent to f that has slope p.

The Legendre transform can also be written

$$g(p) = \sup_x (xp - f(x))$$

The Legendre transform of the Legendre transform gives back the original function (except for a constant). Figure 10 shows this with the points on the y axis representing the legendre transform. The function $G(p, x) = px - g(p)$ is minimized where the lines are tangent to $f(x)$.

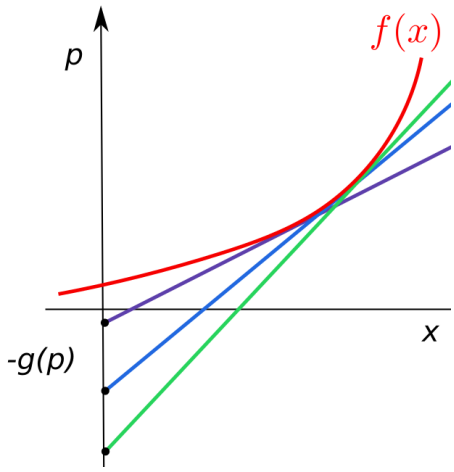

Figure 10: A figure that helps visualize the inverse Legendre transform. The y-intercept is $-g(p)$. The function $G(p, x) = px - g(p)$ is an extremum at a slope p' where $G(p', x) = f(x)$. We can write $f(x) = \sup_p px - g(p)$.

3.4.1 Example: Legendre transform of sqrt(x)

Let's take the Legendre transform of $f(x) = x^{1/2}$. Construct

$$F(x, p) = px - f(x) = px - \sqrt{x}$$

We look for the extremum of this function w.r.t. x

$$\frac{\partial F(x,p)}{\partial x} = p - f'(x) = p - \frac{1}{2}x^{-1/2}$$

Set this to zero. The extremum is at

$$p(x') = \frac{1}{2\sqrt{x'}} \qquad \text{or} \qquad x'(p) = \frac{1}{4p^2}$$

Insert this back into F

$$g(p) = F(x'(p), p) = \frac{p}{4p^2} - \frac{1}{2p} = -\frac{1}{4p}$$

So the Legendre transform is

$$g(p) = -\frac{1}{4p}$$

3.5 Constructing a Hamiltonian from a Lagrangian

Recall our Lagrangian is a function of $\dot{q}, q, t$ or $\mathcal{L}(q, \dot{q}, t)$. Let's for the moment ignore the q, t degrees of freedom so $\mathcal{L}(\dot{q})$. Instead of using $\dot{q}$ as a variable let us use slope $p = \frac{\partial \mathcal{L}(\dot{q})}{\partial \dot{q}}$ as our variable. The Legendre transform gives us the negative of the y intercept of the tangent line with slope p. This is

$$H(p) = \sup_{q'} \left[p\dot{q}' - \mathcal{L}(\dot{q}') \right]$$

with $\dot{q}'$ the special value of $\dot{q}$ where $p = \frac{\partial \mathcal{L}(\dot{q})}{\partial \dot{q}}$. The special $\dot{q}'$ is a function of p.

Now recalling that $\mathcal{L}$ was also a function of q, we can write

$$H(p, q, t) = p\dot{q}(p) - \mathcal{L}(q, \dot{q}(p), t) \tag{15}$$

with the condition for $\dot{q}(p)$

$$p = \frac{\partial \mathcal{L}(q, \dot{q}(p), t)}{\partial \dot{q}}$$

Let's take the partial derivative of equation 15

$$\frac{\partial H}{\partial p} = \dot{q}(p) + p\frac{d\dot{q}(p)}{dp} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}(p)}{dp}$$

The right two terms cancel giving

$$\frac{\partial H}{\partial p} = \dot{q}$$

one of Hamilton's equations.

To find the second of Hamilton's equations we need to use Lagrange's equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

Our definition of H implies that $\frac{\partial H}{\partial q} = -\frac{\partial \mathcal{L}}{\partial q}$, hence Lagrange's equation gives

$$\dot{p} = -\frac{\partial H}{\partial q}$$

and we have recovered both of Hamilton's equations.

3.5.1 Example: Hamiltonian for a Newtonian system in one dimension

For a Lagrangian

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2}\dot{q}^2 - U(q)$$

The Legendre transformation is with respect to the variable $\dot{q}$. For the moment we can ignore q. We consider the function

$$F(\dot{q}, p) = p\dot{q} - \mathcal{L}(\dot{q}) = p\dot{q} - \frac{1}{2}\dot{q}^2 + U$$

The extremum of this function

$$\frac{\partial F}{\partial \dot{q}} = p - \dot{q} = 0$$

occurs for a momentum

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

We then insert this particular value of $\dot{q}(p)$ back into the function F (which we now call H)

$$H(p, q) = p\dot{q}(p) - \mathcal{L}(q, \dot{q}(p), t) = \frac{1}{2}p^2 + U(q)$$

3.6 Geometrical view of a Lagrangian path integral

Consider the integral giving distance along a path that is between points q_1, q_2

$$\int_{q_1}^{q_2} ds$$

where ds gives an infinitesimal path length. For example path lengths in cylindrical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

giving metric elements $g_{rr} = g_{zz} = 1, g_{\theta\theta} = r^2$ and off-diagonal elements that are zero. On the surface of a sphere of radius R

$$ds^2 = R^2 d\phi^2 + R^2 \sin^2\phi d\theta^2$$

If we have a metric $g(q)$ then we can write

$$ds^2 = g_{\mu\nu} dq^\mu dq^\nu$$

where I have used summation notation. We can rewrite this

$$ds = \sqrt{g_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt}} dt$$

Integrating

$$\int_{q_1}^{q_2} ds = \int_{t_1}^{t_2} \sqrt{g_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt}} dt$$

If we minimize the path length then we also minimize a Lagrangian system with

$$L(q, \dot{q}, t) = \sqrt{g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu} \tag{16}$$

Geodesics are then the same as dynamical paths. If we start with a metric we can find a Lagrangian system. The inverse is not so easy. Given a Lagrangian system it may not be straightforward to write the problem in terms of a metric.

Apply Lagrange's equation to 16 one can derive geodesic equations and when re-written using Christofel systems (that depend on derivatives of the metric), this is the geodesic equation seen/used in general relativity.

3.7 Pathological Legendre Transformations

Can it be impossible to make the transformation from a Lagrangian system to a Hamiltonian system? The Legendre transformation of a function can only be done uniquely if the function is convex. A twice differentiable function is convex if its second derivative is always positive.

For example consider a non-convex function that has the same slope at two different points. In this case $p = f'(x)$ is not uniquely inverted! There would be two x values giving the same slope. Another example is $H(I, \theta) = I\omega$. This is a line and so not convex. Oddly I find that $H(I, \theta) = I + aI^2$ is also badly behaved and gives a zero Lagrangian everywhere. So one can find coordinate systems in which the transformation between Hamiltonian and Lagrangian presciptions is not possible.

Are there examples of Lagrangians and Hamiltonians for physics systems that are not convex? Al Shapere's (and Wilczek's) time crystals are an entertaining example of this, see *Classical Time Crystals*, Shapere, A. Wilczek F. 2012, Phys. Rev. Lett. 109, 160402

Consider

$$\mathcal{L}(q, \dot{q}) = \frac{\dot{q}^4}{4} - \frac{\dot{q}^2}{2} - V(q) \tag{17}$$

(see Figure 11) To create a Hamiltonian we need momentum that satisfies

$$p(q_*) = \frac{\partial \mathcal{L}(\dot{q}_*)}{\partial \dot{q}}$$

Inverting this $q_*(p)$, giving us a Hamiltonian

$$H(p, q) = p\dot{q}_*(p) - \mathcal{L}(q, \dot{q}_*(p))$$

For the Lagrangian in equation 17

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}^3 - \dot{q} \tag{18}$$

Giving

$$p = \dot{q}_*^3 - \dot{q}_* \tag{19}$$

It may be useful to compute

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} = 3\dot{q}^2 - 1$$

Equation 19 is not uniquely invertible for $q \in [-1/\sqrt{3}, 1/\sqrt{3}]$.

The Hamiltonian written in terms of $\dot{q}_*$ is

$$H(p,q) = (\dot{q}_*^3 - \dot{q}_*)\dot{q}_* - \frac{\dot{q}_*^4}{4} + \frac{\dot{q}_*^2}{2} + V(q) = \frac{3}{4}\dot{q}_*^4 - \frac{\dot{q}_*^2}{2} + V(q)$$

The Hamiltonian (energy) is minimized at $\dot{q}_* = 1/\sqrt{3}$ or at $\dot{q}_* = 0$. But the non-zero value is precisely that value where the second derivative of the Lagrangian is zero (and so the function not convex) and a position where p is nearly independent of $\dot{q}$ and so $p(q_*)$ not invertible.

Let us look at the equations of motion.

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} &= \frac{\partial \mathcal{L}}{\partial q} \\ \ddot{q}\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} &= \frac{\partial V}{\partial q} \\ \ddot{q}(3\dot{q}^2 - 1) &= \frac{\partial V}{\partial q} \end{aligned}$$

We find that we cannot solve for $\ddot{q}$ at the same location that minimizes the energy. So breakdown of the equations of motion occurs at the same location as time reversal symmetry is broken (and the same location as our Legendre transformation fails). Life gets more interesting in the quantum setting, and perhaps there are ways around these problems making it possible to break time-reversal symmetry in the ground state.

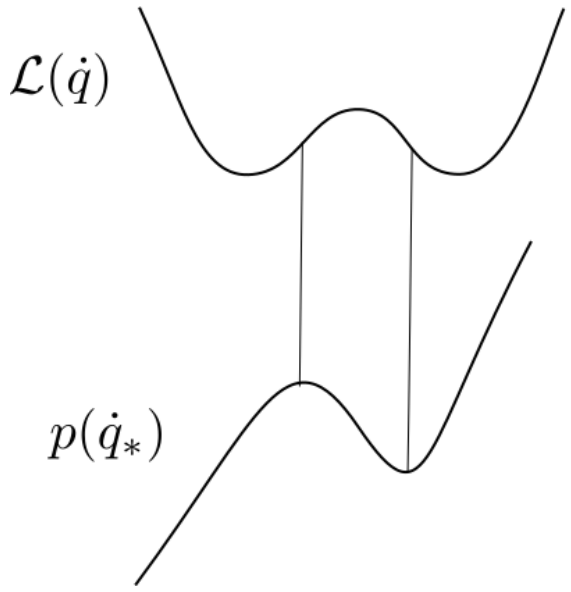

Figure 11: A non convex Lagrangian. The momentum p is not a unique function of $\dot{q}$ giving a multivalued Hamiltonian.

3.8 Phase Space Action

Consider the action

$$S(p,q) = \int (p\dot{q} - H(p,q))dt$$

Variation of S with respect to q gives

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{q}}(p\dot{q} - H) &= \frac{\partial}{\partial q}(p\dot{q} - H) \\ \frac{d}{dt}p &= -\frac{\partial H}{\partial q} \end{aligned}$$

Variation of S with respect to p gives

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{p}}(p\dot{q} - H) &= \frac{\partial}{\partial p}(p\dot{q} - H) \\ 0 &= \dot{q} - \frac{\partial H}{\partial p} \end{aligned}$$

giving Hamilton's equations. The phase-space action yields Hamilton's equations as the extremal condition.

4 Stability of fixed points

The Hamiltonian is a recipe for computing trajectories. We search for points that don't move. These would have $\dot{p} = \dot{q} = 0$ and we can call them *fixed points* and if they are stable fixed points then we refer to them as *equilibrium points*. For a Hamiltonian system in one degree of freedom (2d phase space), fixed points have

$$\dot{p} = 0 \qquad \dot{q} = 0$$

The location (p,q) of fixed points can be found by solving Hamilton's equations equal to zero

$$\frac{\partial H(p,q)}{\partial p} = \frac{\partial H(p,q)}{\partial q} = 0$$

Suppose we have (p_0, q_0) that is a fixed point. We can consider the equations of motion near this point. What are the trajectories like near the fixed point. If the trajectories whisk a particle far away then we can refer to the fixed point as *unstable*. If trajectories remain nearby for all time then the fixed point is *stable*. The fixed point is also an *equilibrium* point. Consider the symmetric matrix

$$\mathbf{M} = \begin{pmatrix} \frac{\partial^2 H}{\partial q^2} & \frac{\partial^2 H}{\partial p \partial q} \\ \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial p^2} \end{pmatrix} \tag{20}$$

evaluated at p_0, q_0. If we expand the Hamiltonian near the fixed point $q = q_0 + x$, $p = p_0 + y$

$$H(x,y) = H(p_0,q_0) + \frac{\partial^2 H(p_0,q_0)}{\partial q^2}\frac{x^2}{2} + \frac{\partial^2 H(p_0,q_0)}{\partial p^2}\frac{y^2}{2} + \frac{\partial^2 H(p_0,q_0)}{\partial p \partial q}xy \tag{21}$$

or

$$H(x,y) = \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where we have dropped the constant term. Our Hamiltonian system can also be written

$$H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^t\mathbf{M}\mathbf{x} \tag{22}$$

where we have defined a vector $\mathbf{x} = (x, y)$ and $\mathbf{x}^t$ is the transpose of $\mathbf{x}$. Or in terms of a dot product

$$H(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}|\mathbf{M}\mathbf{x}\rangle$$

We can also write this in summation notation

$$H(\mathbf{x}) = \frac{1}{2}x_i M_{ij} x_j$$

where repeated indices are summed. Taking the gradient

$$\begin{aligned}
\frac{\partial H}{\partial x_k} &= \frac{\partial}{\partial x_k}\sum_{ij}\frac{1}{2}x_i M_{ij} x_j \\
&= \frac{1}{2}\sum_{jk}\left[\delta_{ik}M_{ij}x_j + \delta_{jk}x_i M_{ij}\right] \\
&= \frac{1}{2}\left[\sum_j M_{kj}x_j + \sum_i M_{ik}x_i\right] \\
&= \frac{1}{2}\left[\sum_j M_{kj}x_j + \sum_i M_{ki}x_i\right] \\
&= \frac{1}{2}\left[\sum_j M_{kj}x_j + \sum_j M_{kj}x_i\right] \\
&= \sum_j M_{kj}x_j
\end{aligned} \tag{23}$$

In vector notation

$$\boldsymbol{\nabla} H = \mathbf{M}\mathbf{x}$$

The matrix is symmetric and that means it has two real eigenvalues. Expanding near the fixed point preserves Hamilton's equations with x acting as a coordinate and y acting as a momentum. So the coordinate transformation to (x, y) is said to be *canonical*.

We can diagonalize the matrix. In a rotated coordinate system we can write the Hamiltonian as

$$H(p,q) = \frac{1}{2}\left(\lambda_1 p^2 + \lambda_2 q^2\right) \tag{24}$$

where λ_1, λ_2 are the eigenvalues. If both eigenvalues are positive or both eigenvalues are negative then the system behaves like a harmonic oscillator and has librations about the fixed point with frequency

$$\omega = \sqrt{\lambda_1 \lambda_2}$$

The fixed point is stable. If the product of the eigenvalues is negative then evolution is exponential with trajectories diverging exponentially along one eigenvector and heading toward the fixed point along the other one. The fixed point is unstable.

Remark Taking into account the units in equation 24 There is only one way to make a frequency from the eigenvalues λ_1, λ_2. This corresponds to the frequency of oscillation about a stable fixed point. For an unstable fixed point, this frequency corresponds to that for exponential behavior near the fixed point.

Remark The harmonic oscillator (figure 1) contains a single stable fixed point at the origin. The pendulum has examples of both types of fixed points (see Figure 2). The unstable fixed point are at $x = \pm\pi$ and the stable ones at $x = 0, 2\pi$.

Question: Why do we ignore the possibility that one of the eigenvalues is zero?

Let us take another look at equation 22. Using Hamilton's equations in the form

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \nabla H$$

with

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we find

$$\dot{\mathbf{x}} = \boldsymbol{\omega}\mathbf{M}\mathbf{x}$$

$$\ddot{\mathbf{x}} = \boldsymbol{\omega}\mathbf{M}\boldsymbol{\omega}\mathbf{M}\mathbf{x}$$

For the 2d system the matrix

$$(\boldsymbol{\omega}\mathbf{M})^2 = -(\det \boldsymbol{M})\mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. If the determinant is positive (two positive or two negative eigenvalues) then the fixed point is stable and the frequency of oscillation is

$$\nu = \sqrt{\det \boldsymbol{M}} = \sqrt{\lambda_1 \lambda_2}$$

If the determinant is negative then solutions are exponentially dependent on time with timescale

$$t_{exp} = \frac{1}{\sqrt{|\det \boldsymbol{M}|}}$$

4.1 Lyapunov and asymptotic stability

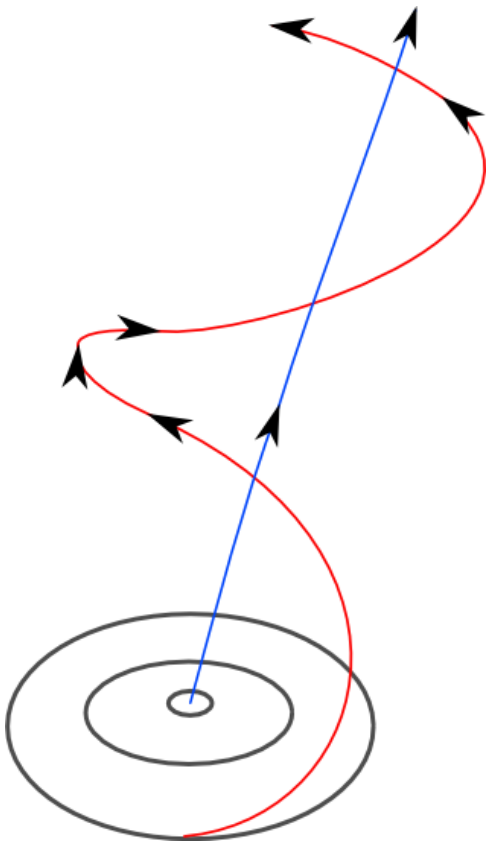

Figure 12: A stable fixed point. The level curves of a 1dimensional harmonic oscillator are shown at the bottom. Trajectories are shown with time in the third dimension. The fixed point is shown with a blue trajectory. Nearby orbits always remain in the vicinity of the stable fixed point.

Stability can be defined more rigorously in the following way. Suppose $\mathbf{x}_f$ is our fixed point and $\phi(t, \mathbf{x}_0)$ a trajectory as a function of time with initial condition $\mathbf{x}_0$. A fixed point

is stable (Lyapunov stable) if for every $\epsilon > 0$ there is a δ such that

$$||\mathbf{x}_f - \phi(t, \mathbf{x}_0)|| < \epsilon$$

for all t and $\mathbf{x_0}$ that satisfy

$$||\mathbf{x}_f - \mathbf{x}_0|| < \delta$$

This means that trajectories never get far away from the fixed point ever, either in future or past. An example of trajectories near a stable fixed point is shown in Figure 12.

A fixed point is asymptotically stable if in addition to being Lyapunov stable, all trajectories converge to that point. Hamiltonian systems do not contain asymptotically stable fixed points because volume in phase space is conserved. However, dissipative systems can contain asymptotically stable fixed points. These are often described as *attracting*.

5 Dimensional rescaling for Hamiltonians

Consider kinetic and potential energy per unit mass. This is appropriate for a low mass particle in orbit about the Sun for example. In this setting we can consider Energy per unit mass. Hence our Hamiltonian would have units of energy per unit mass or velocity^2. Since we are ignoring mass we can consider momentum as having units of velocity.

Consider Hamilton's equations for $H(p, q, t)$

$$\frac{\partial H}{\partial p} = \dot{q} \tag{25}$$

$$\frac{\partial H}{\partial q} = -\dot{p} \tag{26}$$

For a free particle $H = \frac{p^2}{2}$. If I multiply H by a constant $H = \frac{ap^2}{2}$ I would find $\dot{q} = ap$ instead of $\dot{q} = p$. But if I replace dt with $d\tau/a$ I can recover $\dot{q} = ap$. Essentially multiplying the Hamiltonian by a constant a is equivalent to rescaling time by a factor $1/a$. Let's do this more formally. Consider the transformation $\frac{d\tau}{dt} = a$.

$$\dot{q} = \frac{dq}{dt} = \frac{dq}{d\tau}\frac{d\tau}{dt} = a\frac{dq}{d\tau}$$

$$\dot{p} = a\frac{dp}{d\tau}$$

In a new coordinate system but using τ as time instead of t our new Hamiltonian $H(p, q, \tau) = H(p, q, t)a$. Momentum and coordinates are not changed with this rescaling.

A numerical technique known as *regularization* exploits the fact that multiplying a Hamiltonian by time is like rescaling time. Instead of multiply by a constant, a Hamiltonian is multiplied by a function that might depend on coordinates. A new Hamiltonian system

is created with different units of time. For example it is desirable to have short timesteps near a large point mass and large timesteps distant from a point mass. Regularized systems can be used to create more accurate integrators for high eccentricity stars near black holes, for example.

Regard Hamilton's equations again. If I replace q by aQ and p by P/a then Hamilton's equations are unchanged. Rescaling momentum and inversely scaling coordinates by a constant preserves Hamilton's equations which involve derivatives with respect to both p, q. This is counter intuitive as one would think that rescaling length would affect both p, q by the same amount. However if we consider phase space volume as being something that is invariant then it makes more sense.

6 Liouville's Theorem and Phase wrapping

Hamilton's equations treat coordinates and momenta differently. However we can consider the dynamics of the system with a 2N vector $\mathbf{x} = (q_1, q_2....q_n, p_1, p_2...p_n)$ where N is the dimension of $\mathbf{p}$ and $\mathbf{q}$. Our Hamiltonian is $H(\mathbf{x})$. Hamilton's equations involve partial derivatives of H or the gradient operator

$$\boldsymbol{\nabla} H = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2} ... \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, ... \frac{\partial H}{\partial p_n} \right)$$

Using this notation Hamilton's equation can be written as

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \boldsymbol{\nabla} H \tag{27}$$

where the matrix

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \tag{28}$$

has dimension 2Nx2N, and $\mathbf{I}$ is an identity matrix of dimension N. The matrix $\boldsymbol{\omega}$ is antisymmetric so $\omega_{ij} = -\omega_{ji}$. The equations of motion (or Hamilton's equations) in summation notation are

$$\dot{x}_i = \omega_{ij} H_{,j}$$

where shorthand $H_{,j} = \frac{\partial H}{\partial x_j}$.

Consider the divergence of the flow

$$\boldsymbol{\nabla} \cdot \dot{\mathbf{x}} = \partial_i \dot{x}_i = \partial_i \omega_{ij} H_{,j} = \omega_{ij} H_{,ij}$$

where the right hand side I have used summation notation. The antisymmetry of $\boldsymbol{\omega}$ implies that

$$\boldsymbol{\nabla} \cdot \dot{\mathbf{x}} = 0$$

Let us show this when we have 1 degree of freedom and a 2-dimensional phase space

$$\dot{\mathbf{x}} = (\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right)$$

$$\boldsymbol{\nabla} \cdot \dot{\mathbf{x}} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial^2 H}{\partial p \partial q} - \frac{\partial^2 H}{\partial q \partial p} = 0$$

When $\boldsymbol{\nabla} \cdot \dot{\mathbf{x}} = 0$, the flow is *incompressible* in phase space, and we have proved Liouville's theorem.

6.1 The meaning of incompressible flow

Let us expand on the connection between being divergence free and being incompressible. Consider a distribution of particles in phase space. The number of particles per unit volume would be $\rho(\mathbf{p}, \mathbf{q})$ or $\rho(\mathbf{x})$. This the number of particles at position $\mathbf{x}$ in a region with volume $dV = dx^{2N} = dp^N dq^N$. We can consider the flux $\rho \mathbf{v}$ of particles out of each surface of a box that has volume dV. Integrating the flux over all surfaces of a box gives us the number of particles leaving the box per unit time.

$$\frac{dN}{dt} = \int_S \rho \mathbf{v} d\mathbf{A}$$

over the surface, S, of the box. Here the velocity $\mathbf{v} = \dot{\mathbf{x}}$. Using Gaus' law we can rewrite this as

$$\frac{dN}{dt} = \int_V \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) dV$$

If particles are leaving the box then the number density in the box must decrease.

$$\frac{dN}{dt} = -\int \frac{\partial \rho}{\partial t} dV$$

We find a conservation law

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

which can be recognized as conservation of mass for a fluid flow. Consider a bowling ball moving through water. As the bowling ball moves past an observer, it would appear that the density at some location is changing. However the bowling ball is not being compressed. The problem is that the above equation is written respect to a fixed coordinate system and we need to think about the density in a volume element that is moving with the fluid.

We can rewrite this equation as

$$\begin{aligned} \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \rho &= -\rho \boldsymbol{\nabla} \cdot \mathbf{v} \\ \frac{D\rho}{Dt} &= -\rho \boldsymbol{\nabla} \cdot \mathbf{v} \end{aligned} \tag{29}$$

where I have used

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}$$

is the advective derivative or that moving with the fluid. Consider $\rho(\mathbf{x}, t)$ as it varies

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial x}\dot{x} + \frac{\partial \rho}{\partial y}\dot{y} + \frac{\partial \rho}{\partial z}\dot{z} + \frac{\partial \rho}{\partial t} \\ &= \mathbf{v} \cdot \boldsymbol{\nabla}\rho + \frac{\partial \rho}{\partial t} \end{aligned}$$

The equation 29 implies that when $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ then $\frac{D\rho}{dt} = 0$ and the density of a distribution of particles remains fixed even as the distribution of particles is deformed.

6.2 Phase wrapping

If we start with a bunch of evenly distributed particles and look at the area covered by then for a 2D phase space (1 degree of freedom Hamiltonian) then the area covered by these particles remains constant.

For a harmonic oscillator the period of a orbit is independent of the amplitude. The phase flow could cause a distribution of particles originally located in a small region of phase space distant from the origin to rotate about the origin. The area and shape of the distribution would remain content but the central value of $x, \dot{x}$ for the distribution would move along the orbit (see Figure 13). The harmonic oscillator $H(p, q) = \frac{1}{2}(p^2 + \omega^2 q^2)$ can also be written

$$H(I, \phi) = I\omega$$

which has $\dot{\phi} = \omega$ independent of amplitude or I. On the level curve plot I sets the distance from the origin.

However in many oscillating systems the period of the orbit depends on the amplitude. This is true for the pendulum and even at moderate amplitude for the anharmonic oscillator with a quartic potential $U(x) = x^2 + ax^4$. The area enclosed by the distribution remains constant as the flow is divergence-free. A distribution of particles originally in a small region in phase space is sheered out because particles have different amplitudes and so different oscillation frequencies. This kind of system can be described with a Hamiltonian

$$H(I, \phi) = I\omega + aI^2$$

giving $\dot{\phi} = \omega + aI$. The period depends on the distance from the origin in Figure 14a or the I momentum. See Figure 14 for an illustration. This phenomena is called *phase wrapping* because particles with different amplitudes diverge in orbital *phase*.

Remark Phase wrapping is a nice interpretation of shell like features sometimes seen in the outskirts of elliptical galaxies (see Figure 15).

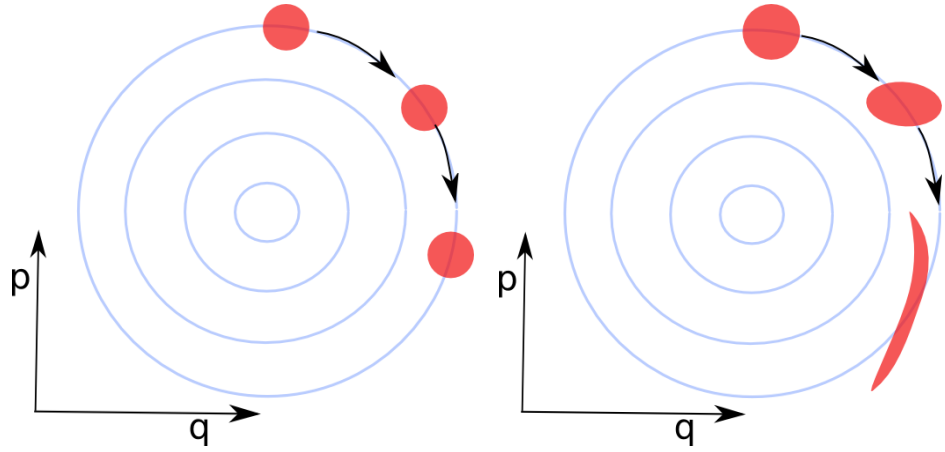

Figure 13: The distribution of particles at three different times are shown as red regions. On left: the flow of a harmonic oscillator $H(q,p) = \frac{1}{2}(p^2+q^2)$. There is no phase wrapping because the period is independent of amplitude. On right: for an anharmonic oscillator where the period depends on the amplitude. Here the amplitude of oscillation is dependent on the radius from the origin.

Remark Liouville's theorem implies that Hamiltonian phase flows (for 2d phase space) are area preserving.

Remark No where did we assume that H was time independent.

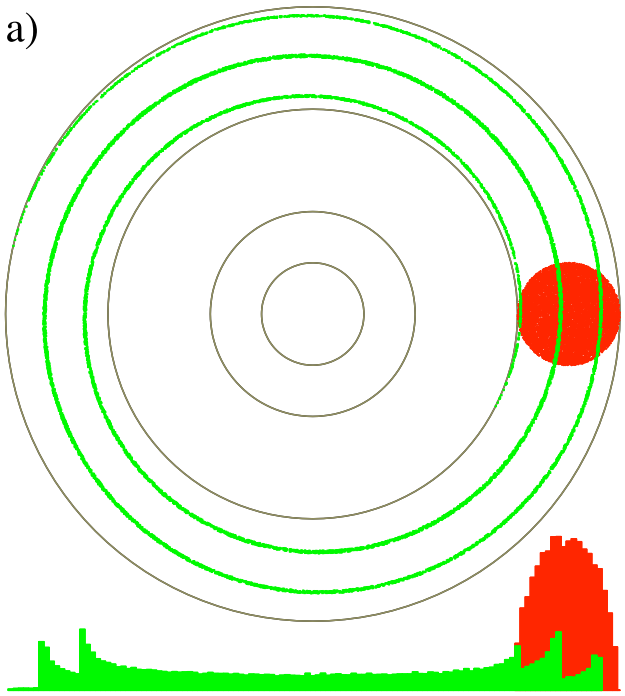

Figure 14: Particles are initially located in the red region but after a period of time are distributed in the green region. Level curves for the dynamical system are circles but the period of oscillation depends on amplitude. The particle distribution is projected downward on the bottom, showing a series of caustics. The process is sometimes described as phase wrapping as the differences in phases of the particles becomes larger with time. The projection is what might be seen in coordinate space. The peaks correspond to extreme positions of radius where the particles spend more time. For example, stars in Keplerian orbit spend more time at apocenter than at pericenter.

Figure 15: Image of a galaxy showing stellar shells. In the past a small galaxy was tidally disrupted by a larger one. Stars in the small galaxy were originally distributed in a small region in phase space. Later on stars are primarily seen at apocenters of their orbits and so show a shell-like distribution in the background larger galaxy. This image is from http://apod.nasa.gov/apod/ap110226.html

7 Some non-conservative two-dimensional dynamical systems

So far we have only looked at Hamitonian (conservative systems). What happens if we add a little bit of damping? The system is not longer area preserving.

7.1 Example: Damped pendulum

Consider a damped pendulum model

$$\ddot{\phi} + \alpha\dot{\phi} + \sin\phi = 0 \tag{30}$$

with $\alpha > 0$, the damping coefficient. We multiply the equation by $\dot{\phi}$

$$\ddot{\phi}\dot{\phi} + \alpha\dot{\phi}^2 + \dot{\phi}\sin\phi = 0$$

$$\frac{d}{dt}\left[\frac{\dot{\phi}^2}{2} - \cos\phi\right] = -\alpha\dot{\phi}^2$$

The left term we notice is the energy of the undamped pendulum.

$$\frac{dE}{dt} = -\alpha\dot{\phi}^2$$

This equation holds if we replace $\cos\phi$ with a general potential $U(\phi)$. As long as $\alpha > 0$ we find that energy is always decreasing. The fixed points of the problem are at same location as for the pendulum ($\dot{\phi} = 0, \phi = 0, \pi$) and their stability is also the same as for the pendulum model. This (and thinking about the energy) makes it straightforward to figure out which direction trajectories go. To help with the phase portrait, we can describe this as a two-dimensional dynamical system with $\mathbf{x} = (\phi, y)$

$$\begin{aligned} \dot{\phi} &= y \\ \dot{y} &= -\sin\phi - \alpha y \end{aligned} \tag{31}$$

or

$$\begin{pmatrix} \dot{\phi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(y,\phi) \\ g(y,\phi) \end{pmatrix} \tag{32}$$

with $f(y,\phi) = y$ and $g(y,\phi) = -\sin\phi - \alpha y$.

How do we know that this system does not preserve area (is compressible)? Using $\mathbf{x} = (\phi, y)$ we can compute $\boldsymbol{\nabla}\cdot\dot{\mathbf{x}}$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\dot{\mathbf{x}} &= \frac{\partial\dot{\phi}}{\partial\phi} + \frac{\partial\dot{y}}{\partial y} \\ &= \frac{\partial f}{\partial\phi} + \frac{\partial g}{\partial y} \\ &= \frac{\partial}{\partial\phi}y + \frac{\partial}{\partial y}(-\sin\phi - \alpha y) \\ &= -\alpha y \end{aligned}$$

This you will recognize is the same as dE/dt the rate that energy is lost. For $\alpha \neq 0$ the system is not incompressible. Phase space volume is *not* conserved. Our map is *not* an area preserving map.

7.1.1 Fixed Points

Consider the Jacobian

$$A = \begin{pmatrix} \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial \phi} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\phi & -\alpha \end{pmatrix}$$

The Jacobian is useful as near the fixed point we can approximate the dynamics as a linear system with

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where $\mathbf{x} = (y - y_*, \phi - \phi_*)$ and y_*, ϕ_* are the coordinates of the fixed point. For a linear system a general solution is

$$\mathbf{x} = c_1 e^{\lambda_1 t}\hat{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\hat{\mathbf{v}}_2$$

where λ_1, λ_2 are eigenvalues of A, $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A and c_1, c_2 are constants.

Eigenvalues of the Jacobian matrix at the fixed points can be used to determine the stability of the fixed points. Recall that the trace gives the sum of the eigenvalues and the determinant the product of the two eigenvalues. The characteristic equation can be written

$$\det A - (\mathrm{tr}A)\lambda + \lambda^2 = 0$$

has roots

$$\lambda_\pm = \frac{1}{2}\left[\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right]$$

Eigenvalues are both complex (leading to rotation about the fixed point) when

$$(\mathrm{tr}A)^2 - 4\det A < 0$$

Fixed point is a saddle point if both eigenvalues are real and one is positive and the other negative.

For our damped pendulum $\mathrm{tr}A = -\alpha$ and $\det A = \cos\phi$ which is ± 1 depend on the fixed point.

$$\lambda_\pm = \frac{1}{2}\left[-\alpha \pm \sqrt{\alpha^2 - 4\cos\phi}\right]$$

At the fixed point with $\cos\phi = 1$ ($\phi = 0$) we have circulation as long as $\alpha < 2$, and in this case both eigenvalues are complex. With $\alpha > 0$ we expect orbits to spiral in (*stable spiral*). If α is too large ($\alpha > 2$) then we have two negative eigenvalues, orbits just exponentially drop in without any rotation and the point is just a *stable node*.

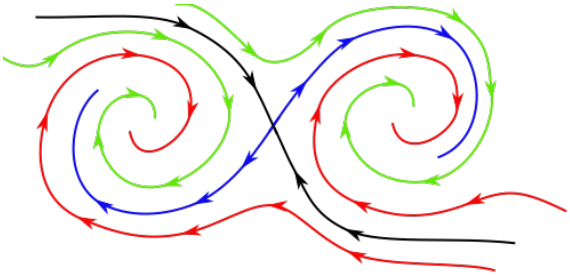

Figure 16: Trajectories for a damped pendulum (Equation 30). All trajectories eventually approach a fixed point.

At the other fixed point (with $\cos\phi = -1$, $\phi = \pi$), both eigenvalues are real and one is always positive, the other negative and so it is always a *saddle*.

There are two fixed points, one is always a *saddle* and the other a *sink* (see Figure 16).

7.1.2 Appearance of a limit cycle

A slightly more complicated model has the damped pendulum

$$\ddot{\phi} + \alpha\dot{\phi} + \sin\phi = I \tag{33}$$

with constant I, corresponding to a driven torque. The equations of motion on the plane

$$\begin{aligned} \dot{\phi} &= y \\ \dot{y} &= I - \sin\phi - \alpha y \end{aligned} \tag{34}$$

Fixed points are at $y = 0$, $\sin\phi = I$, so they only exist when $|I| < 1$. The Jacobian is the same as when $I = 0$. $\mathrm{tr}A = -\alpha$ and $\det A = \cos\phi = \pm\sqrt{1 - I^2}$ at the fixed points. We have a situation similar to above where we can have a stable node or a stable spiral for one fixed point where the other is a saddle node.

The fixed point occurs where $I = \sin\phi$ so if $I > \pm 1$ then there is no fixed point. However, there can be a stable cycle.

For $I > 1$ we can show that there is a stable cycle. We note that there is a function $y_s(\phi)$

$$y_s(\phi) = \frac{1}{\alpha}\left[I - \sin\phi\right] \tag{35}$$

where $\dot{y}$ is zero. Above this line on the cylinder, $\dot{y} < 0$ and below it $\dot{y} > 0$. This means that trajectories above and below the $y_s(\phi)$ curve approach this curve (see Figure 17). All trajectories have $\dot{\phi} > 0$ as $I > 1$ so all trajectories move to the right on the y, ϕ plane.

Take a point y, $\phi = 0$ and consider its trajectory when it reaches $\phi = 2\pi$. Consider the map $G(y)$ which takes the initial y value and returns the y value at $\phi = 2\pi$. A point y_a, $\phi = 0$ starting above the curve y_s must have $G(y_a) < y_a$. A point y_b, $\phi = 0$, starting below the curve y_s must have $G(y_b) > y_b$. At some point the function $G(y)$ must cross the line $h(y) = y$ and $G(y_*) = y_*$ and this orbit corresponds to a periodic orbit (see Figure 18). Since orbits above it move downwards and orbits below it move upwards, it's also a *limit cycle*.

Notes: in regime of high damping $\alpha > 1$ this system displays hysteresis. $\langle y \rangle$ is above zero when set by the cycle or equal to zero as set by the fixed point. The dynamical system is used to model a non-linear electronic circuit and Josephson junctions. Limit cycles are examples of phenomena that arise in non-conservative 2 dimensional dynamical systems and not in conservative ones. There are number of ways to prove existence and non-existence of cycles. Also relevant is the Poincaré Dixon theorem.

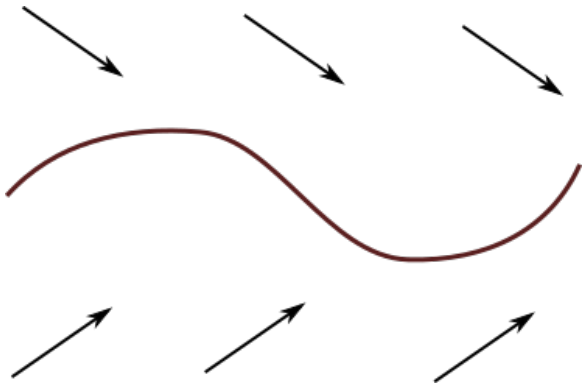

Figure 17: For $I > 1$ all trajectories approach $y_s(\phi)$ (equation 35) for the damped torqued pendulum.

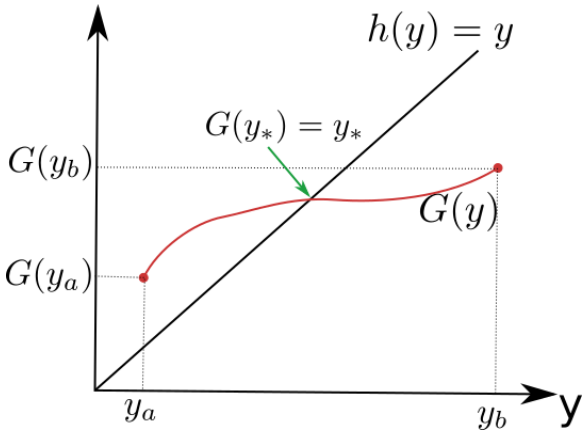

Figure 18: The existence of a periodic orbit for the damped torqued pendulum (equation 33).