Astronomy 142 Workshop #4: Solutions

1. ***The CNO Bi-Cycle.*** *The complete CNO cycle of nuclear reactions is as follows:*

$$^{12}_{6}\text{C} + ^{1}_{1}\text{H} \Rightarrow ^{13}_{7}\text{N} + ^{0}_{0}\gamma$$

$$^{13}_{7}\text{N} \Rightarrow ^{13}_{6}\text{C} + \underline{^{0}_{1}\bar{e}} + \underline{^{0}_{0}\nu_e}$$

$$^{13}_{6}\text{C} + \underline{^{1}_{1}\text{H}} \Rightarrow ^{14}_{7}\text{N} + \underline{^{0}_{0}\gamma}$$

$$^{14}_{7}\text{N} + ^{1}_{1}\text{H} \Rightarrow ^{15}_{8}\text{O} + \underline{^{0}_{0}\gamma}$$

$$^{15}_{8}\text{O} \Rightarrow ^{15}_{7}\text{N} + ^{0}_{1}e^{+} + \underline{^{0}_{0}\nu_e}$$

$$^{15}_{7}\text{N} + \underline{^{1}_{1}\text{H}} \Rightarrow ^{12}_{6}\text{C} + \underline{^{4}_{2}\text{He}} \qquad \text{(99.96\% of the time)}$$

$$\Rightarrow ^{16}_{8}\text{O} + ^{0}_{0}\gamma \qquad \text{(0.04\% of the time)}$$

$$^{16}_{8}\text{O} + ^{1}_{1}\text{H} \Rightarrow ^{17}_{9}\text{F} + ^{0}_{0}\gamma$$

$$^{17}_{9}\text{F} \Rightarrow \underline{^{17}_{8}\text{O}} + \underline{^{0}_{1}\bar{e}} + ^{0}_{0}\nu_e$$

$$\underline{^{17}_{8}\text{O}} + ^{1}_{1}\text{H} \Rightarrow \underline{^{14}_{7}\text{N}} + ^{4}_{2}\text{He}$$

a. *Fill in the blanks in the table of reactions with the missing names of the reacting particles.*

   See above.

b. *One part of the bi-cycle is indicated on the reaction table. Find another catalytic cycle among the reactions, and label it just as the first part is.*

   See above. The new cycle, which we may call the "NO cycle," uses $^{14}_{7}\text{N}$ as a catalyst.

c. *What are the overall reactions associated with each of the two catalytic cycles? How does the energy released in each overall reaction compare with that released in the pp chain?*

   Both cycles add up to the same overall reaction:

$$4\left(^{1}_{1}\text{H}\right) \rightarrow ^{4}_{2}\text{He} + 2\left(^{0}_{1}\bar{e}\right) + 2\left(^{0}_{0}\nu_e\right) + 3\left(^{0}_{0}\gamma\right) \quad .$$

   Apart from the number of (massless) photons, this is also the same as the pp chain, and thus involves the same energy release.

2. *The flux of pp-chain electron neutrinos from the Sun has been measured to be about half the flux that is expected at the temperature inferred for the Sun's center,* $T_{c\odot} = 15.7\times10^{6}\ \text{K}$.

a. *What would the Sun's central temperature have to be, in order to produce pp-chain electron neutrinos at half the rate that applies to* 15.7×10^{6} K *?*

As we learned in lecture, that the rate of neutrino production is proportional to the fusion probability $P(T)=D\exp\left(-\left[T_0/T\right]^{1/3}\right)$, where $T_0=(3/2)^3\left(4\pi^2 q_p^2/h\right)^2 m/k=3.86\times10^{10}$ K A reduction by a factor of two in the neutrino flux goes with a reduction by a factor of two in this probability:

$$\frac{P(T)}{P(T_{c\odot})}=\frac{e^{-(T_0/T)^{1/3}}}{e^{-(T_0/T_{c\odot})^{1/3}}}=\frac{1}{2} \quad .$$

Solve this for T:

$$-\left(\frac{T_0}{T}\right)^{\frac{1}{3}}+\left(\frac{T_0}{T_{c\odot}}\right)^{\frac{1}{3}}=-\ln 2$$

$$T=\frac{T_0}{\left(\left[\frac{T_0}{T_{c\odot}}\right]^{\frac{1}{3}}+\ln 2\right)^3}=1.35\times10^{7}\text{ K} \quad .$$

This is smaller than the accepted value by $\left(T_{c\odot}-T\right)/T_{c\odot}=14\%$.

b. *Review briefly the means by which we inferred the Sun's central temperature (in lecture on 31 January). By how much would the accepted values of other measurable properties of the Sun need to be in error for the temperature to be off by this much? Is it likely that such differences could be within present observational uncertainties?*

Here is the formula we got for the temperature, and expressions we got for the main ingredients of this formula:

$$T_c=\frac{P_c\bar{m}_c}{\rho_c k}, \quad P_c=19\frac{GM_\odot^2}{R_\odot^4}, \quad \rho_c=25\frac{M_\odot}{R_\odot^3} \quad .$$

So to change the inferred temperature we would need to infer a mistake in our assessment of the pressure or the typical particle mass smaller, or the density larger, or some combination of those things. We don't have much choice about the particle mass (as we'll see next Tuesday). To have made a mistake in the pressure or density we'd need to have made an error in the factors of 19 and 25 in the expressions for P_c and ρ_c. These factors come from (correct) solutions to the equations of stellar structure. Needless to say there's very little uncertainty in the values of the mass and radius of the Sun. So the most likely place to look for an error is in the basic physics that give the equations of stellar structure: hydrostatic equilibrium, ideal-gas behaviour, *etc*. But this physics does a good job of explaining all other aspects of the Sun and other stars. Thus it is considered very unlikely that the temperature at the center of the Sun is 14% smaller than we always thought.

3. *The quantum-mechanical, or de Broglie, wavelength corresponding to a particle with momentum* $p_x = mv_x$ *is* $\lambda = h/p_x$.

   a. *Present a brief argument to show that degeneracy pressure is important when a particle is confined to a space similar in size to, or smaller than, its de Broglie wavelength.*

      By the uncertainty principle, $\Delta x \geq h/\Delta p_x$; the momentum uncertainty (and pressure) is larger the smaller Δx is, so take the equality. Then, if the typical momentum that arises is comparable to the momentum uncertainty, i.e. $p_x \approx \Delta p_x$, then $\Delta x = h/p_x = \lambda$.

   b. *Show from the relative sizes of their de Broglie wavelengths that electron degeneracy pressure can be larger than electron thermal pressure – that is, the electrons are degenerate – while the nuclei that are mixed in with the electrons still behave as an ideal gas.*

      The masses of the proton and neutron are approximately equal, and are 1836 times as large as the electron's mass. Heavier nuclei exceed the electron mass by larger factors. For similar speeds, then, the de Broglie wavelengths of the simple baryons, $\lambda_p = h/m_p v$, are a factor of 1836 smaller than the electron de Broglie wavelengths, $\lambda_e = h/m_e v$. Under such conditions you'd have to confine a proton to a space 1836 times as small as that of an electron to see an equivalent display of degeneracy pressure. But in real gases, in which the protons and electrons are mixed, the confinement spaces are the same for protons and electrons, so it's possible for the electrons to be degenerate while the protons are still far from degenerate, and still thus behave as an ideal gas.

4. **Brown dwarfs.** *Consider a star of such very low mass as to be only marginally capable of thermonuclear heat production. Under the assumptions the star is all hydrogen (Z = A = 1), that gravity is balanced by nonrelativistic electron degeneracy pressure, and that protons, at the same pressure and temperature as the electrons, act as an ideal gas, derive the equation relating the star's central temperature* T_c *to its total mass M. If* $T_c \geq 3\times10^6$ K *is required to sustain the p-p chain fusion reactions, what, therefore, is the minimum mass of a luminous star? Express your numerical answer in solar masses, and compare it to the mass of Jupiter (*2×10^{30} *gm).*

   "Stars" with mass less than this minimum never undergo fusion energy generation; such objects are called brown dwarfs.

   As we saw in lecture today, the central pressure and mass density in a body supported by electron degeneracy pressure are given by

$$P_c = 0.77\frac{GM^2}{R^4} \qquad \text{and}$$

$$\rho_c = 1.43\frac{M}{R^3} \quad .$$

   Since $Z = A = 1$ by assumption, the protons have the same pressure and temperature as the electrons, and will behave as an ideal gas. We thus have an additional expression for P_c:

$$P_c = \frac{\rho_c k T_c}{m_p} \quad ,$$

   so that

$$T_c = \frac{m_p P_c}{k\rho_c} = \frac{0.77GMm_p}{1.43kR} \quad .$$

Using the mass – radius relation derived in lecture (with Z/A = 1),

$$R = 0.114\frac{h^2}{Gm_e m_p^{5/3}}M^{-1/3} \quad ,$$

we can eliminate R from the expression for T_c:

$$\begin{aligned} T_c &= \frac{0.77GMm_p}{1.43k}\frac{Gm_e m_p^{5/3}}{0.114h^2}M^{1/3} = 4.72\frac{G^2 m_e m_p^{8/3}}{h^2 k}M^{4/3} \\ &= 1.24\times 10^{-36}\, M^{4/3}\ \mathrm{K\ gm^{-4/3}} \quad . \end{aligned}$$

We know that T_c needs to be greater than about three million degrees to ignite the *p-p* chain of fusion reactions. From this expression, therefore, a star must have a mass of

$$M \ge 6\times 10^{31}\ \mathrm{gm} = 0.03M_\odot \quad ,$$

to undergo fusion, and thus to be luminous in the long term. This limit corresponds to a body a factor of about 30 more massive than Jupiter (0.001 $M_\odot$), and about a factor of 30 *less* massive than the Sun.

Our result is pretty close to the present, accepted value of 0.08 $M_\odot$ produced by much more detailed calculations. The reason we didn't get exactly the right answer is that we ignored the Coulomb interaction between electrons; to include it is difficult, but produces the result just quoted. The important thing to notice here is that stars cannot be made with arbitrarily small masses, because of electron degeneracy pressure.

5. *Beginning with the relativistic form for electron degeneracy pressure, and the expressions for central pressure from weight and central density in a relativistic-degenerate equation of state:*

$$P_e = 0.123hcn_e^{4/3} \quad , P_c = 11\frac{GM^2}{R^4}, \quad \textit{and} \quad \rho_c = 12.9\frac{M}{R^3} \quad ,$$

substitute $n_e = Z\rho/Am_p$ *and manipulate to obtain an expression for the electron degeneracy pressure in a star made of material with nuclear charge Z and mass number A, and an expression for the (Chandrasekhar) maximum mass of such a star. You will thus fill in the steps left out in arriving at the results on slide 11 of yesterday's lecture.*

Calculate the maximum mass of a carbon white dwarf, expressing your answer in solar masses.

We substitute $n_e = Z\rho/Am_p$ into $P_e = 0.123hcn_e^{4/3}$ and duly obtain

$$P_e = 0.123\frac{hc}{m_p^{4/3}}\left(\frac{Z}{A}\right)^{4/3}\rho^{4/3} \quad .$$

The Chandrasekhar mass is that for which the relativistic electron pressure balances the pressure due to the weight, i.e.

$$11\frac{GM_{\rm Ch}^2}{R^4} = 0.123\frac{hc}{m_p^{4/3}}\left(\frac{Z}{A}\right)^{4/3}\left(12.9\frac{M_{\rm Ch}}{R^3}\right)^{4/3} \quad ;$$

the Rs cancel out, and we are left with

$$M_{\rm Ch}^{2/3} = 0.340\frac{hc}{Gm_p^2}\left(\frac{Z}{A}\right)^{4/3} m_p^{2/3} \quad ,$$

or

$$M_{\rm Ch} = 0.198\left(\frac{hc}{Gm_p^2}\right)^{3/2}\left(\frac{Z}{A}\right)^2 m_p$$
$$= 1.73\times10^{57}\, m_p = 2.88\times10^{33}\ \text{gm} = 1.44 M_\odot \quad ,$$

for Z/A = 0.5, as is the case for carbon.

6. I*n an all sky optical proper motion survey you detect a fast moving star that is very blue, bluer than an A star. What type of object is this likely to be? Explain why. In a hypothetical near-infrared all sky proper motion survey you pick up an object that is moving fast that is very red, redder than an M star. What type of object is this likely to be? Explain why.*

Blue stars can be main sequence stars, giants or white dwarfs. However the proper motion depends on the distance to the star as well as how fast it is moving. Nearby stars tend to have high proper motions. So a proper motion survey will find high proper motion stars that are either halo stars (moving fast but far away) or are nearby and intrinsically faint. A blue high proper motion star is most likely a white dwarf, though it might be a low metallicity giant of some sort in the halo. Likewise a red proper motion survey will pick up brown dwarfs, M stars and red halo giants.

7. *Based on APP 10.13. The radius of a 1 solar mass white dwarf is estimated to be 10^9cm.*

a. Estimate the thermal energy stored in the white dwarf if it has a temperature of 10^7K?

We can estimate $E \sim NkT$ with N the number of particles in the star. This gives

$$N \sim \frac{M}{\bar{m}} \sim \frac{2\times10^{33}\,\text{g}}{0.5\times1.7\times10^{-24}\,\text{g}} \sim 2\times10^{57}$$
$$E \sim 2\times10^{57}\times1.4\times10^{-16}\,\text{erg/K}\times10^{7}\,\text{K} \sim 3\times10^{48}\,\text{erg}$$

b. Assuming the surface radiates as a 10^4K black body estimate its cooling timescale, $t_{cool} \sim E\,(dE/dt)^{-1}$.

$$L = \frac{dE}{dt} = 4\pi R^2 \sigma T^4 \sim 12 \times (10^9\,\mathrm{cm})^2 5.7 \times 10^{-5} (10^4\,\mathrm{K})^4 \sim 7 \times 10^{30}\,\mathrm{erg/s}$$

$$t_{cool} \sim \frac{E}{L} \sim \frac{3 \times 10^{48}\,\mathrm{erg}}{7 \times 10^{30}\,\mathrm{erg/s}} \sim 4 \times 10^{17}\,s \sim 10^{10}\,\mathrm{years}$$

This is about a Hubble time. White dwarfs should stay luminous for quite a while.

8. Web assignment: At 2 microns mostly what is seen in the Milky Way is star light. The stellar disk is thicker than the gas disk so the HI looks thinner than the 2 micron image. At 100 microns what is seen is relatively cool dust which reradiates light absorbed from stars but at longer wavelengths. The Galactic center is 0,0 degrees galactic latitude and longitude. The Galactic anticenter is 180,0 degrees longitude, latitude. Using Ned's coordinate calculator I find that the Galactic center is at RA=17h45m37.22141s DEC=-28d56m10.2289s epoch J2000. The coordinate calculator also gives you the extinction through the galaxy in the direction toward the Galactic center. They estimate an extinction of hundreds of magnitudes! Note the extinction estimates are only an estimate when the gas density is so high.