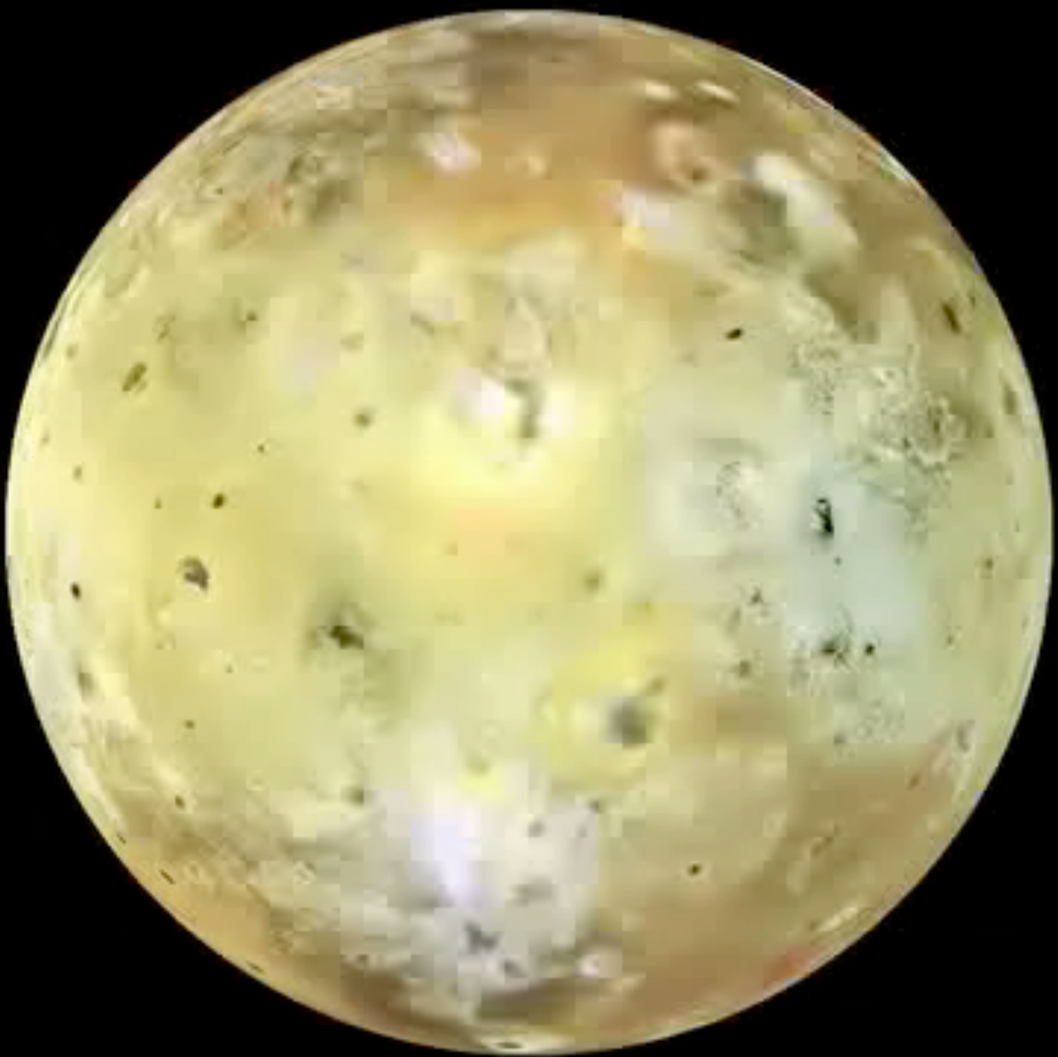


Tidal evolution

- Tidal deformation
- Rotational deformation
- Tidal torques
- Orbit decay and spin down rates
- Tidal circularization
- Hot Jupiters
- Precession
- Spin-Orbit resonance

Tidal and rotational deformations

- Give information on the interior of a body, so are of particular interest
- Tidal forces can also be a source of heat and cause orbital evolution
- Almost all regular satellites in the solar system are tidally evolved – almost all are tidally locked
- Exoplanets around M stars tend to be tidally locked
- Hot Jupiter exoplanets are tidally locked

Potentials of non-spherical bodies

- External to the planet the gravitational potential is zero
- Separate in r, ϕ, θ
- Can expand in spherical harmonics with general solution

$$\nabla^2 \Phi = 0$$

$$\Phi(r,\phi,\mu) = \sum_n \left(A_n r^n + B_n r^{-(n+1)}\right) S_n(\mu,\phi)$$

- Given axisymmetry the solutions are only a function of $\mu = \cos\theta$ and are Legendre Polynomials P_n

$$\begin{aligned} P_1(\mu) &= \mu \\ P_2(\mu) &= \frac{1}{2}(3\mu^2 - 1) \end{aligned}$$

colatitude: θ, is 0 at +z, is π along -z and is $\pi/2$ along equator

Potential External to a non-spherical body

- For a body with constant density γ body, mean radius C and that has an equatorial bulge

$$R(\theta) = C(1 + \epsilon_2 P_2(\cos(\theta))$$

- Has exterior gravitational potential

$$\Phi(r,\theta) = -\frac{4}{3}\pi C^3 \gamma G \left(\frac{1}{r} + \frac{3}{5}\frac{C^2}{r^3}\epsilon_2 P_2(\cos\theta) \right)$$

- And internal potential

$$\Phi(r,\theta) = -\frac{4}{3}\pi C^3 \gamma G \left[\frac{3C^2 - r^2}{2C^3} + \frac{3}{5}\frac{r^2}{C^3}\epsilon_2 P_2(\cos\theta) \right]$$

Potential external to a non-spherical body

- Quadrupole component only

$$\Phi_2 \sim -\frac{GMR^2}{r^3}\epsilon_2 P_2(\cos\theta)$$

- Form useful for considered effect of rotation and tidal forces on shape and external potential
- Match surface to tidal deformation potential or match surface to effective centrifugal potential for rotational deformation

Rotational deformation

- Define an effective potential which includes a centripetal term that can be written like this:

$$\Phi_{cf} = \frac{\Omega^2 r^2}{3}[P_2(\cos\theta) - 1]$$

factor of 3 from definition of P_2

- Flattening depends on ratio of centrifugal to gravitational acceleration

$$q = \frac{\Omega^2 R^3}{G m_p}$$

- Exterior to body $\Phi(r,\theta) = -\frac{GM}{r}\left[1 - \sum_n J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta)\right]$

- Taking J_2 term only and assuming isopotential surface

$$gh + J_2 \frac{G m_p}{R} + \frac{\Omega^2 R^2}{3} = 0$$

$$f = \frac{3}{2} J_2 + \frac{q}{2}$$

rewrite this in terms of difference between pole and equatorial radius, f and spin q parameter

Rotational deformation

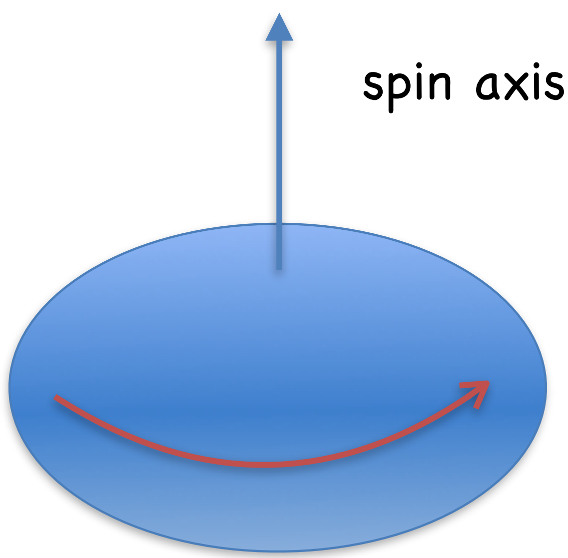
- It is also possible to relate J_2 to moments of inertia

$$J_2 \approx \frac{C - A}{Ma^2}$$

- Where C,A are moments of inertia and a is a long semi-axis of the body

oblateness

$$f = \frac{r_{equatorial} - r_{polar}}{r_{equatorial}}$$



Body becomes oblate (pancake) shaped due to centrifugal force from spin

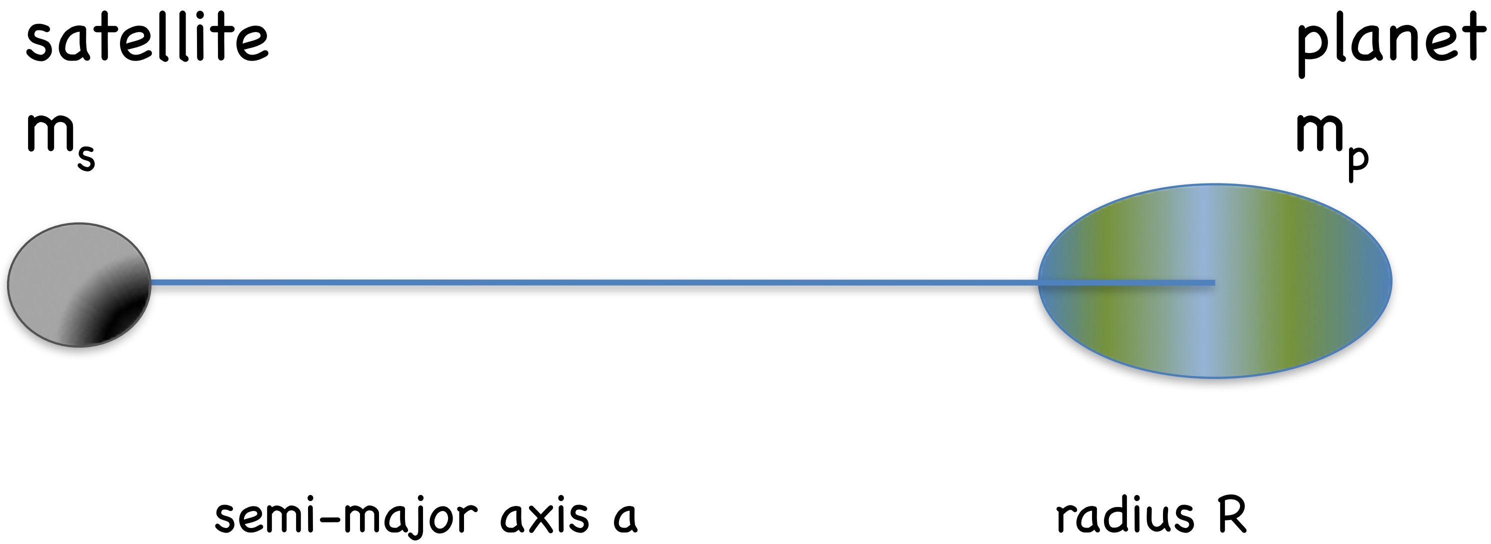
Rotational deformation

- Assume surface is a zero potential equilibrium state
- Balance J_2 term with centrifugal term to relate oblateness ($f=\delta r/r$) to moments of inertia (J_2) and rotation

$$f = \frac{3}{2}J_2 + \frac{q}{2}$$

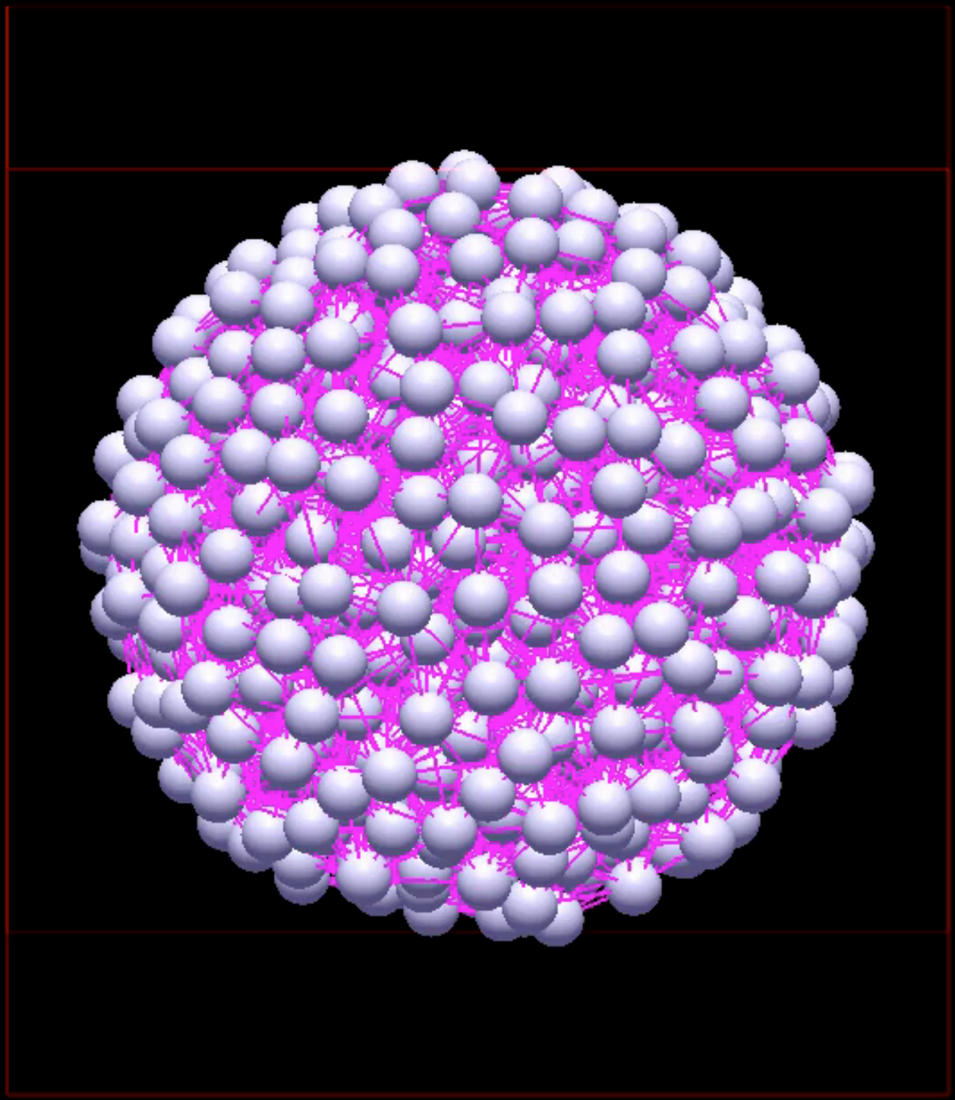
- Observables from a distance: oblateness f, spin parameter, q
- Resulting constraint on moments of inertia which gives information on the density distribution in the body
- J_2 can also be measured from a flyby or by observing precession of an orbit
- A variant of this formula is called the Darwin-Radau relation

Tidal Deformation



object is prolate rather than oblate
cigar rather than pancake

Mass Spring Model



Gravity N-body
+ inter-particle spring forces

github:
hannorein/rebound
aquillen/

Elastic modulus computed from number density, lengths and strengths of springs

Viscosity computed from spring damping rates

Tidal Deformation

- Tidal deformation by an external object
- In equilibrium body deforms into a shape along axis to perturber

$$R(\theta) = C(1 + \epsilon_2 P_2(\cos(\theta))$$

- Surface is approximately an equipotential surface
- That of body due to itself must balance that from external tide.
- Tidal potential V_T
- Body assumed to be in hydrostatic equilibrium
- Height x surface acceleration is balanced by external tidal perturbation

$$gh \sim V_T$$

$$h = \epsilon C$$

C = mean radius

$$g = \frac{Gm_p}{C^2}$$

surface acceleration

Tidal force due to an external body

- External body with mass m_s, and distance to planet of a
- Via expansion of potential the tidal force from satellite onto planet is

$$\Phi(\psi) = -G\frac{m_s}{a^3}R_p^2 P_2(\cos\psi)$$

- Where ψ is defined as along satellite planet axis
- If we assume constant density then this can be equated to potential term of a bulging body to find ε the size of the equatorial deformation from spherical
- More sophisticated can treat core and ocean separately each with own density and boundary and using the same expressions for interior and exterior potential

Planet's potential

equipotential surface h(ψ) and acceleration $g = \frac{GM_p}{R_p^2}$

$gh(\psi)$ acceleration times height above mean surface is balanced by tidal potential at the surface

$$= \frac{Gm_s}{a^3} R_p^2 P_2(\cos\psi)$$

for these to balance h(ψ) must be proportional to P_2 (cos ψ)

$$h \sim R_p \frac{m_s}{M_p} \left(\frac{R_p}{a}\right)^3$$

The potential generated by the deformed planet, exterior to the planet

$$V_p \sim \frac{GM_p}{r^3} R_p^2 P_2(\cos\psi) \times \frac{h}{R_p}$$

Love number k_2

$$V_p \sim \frac{Gm_s}{a}\left(\frac{R_p}{a}\right)^5 P_2(\cos\psi) \quad = k_2 \frac{Gm_s}{a}\left(\frac{R_p}{a}\right)^5 P_2(\cos\psi)$$

Details about planet's response to the tidal field is incorporated into the unitless Love number

Love numbers

Surface deformation $$\frac{h}{R} = h_2 \frac{V_T(R)}{g}$$

Potential perturbation $$V_2(r,\theta) = -k_2 V_T(R) \left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$

h_2 , k_2 are Love numbers
They take into account structure and strength of body

$$h_2 = \frac{5/2}{1+\tilde{\mu}} \qquad k_2 = \frac{3}{5} h_2$$

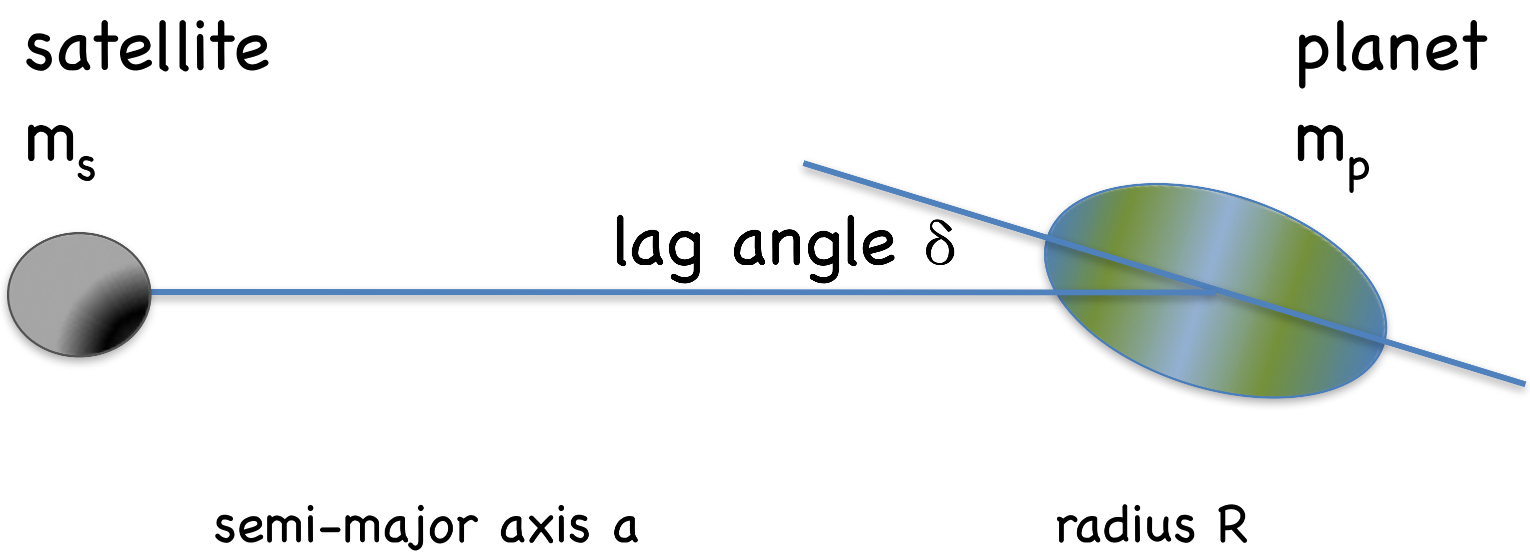
For uniform density bodies $$\tilde{\mu} \equiv \frac{19\mu}{2\rho g R}$$ μ is rigidity

dimensional quantity ratio of elastic and gravitational forces at surface

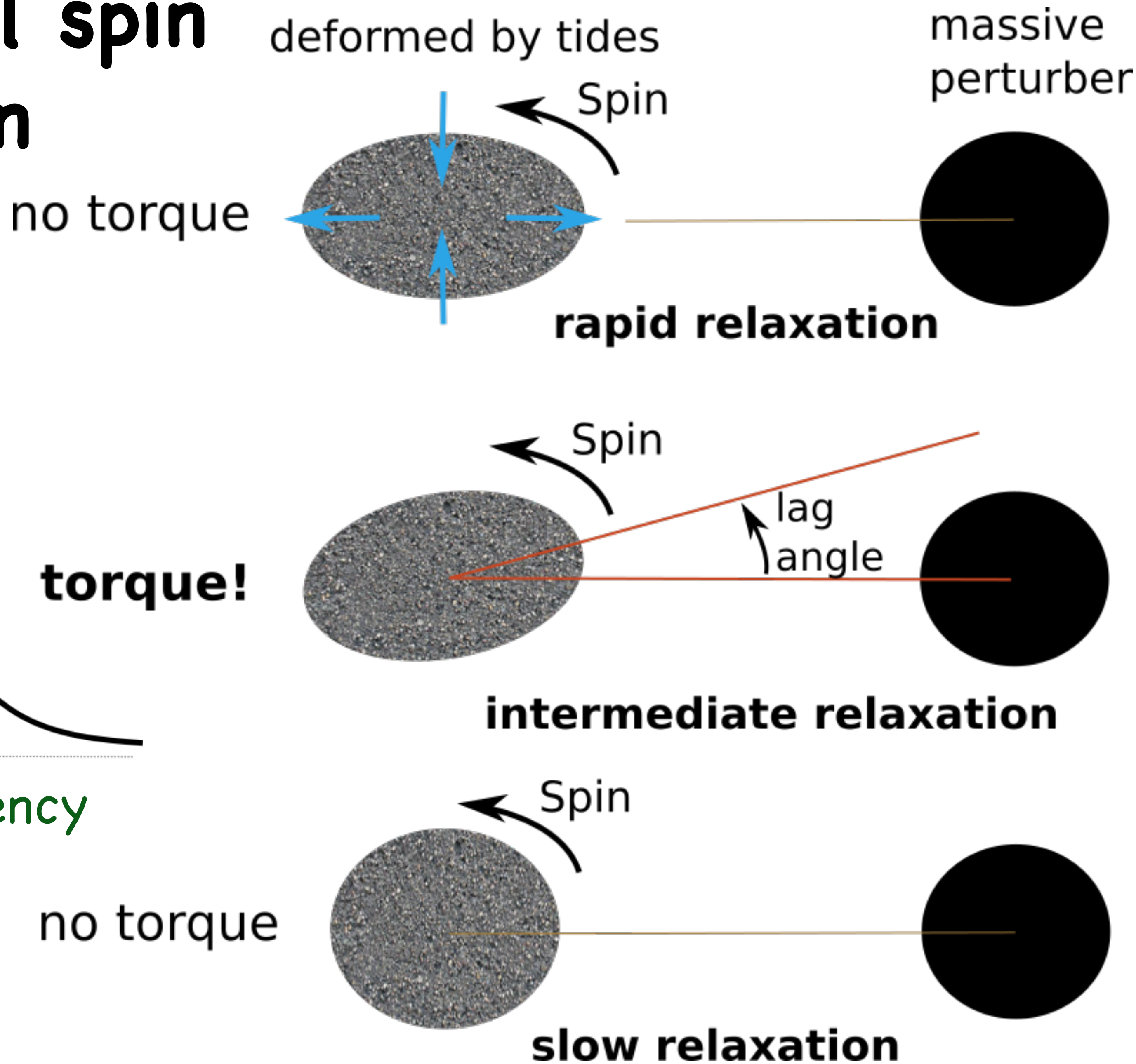
Love numbers

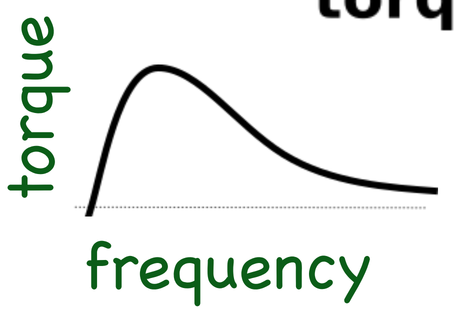
- h_2 = 5/2, k_2=3/2 for a uniform density fluid
- These values sometimes called the equilibrium tide values. Actual tides can be larger- not in equilibrium (sloshing)
- For stiff bodies h_2,k_2 are inversely proportional to the rigidity μ (which is like the shear modulus) and Love numbers are smaller than for a fluid or gaseous body
- Constraints can be made on the stiffness of the center of the Earth based on tidal response

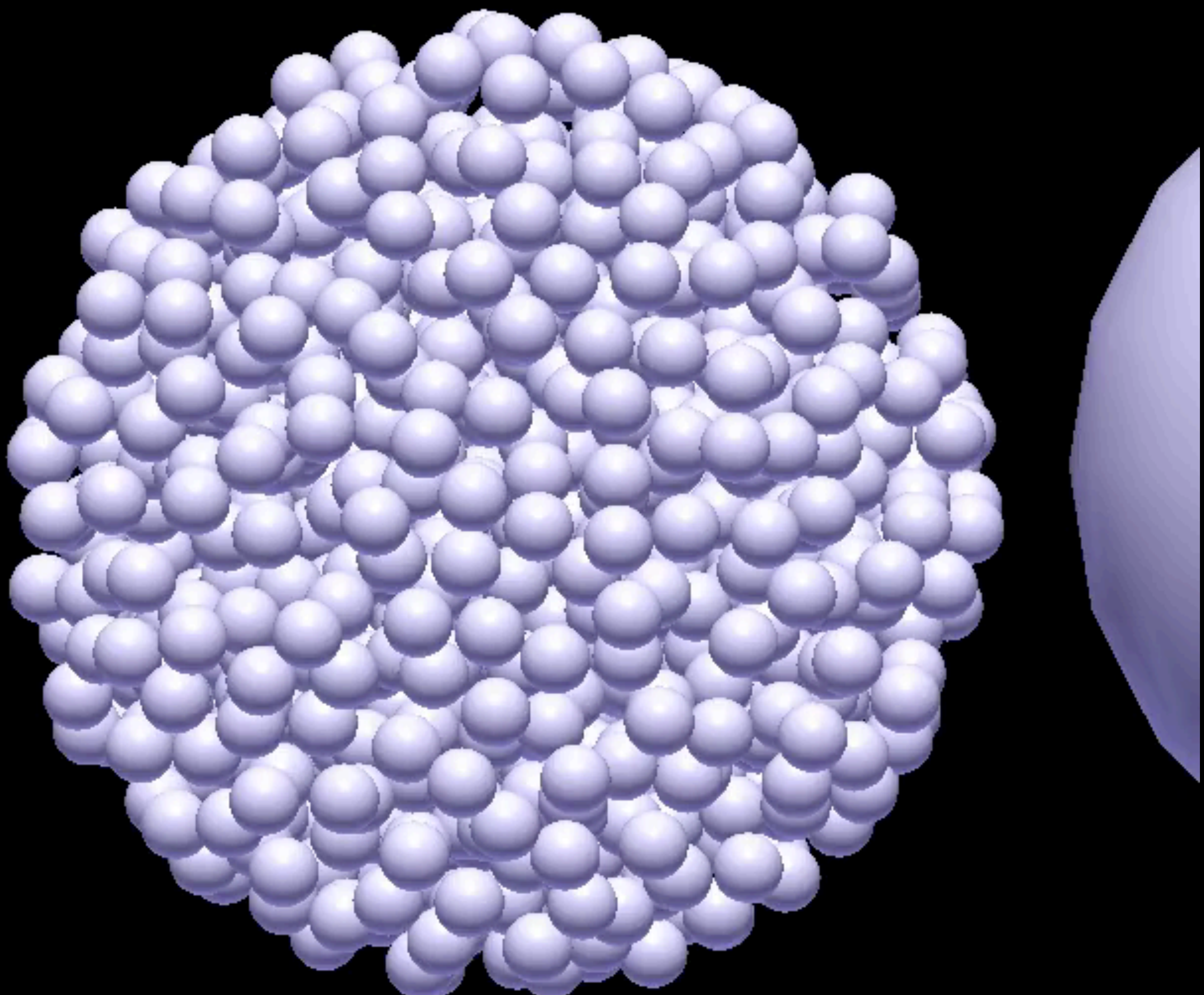
Tidal torque



Tidal spin down







movie of spin up — toward tidal lock

Tidal Torques

- Dissipation function Q = E/dE, energy divided by energy lost per cycle; Q is unitless
- For a driven damped harmonic oscillator the phase shift between response and driving frequency is related to Q: $\sin\delta = -1/Q$
- Torque on a satellite depends on difference in angle between satellite planet line and deformation angle of satellite

$$\Gamma = -m_s \frac{\partial \Phi}{\partial \psi}$$

and this is related to energy dissipation rate Q

Tidal Torques

- Torque on satellite is opposite to that on planet
- However rates of energy change are not the same (energy lost due to dissipation)
- Energy change for rotation is $\Gamma\Omega$ (where Ω is rotation rate and Γ is torque)
- Energy change on orbit is Γn where n is mean motion of orbit.
- Angular momentum is fixed (dL/dt =0)so we can relate change of spin to change of mean motion

$$L = I\Omega + \frac{m_p m_s}{m_p + m_s} a^2 n \qquad \dot{\Omega} = -\frac{1}{2I}\frac{m_p m_s}{m_p + m_s} n a \dot{a}$$

Tidal Torques

$$E = \frac{1}{2} I\Omega^2 - \frac{Gm_p m_s}{2a}$$

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2}\frac{m_p m_s}{m_p + m_s} n^2 a\dot{a}$$

$$\dot{E} = \frac{1}{2}\frac{m_p m_s}{m_p + m_s} n a\dot{a}(\Omega - n)$$

- Subbing in for dΩ/dt
- As dE/dt is always negative the sign of (Ω-n) sets the sign of da/dt
- If satellite is outside synchronous orbit it moves outwards away from planet (e.g. the Moon) otherwise moves inwards (Phobos)

Orbital decay and spin down

- Derivative of potential w.r.t ψ depends on

$$\frac{\partial P_2(\cos\psi)}{\partial\psi} = -\frac{3}{2}\sin 2\psi$$

- Torque depends on this derivative, the angle itself depends on the dissipation rate Q

$$\dot{a} = \mathrm{sign}(\Omega_p - n)\frac{3k_2}{Q_p}\frac{m_s}{m_p}\left(\frac{C_p}{a}\right)^5 na \qquad C_p = R_p$$

- where k_2 is a Love number used to incorporate unknowns about internal structure of planet
- Computing the spin down rate

$$\dot{\Omega}_p = -\mathrm{sign}(\Omega_p - n)\frac{3k_2}{2\alpha_p Q_p}\frac{m_s^2}{m_p(m_p + m_s)}\left(\frac{C_p}{a}\right)^3 n^2$$

- α_p describes moment of inertia of planet $I=\alpha_p m_p R_p^2$

Orbital decay

$$\dot{a} = \text{sign}(\Omega_p - n)\frac{3k_2}{Q_p}\frac{m_s}{m_p}\left(\frac{C_p}{a}\right)^5 na$$

- Tidal timescales for decay tend to be strongly dependent on distance.
- Quadrupolar force drops quickly with radius.
- Strong power of radius is also true for gravitational wave decay timescale
- Direction of orbit drift depends on whether spin rate is faster or slower than mean motion.

Mignard A parameter

Satellite tide on planet
Planet tide on satellite

How to estimate which one is more important if both are rotating?

$$A \equiv \left(\frac{m_p}{m_s}\right)^2 \left(\frac{R_s}{R_p}\right)^5 \frac{k_{2s}}{k_{2p}} \frac{Q_p}{Q_s}$$

A way to judge importance of tides dissipated in planet vs those dissipated in satellite

Spin down leads to synchronous rotation
When both bodies are tidally locked tidal dissipation ceases

Eccentricity damping or Tidal circularization

Neglect spin angular momentum, only consider orbital angular momentum that is now conserved conserved

$$e^2 = 1 + \frac{2EL_{orb}^2}{G^2}\frac{(m_p + m_s)}{(m_p m_s)^3} \qquad \dot{e} = -\frac{\dot{E}}{2eE}(1-e^2) \approx -\frac{\dot{E}}{2eE}$$

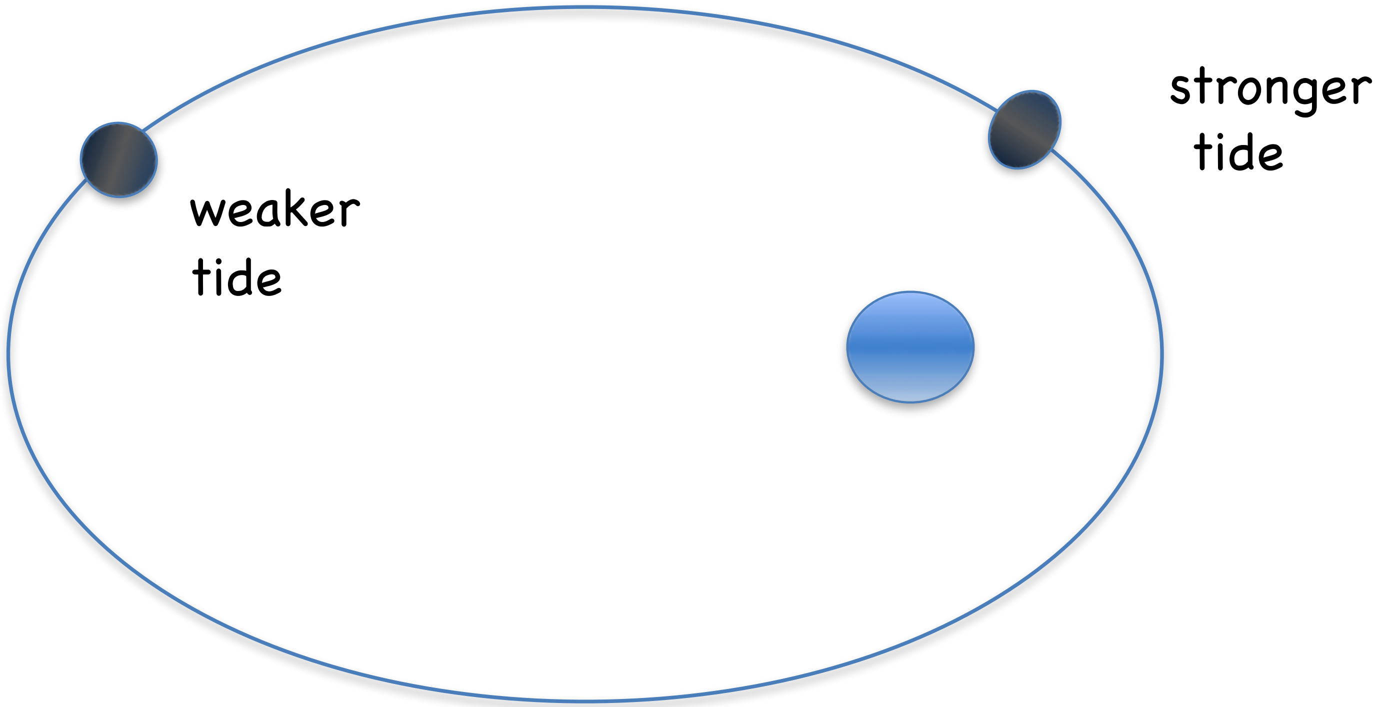
- As dE/dt must be negative then so must de/dt

  Eccentricity is damped

- This relation depends on (dE/dt)/E so is directly dependent on Q as were our previously estimated tidal decay rates

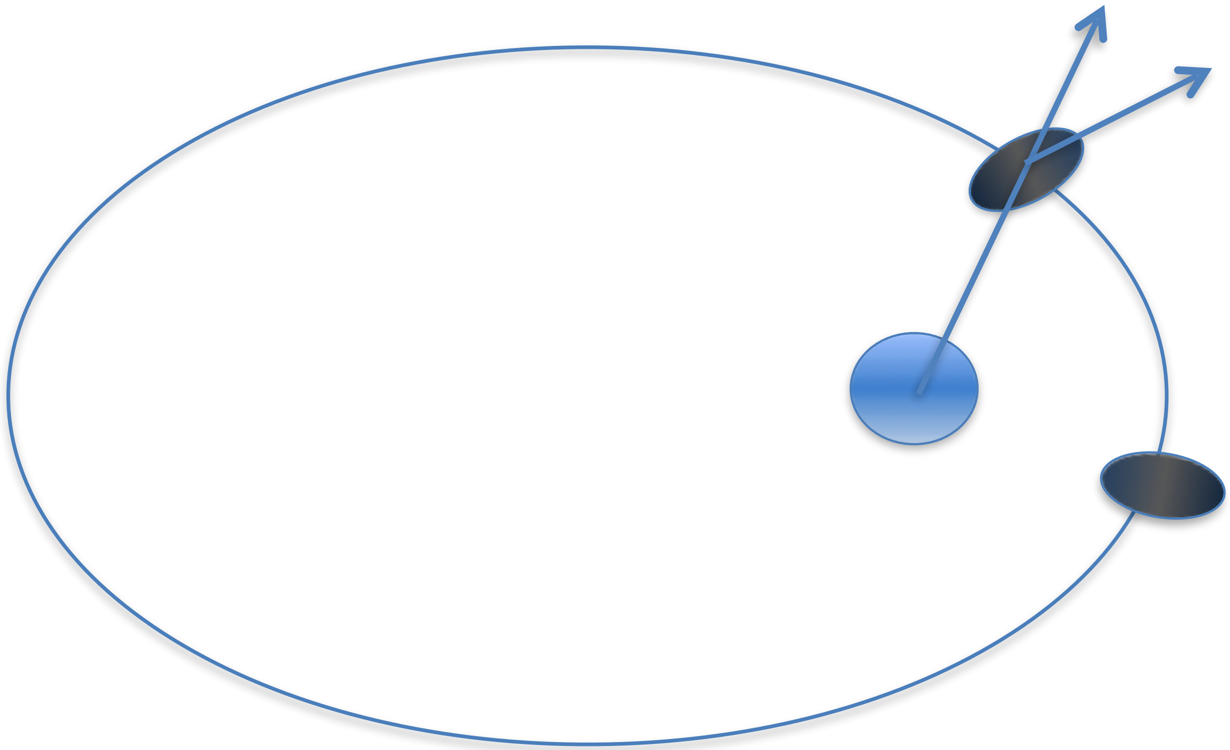
Eccentricity can increase if one body is spinning, but depending on Mignard A parameter. Moon's eccentricity is currently increasing.

Tidal circularization (amplitude or radial tide)



- Distance between planet and satellite varies with time
- Tidal force on satellite varies with time
- Amplitude variation of body response has a lag giving energy dissipation

Tidal circularization (libration tide)



- In an elliptical orbit, the angular rotation rate of the satellite is not constant. With synchronous rotation there are variations in the tilt angle w.r.t to the vector between the bodies
- Lag gives energy dissipation

Tidal circularization

Consider the gravitational potential on the surface of the satellite as a function of time
The time variable components of the potential from the planet due to planet to first order in e

$$V_s = -\frac{G m_p C_s^2}{a^3}\left[3e\cos nt P_2(\cos\beta) + 3e\sin^2\theta\sin 2\phi\sin nt\right]$$

radial tide
tide

libration

β angle between position point in satellite and the line joining the satellite and guiding center of orbit

θ azimuthal angle from point in satellite to satellite/planet plane
φ azimuthal angle in satellite/planet plane

Response of satellite

For a stiff body: The satellite response depends on the surface acceleration, can be described in terms of a rigidity in units of g

$$h \sim \frac{m_p C_s^4}{a^3} \frac{e}{\tilde{\mu}}$$

Total energy is force x distance (actually stress x strain integrated over the body)

$$F \sim \frac{G m_p C_s}{a^3} e m_s$$

$$W \sim Fh \sim \frac{G m_p^2}{a} \left(\frac{C_s}{a} \right)^5 \frac{e^2}{\tilde{\mu}}$$

This is energy involved in tide

Using this energy and dissipation factor Q we can estimate eccentricity damping rate de/dt

$$\frac{\dot{E}}{E} \sim \frac{n}{Q_s}$$

Eccentricity damping or Tidal circularization

- Eccentricity decay rate for a stiff homogenous incompressible spherical satellite

$$\tau_e = \frac{4}{63}\frac{m_s}{m_p}\left(\frac{a}{C_s}\right)^5 \frac{\mu_s Q_s}{n}$$

- where μ is a rigidity. This is a sum of libration and radial tides.
- A high power of radius.
- Similar timescale for fluid bodies but replace rigidity with the Love number. Decay timescale rapidly become long for extra solar planets outside 0.1 AU

Tidal forces and resonance

- If tidal evolution of two bodies causes orbits to approach, then capture into resonance is possible
- Once captured into resonance, eccentricities can increase (Galilean satellites!)
- Hamiltonian is time variable, however in the adiabatic limit volume in phase space is conserved.
- Assuming captured into a fixed point in Hamiltonian, the system will remain near a fixed point as the system drifts. This causes the eccentricity to increase
- Tidal forces also damp eccentricity
- Either the system eventually falls out of resonance or reaches a steady state where eccentricity damping via tide is balanced with the increase due to the resonance

Time delay vs phase delay

Prescriptions for tidal evolution

- Darwin-Mignard tides: constant time delay
- Darwin-Kaula-Goldreich: constant phase delay

constant phase angle $\delta \sim 1/Q$

constant delay

spinning object (not locked) $\delta = (\Omega - n)\Delta t$

synchronously locked $\delta = n\Delta t$

If Δt is fixed, then phase angle and Q (dissipation) changes with n (frequency)

Lunar studies suggest that dissipation is a function of frequency (see work by Efroimsky, Williams, Makarov)

Using a complex compliance to relate stress and strain. The analogy is a complex frequency dependent impedance relating voltage and current.

Tides expanded in eccentricity

For two rotating bodies, as a function of both obliquities and tides on both bodies

The Mignard evolutionary equations are:

$$\frac{dX}{dt}=\frac{C_X}{X^7}\left[-\frac{f_0}{\beta^{15}}(1+A)+\left(UX^{3/2}\cos i+A\frac{\omega'}{n}\cos(I')\right)\frac{f_1}{\beta^{12}}\right]$$

$$\frac{de}{dt}=\frac{C_e}{X^8}\left[-\frac{f_3}{\beta^{13}}(1+A)+\left(UX^{3/2}\cos i+A\frac{\omega'}{n}\cos(I')\right)\frac{f_4}{\beta^{10}}\right]$$

$$\frac{di}{dt}=-\frac{C_i}{X^{13/2}}\sin i\left(U+\frac{A}{X^{3/2}}\right)\frac{f_2}{\beta^{10}}$$

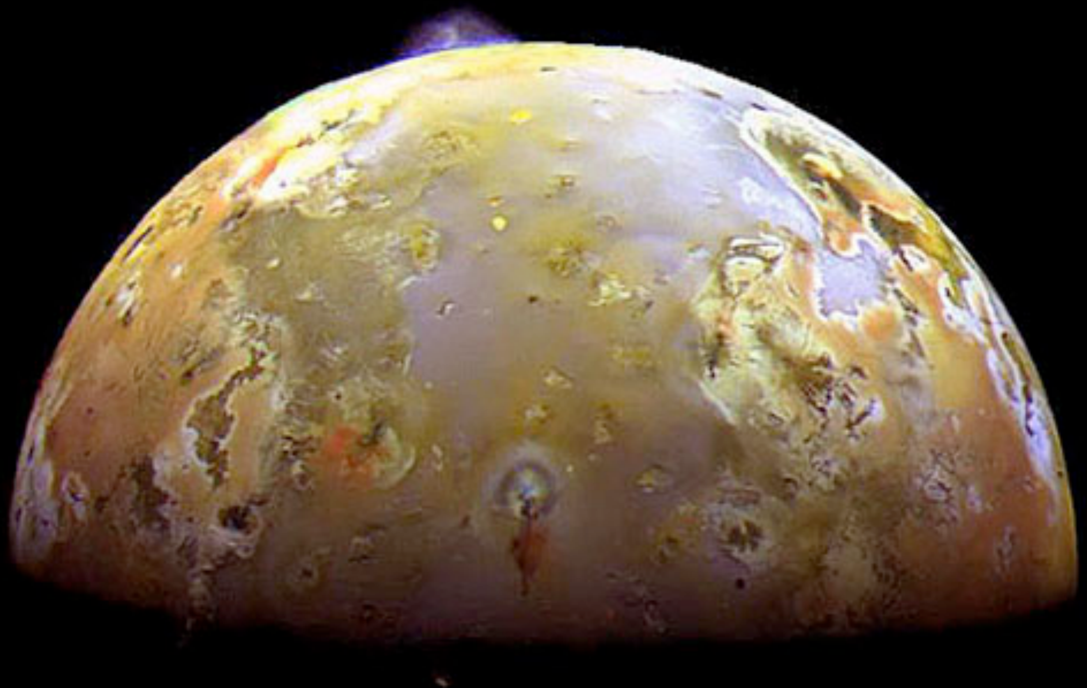
$$\frac{dU}{dt}=-C_{U,0}\left[U-\frac{n_\odot}{n_G}\right]+C_{U,1}\frac{1}{X^{15/2}}\left[\frac{f_1}{\beta^{12}}-UX^{3/2}\cos i\frac{f_2}{\beta^9}\right]\cos i$$
$$-C_{U,2}U(\sin i)^2\frac{1}{X^6}\frac{f_2}{\beta^9}$$

$$\frac{d(\omega/n_G)}{dt}=\frac{C_{\omega,0}}{X^6}\left[-\frac{f_2}{\beta^9}\left[\left(\frac{\omega}{n_G}\right)^2\frac{2+(\sin i)^2}{2}+U^2\frac{3(\cos i)^2-1}{2}\right]\right.$$
$$\left.+2U\cos i\frac{f_1}{\beta^{12}}\frac{1}{X^{3/2}}\right]+C_{\omega,1}\left[2\frac{n_\odot}{n_G}U-\left(\frac{\omega}{n_G}\right)^2-U^2\right]$$

where $\beta=\sqrt{1-e^2}$, $U=(\omega/n_G)\cos(I)$, the grazing mean motion $n_G=\sqrt{GM/R_E^3}$, G is Newton's constant, $X=a/R_E$, with R_E the radius of the Earth, a is the semimajor axis of the lunar orbit, e the orbital eccentricity, n is the orbital mean motion, i is the orbital inclination to the ecliptic, I is obliquity, ω is rotation rate, m is the mass of the Moon, M is the mass of the Earth, $\mu=(1/m+1/M)^{-1}$ is the reduced

Tidal heating of Io

Peale et al. 1979 estimated that if Qs~100 as is estimated for other satellites that the tidal heating rate of Io implied that it's interior could be molten. This could weaken the rigidity and so lead to a runaway melting events. Io could be the most "intensely heated terrestrial body" in the solar system



Morabita et al. 1979 showed Voyager I images of Io displaying prominent volcanic plumes. Voyager I observed 9 volcanic erruptions. Above is a Gallileo image

Tidal evolution

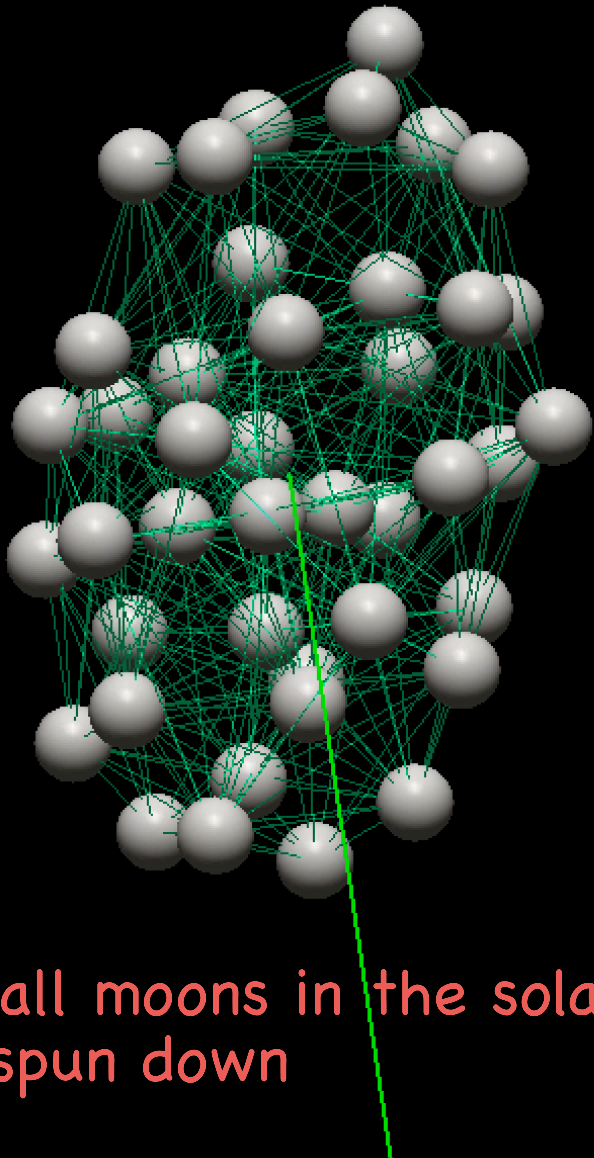
- The Moon is moving away from Earth.
- The Moon/Earth system may have crossed or passed resonances leading to heating of the Moon, its current inclination and affected its eccentricity (Touma et al.)
- Satellite systems may lock in orbital resonances
- Estimating Q is a notoriously difficult problem as currents and shallow seas may be important
- Q may not be geologically constant for terrestrial bodies (in fact: can't be for Earth/moon system)
- Evolution of satellite systems could give constraints on Q (work backwards)

Consequences of tidal evolution

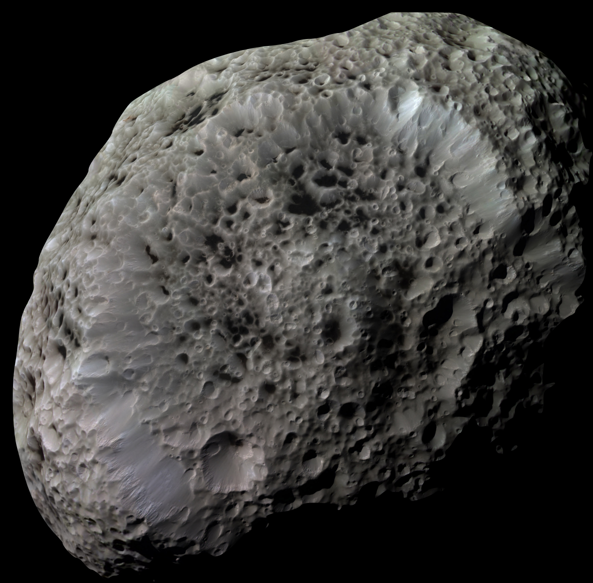
- Inner body experiences stronger tides usually
- More massive body experiences stronger tidal evolution
- Convergent tidal evolution (Io nearer and more massive than Europa)
- Inner body can be pulled out of resonance w. eccentricity damping (Papaloizou, Lithwick & Wu, Batygin & Morbidelli) possibly accounting for Kepler systems just out of resonance
- Alternatively spin resonance lifting obliquity can enhance dissipation (Millholand & Laughlin)

Tumbling

Because of elongated shapes, tumbling was predicted for Pluto's Minor Satellites (Showalter+2015,Correia+2015)



At slow spins, spin orbit resonances overlap (Wisdom), causing chaotic spin if eccentricity is not low — Hyperion tumbles! Tidal lock is foiled



Almost all moons in the solar system have tidally spun down

Static vs Dynamic tides

- We have up to this time considered very slow tidal effects
- Internal energy of a body Gm^2/R
- Grazing rotation speed is of order the low quantum number internal mode frequencies $(Gm/R^3)^{1/2}$
- During close approaches, internal oscillation modes can be excited
- Energy and angular momentum transfer during a pericenter passage depends on coupling to these modes (e.g. Press & Teukolsky 1997)
- Tidal evolution of eccentric planets or binary stars
- Tidal capture of two stars on hyperbolic orbits

Energy Deposition during an encounter

following Press & Teukolsky 1977

$$\dot{E} = \int dV \, \rho \mathbf{v} \cdot \nabla \Phi$$

dissipation rate depends on motions in the perturbed body

$$\Phi(\mathbf{r}, t) = \frac{GM}{|\mathbf{r} - \mathbf{R}(t)|}$$

perturbation potential

$$\mathbf{v} = \frac{\partial \xi}{\partial t}$$

velocity w.r.t to Lagrangian fluid motions

$$\Delta E = \int dt \dot{E}$$

total energy exchange

Describe both potential and displacements in terms of Fourier components

Describe displacements in terms of a sum of normal modes

Because normal modes are orthogonal the integral can be done in terms of a sum over normal modes

Energy Deposition due to a close encounter

Potential perturbation described in terms of a sum over normal modes

$$\nabla\Phi = \sum A_n(\omega)\xi_n$$

$$\Delta E \propto \sum |A(\omega)|^2$$

energy dissipated depends on "overlap" integrals of tidal perturbation with normal modes

dimensionless expression

$$\eta \equiv \sqrt{\frac{M_*}{M_* + m}}\sqrt{\frac{d^3}{R_*^3}}$$

d=pericenter

internal binding energy of star

strong dependence on distance of pericenter to star

$$\delta E = \left(\frac{GM_*^2}{R_*}\right)\left(\frac{m}{M_*}\right)^2 \sum_l \left(\frac{R_*}{d}\right)^{2l+2} T_l(\eta)$$

multipole expansion

to estimate dissipation and torque, you need to sum over modes of the star/planet, often only a few modes are important

Capture and circularization

- Previous assumed no relative rotation. This can be taken into account (e.g., Invanov & Papaloizou 04, 07 …)
- "quasi-synchronous state" that where rotation of body is equivalent to angular rotation rate a pericenter
- Excited modes may not be damped before next pericenter passage leading to chaotic variations in eccentricity (work by R. Mardling)

Tidal predictions taking into account normal modes of a planet

- Work by Jennifer Meyer and Jack Wisdom on this (and Lau?)

Hot Jupiters

- Critical radius for tidal circularization of order 0.1 AU
- Rapid drop in mean eccentricity with semi-major axis. Can be used to place a limit on Q and rigidity of these planets using the circularization timescale and ages of systems
- Large eccentricity distribution just exterior to this cutoff semi-major axis
- Large sizes of planets found via transit surveys a challenge to explain.

Possible Explanations for large hot Jupiter radii

- They are young and still cooling off
- They completely lack cores?
- They are tidally heated via driving waves at core boundary rather than just surface (Ogilvie, Lin, Goodman, Lackner)
- Gravity waves transfer energy downwards
- Obliquity tides. Persistent misalignment of spin and orbital angular momentum due to precessional resonances (Cassini states) (A possibility for accounting for oceans on Europa?)
- Evaporation of HeI (Hanson & Barman) Strong fields limit loss of charged Hydrogen but allow loss of neutral He leading to a decrease in mean molecular weight
- Ohmic dissipation (e.g., Batygin)
- Kozai resonance (e.g., Naoz)

Cooling and Day/Night temperatures

- Radiation cooling timescale sets temperature difference
- If thermal contrast too large then large winds are driven day to night (advection)
- Temperature contrast set by ratio of cooling to advection timescales (Heng)

Quadrupole moment of non-round bodies

- For a body that is uneven or lopsided like the moon consider the ellipsoid of inertia (moment of inertial tensor diagonalized)
- Euler's equation, in frame of rotating body

$$\Gamma = \dot{L} + \omega \times L = I\dot{\omega} + \omega \times L$$

- Set this torque to be equal to that exerted tidally from an exterior planet

Gravitational Potential interaction between a point mass and an extended axisymmetric mass

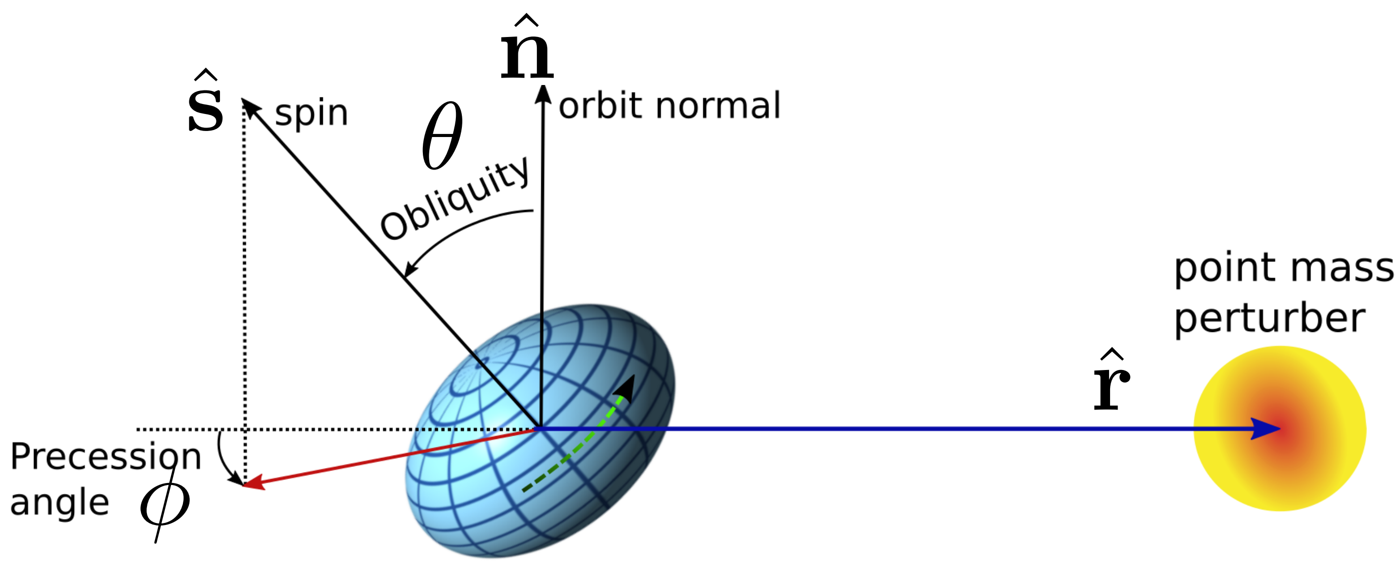
Quadrupole moment matrix
Moments of Inertia matrix
MacCullough's formula

$$\left.\right\} \quad U_{interaction} = GM \int \frac{\rho(\mathbf{r}')d^3r'}{|\mathbf{r} - \mathbf{r}'|} = 3(C-A)\frac{GM}{r^3}(\hat{\mathbf{s}}\cdot\hat{\mathbf{r}})^2$$

$$U_2 = -\frac{Gm}{2r^3}(\mathrm{tr}I - 3I_{zz})$$

where I_{zz} is the moment of inertia about the axis pointing to the perturber

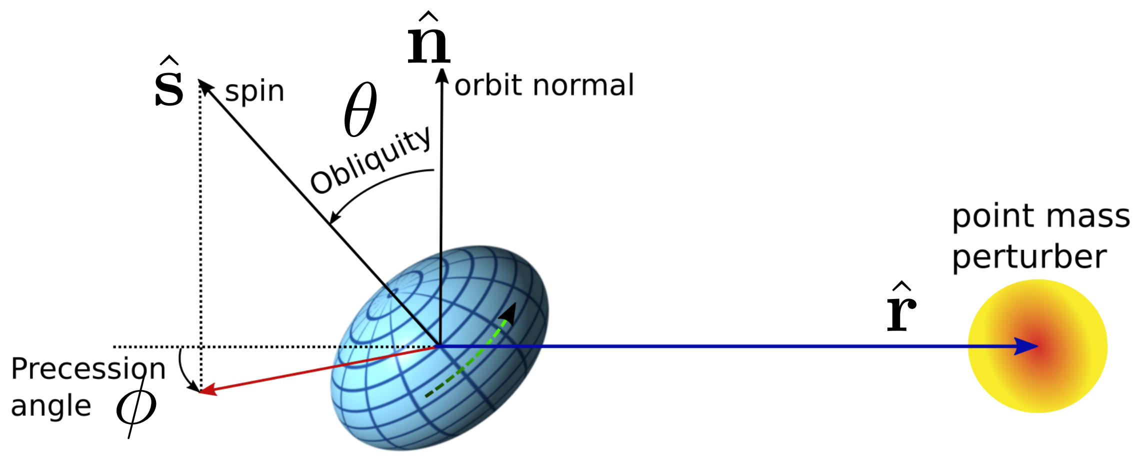
spin axis is axis of symmetry $\hat{\mathbf{s}} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$



MacCullagh's formula for the torque on a non-round (but axisymmetric) body caused by a distant point mass

moments of inertia

Instantaneous torque $$\mathbf{T} = 3(C - A)\frac{GM_*}{r^3}(\hat{\mathbf{r}} \cdot \hat{\mathbf{s}})(\hat{\mathbf{r}} \times \hat{\mathbf{s}})$$



This is MacCullagh's formula for a axisymmetric body

Precession of a planet that has obliquity

You need to average the instantaneous torque about the orbit

The precession rate is

$$\dot{\phi} = -\frac{3}{2}\frac{(C-A)n^2}{C\omega^2}\cos\theta$$

θ : obliquity
n: mean motion
ω: spin rate
C,A: moments of inertia

Saturn's precession
Earth's precession

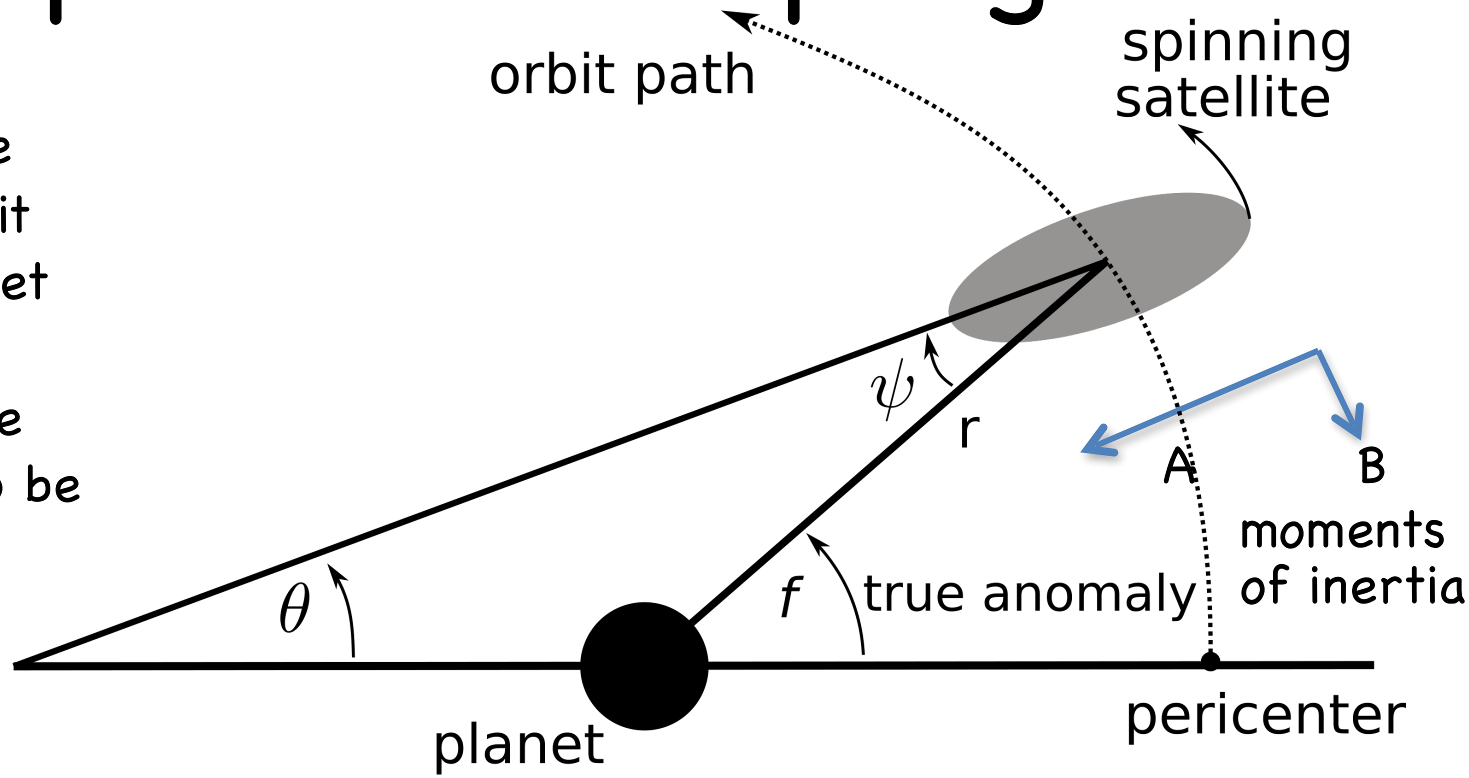
On planetary precession

- Ring systems precess with the planet, and making the precession rate faster
- Slow orbit resonances can tilt planet's obliquities
  (e.g. a body of work by Bill Ward)
- Mars's obliquity is though to be chaotic, whereas the Moon might stabilize the Earth's obliquity
- Saturn and Jupiter may have been tilted over by spin resonances
- Satellites of Pluto also may have been tilted over.

Spin orbit coupling

spin of satellite
w.r.t to its orbit
about the planet

Fixed reference
frame taken to be
that of the
satellite's
pericenter



- A,B,C are from moment of inertia tensor
- Introduce a new angle related to mean anomaly M of object
- A spin resonant angle

$$C\ddot{\theta} - \frac{3}{2}(B-A)\frac{Gm_p}{r^3}\sin 2\psi = 0$$

$$\gamma = \theta - pM \qquad p = i/j$$

$$\ddot{\gamma} = \ddot{\theta}$$

p is integer ratio

Spin orbit resonance

$$\ddot{\gamma} + \frac{3n^2}{2}\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3 \sin(2\gamma + 2pM - 2f) = 0$$

- Now can be expanded in terms of eccentricity e of orbit using low-eccentricity expansions
- If near a resonance then the angle γ is slowly varying and one can average over other angles to give an equation that is that of a pendulum

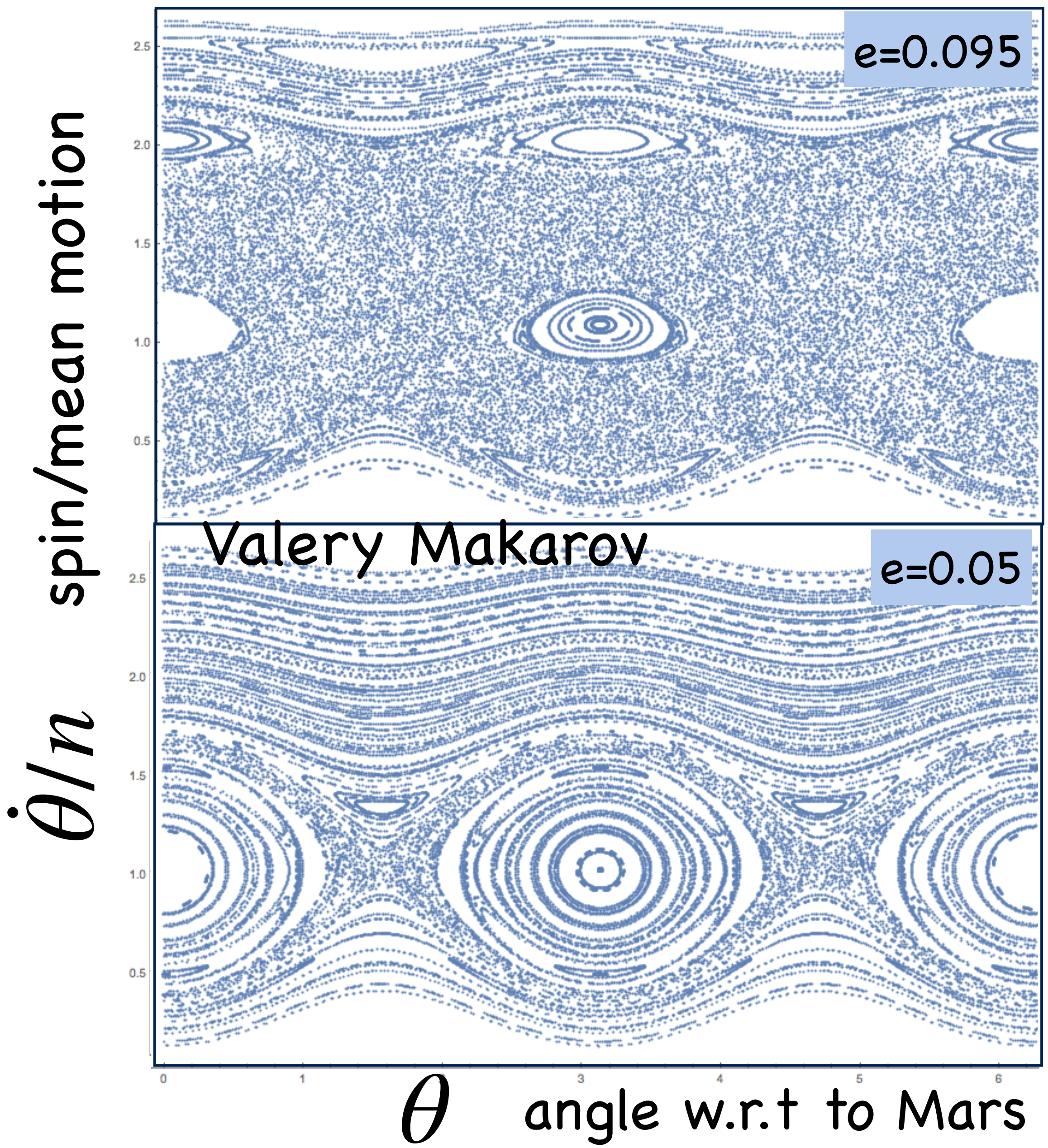
$$\ddot{\gamma} \sim -\frac{\omega_0^2}{2}\sin 2\gamma$$

with libration frequency

$$\omega = n\left[3\frac{(B-A)}{C}H(p,e)\right]^{\frac{1}{2}}$$

coefficients depend on eccentric and which angle/resonance is considered

Phobos Poincaré sections at apocenter



At high eccentricity Chaotic spin everywhere

At lower eccentricity Chaotic regions still exist

Valery Makarov, James Kwiecinski, Michael Efroimsky

Spin Orbit Resonance

Libration frequency $$\omega = n\left[3\frac{(B-A)}{C}H(p,e)\right]^{\frac{1}{2}}$$

See for example, Celletti's book

$$H(1,e) = 1 - \frac{5e^2}{2} + \frac{e^4}{16}$$

Spin synchronous

$$H\left(\frac{3}{2},e\right) = \frac{7e}{2} - \frac{123e^3}{16}$$

3:2 spin orbit resonance, like mercury, 3 spin periods per 2 orbit periods

$$H\left(\frac{1}{2},e\right) = -\frac{e}{2} + \frac{e^3}{16}$$

2:1 spin orbit resonance

Spin Orbit Resonance and Dynamics

- Tidal force can be expanded in Fourier components
- Contains high frequencies if orbit is not circular
- Eccentric orbit leads to oscillations in tidal force which can trap a spinning non-spherical body in resonance
- If body is sufficient lopsided then motion can be chaotic
- All objects must chaotically tumble prior to reaching tidal lock (Wisdom 1987)

Notes on spin-orbit resonance capture

- Mercury is in a 3:2 resonance but is only at moderate eccentricity
- Tidal dissipation decreases eccentricity
- Tidally dissipation also reduces volume in phase space
- Taking into account only the dominant frequency, capture into the 3:2 resonances is unlikely
- Either eccentricity was higher in the past or high frequency terms aid in capture
- There have been few simulations of the capture process — in other settings we found that additional spin resonances might be important (libration resonance, NPA resonance ...)

Reading

- M+D Chap 4, 5, Cellitti's book