Today in Astronomy 142

- Approximations, scaling relations, and characteristic scales in astrophysics.
- Order of Magnitude estimates
- Averages

At right: An optical image from the Hubble Space Telescope of the debris disk around Fomalhaut (Kalas et al. 2005) The central star is occulted by a coronagraph mask. The light detected is starlight scattered by dust particles in the ring. The ring is not centered around the star but offset so has an eccentricity of 0.1.
Approximations in the equations of astrophysics

- Astrophysical objects, be they planets, stars, nebulae or galaxies, are all very complex compared to the physical systems you have met in your physics classes.
- In order to simplify the relevant systems of equations that describe these objects to the point that they can be solved, astrophysicists have to employ approximations to the functions involved. The approximations used in introductory treatments of the subjects are often very crude, but can still be useful in illuminating the general operating features of astrophysical systems.
- Good simple approximations can often be obtained from power-series representations of elementary functions.
Common power-series representations and first-order approximations

Suppose $x < 1$. Then $x^2, x^3, \text{etc.}$ are even smaller. A **first-order** approximation is one in which we ignore power-series terms of higher power than $x^1$. Examples:

- **Small-angle approximation**

\[
\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \approx x
\]

\[
\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!} = 1 - \frac{x^2}{2} + \frac{x^4}{120} - \ldots \approx 1
\]

\[
\tan x = \sum_{i=0}^{\infty} \frac{2^{2i+2} (2^{2i+2} - 1) B_i x^{2i+1}}{(2i+2)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \approx x
\]

\[
\arctan x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{2i + 1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \approx x
\]

\[
e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \ldots \approx 1 + x
\]

\[
\ln(1 + x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{i+1}}{i+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \approx x
\]

\[
(1 + x)^n = \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)!} x^i = 1 + nx + \frac{n(n-1)}{2} x + \ldots \approx 1 + nx
\]
Expansions

Expansions can be found using a Taylor series

\[ f(a + x) = \sum f^n(x) \bigg|_{x=a} \frac{x^n}{n!} \]

Example

\[ f(x) = \log_{10}(1 + x) \]
Expansions

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Example

\[ f(x) = \log_{10}(1 + x) \]

Let \( y = \log_{10}(1 + x) \) \hspace{1cm} 10^y = 1 + x \hspace{1cm} 10 = e^{\ln 10} \hspace{1cm} e^{\ln 10y} = 1 + x \]

\[ \frac{dy}{dx} = \frac{10^{-y}}{\ln 10} \quad \frac{dy}{dx} \bigg|_{x,y=0} = \frac{1}{\ln 10} \]

\[ f(x) \sim 0 + \frac{x}{\ln 10} \]
Example first-order approximations

Find approximations to first order in $x$ for:

\[ \sqrt{e^x \cos x} \]

\[ \frac{1}{e^x - 1} \]

\[ \frac{4^n \tan x}{(2 + x)^n (2 - x)^n} \]

\[ \frac{e^{ix} - e^{-ix}}{2i} \]
Example first-order approximations

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**Answers:**

\[ \approx \sqrt{(1 + x)1} \approx 1 + \frac{x}{2} \]
\[ \approx \frac{1}{1 + x - 1} = \frac{1}{x} \]
\[ = \frac{4^n \tan x}{(4 - x^2)^n} \approx \frac{4^n \tan x}{4^n} \approx x \]
\[ \approx \frac{1 + ix - 1 - (-ix)}{2i} = x \]
Example first order approximations

Another reason first order approximations are useful is that they help you estimate errors.

1. \( m = -2.5 \log_{10} (1 + x) \)
   What is this to first order in \( x \)?

2. Does anybody recognize this function?
Example first order approximations

Another reason first order approximations are useful is that they help you estimate errors

1. To first order in $x$

$$m = -2.5 \log_{10}(1 + x) \sim -2.5 \left(0 + \frac{x}{\ln 10}\right)$$

2. This function describes astronomical magnitudes!

The first order form is useful to estimate what error in magnitude measurement is caused by an error in a flux measurement
Errors in magnitude

Suppose you measure a flux of $f$ and a magnitude of $m$

$$m = -2.5 \log_{10} f + \text{constant}$$

Suppose you have an error in your flux measurement. You could have measured a flux of $f(1+x)$, where $x$ is the fractional error in your flux measurement. What error in the magnitude would you have?

$$m' = -2.5 \log_{10} [f(1+x)] + \text{constant}$$
$$= -2.5 \log_{10} f - 2.5 \log_{10} (1+x) + \text{constant}$$
$$dm = m' - m = -2.5 \log_{10} (1+x)$$
$$\sim \frac{-2.5x}{\ln 10} \sim \frac{-2.5x}{2.3} \sim -1.08x$$
error in magnitudes is approximately equal to fractional error in flux
Errors in Magnitude

Question: A photometric measurement of a star gives you a magnitude of $12.0 \pm 0.1$ magnitudes. What is the uncertainty in flux (as a percentage)?
Errors in Magnitude

Question: A photometric measurement of a star gives you a magnitude of 12.0 ± 0.1 magnitudes. What is the uncertainty in flux (as a percentage)?

Answer: About 10% (more accurately 1.08x0.1)
Scaling relations

Sometimes the difference between results under different approximations or assumptions takes the form of common function of some key physical parameters times different factors that are independent of these parameters. In this case the cruder approximation gives us a useful scaling relation.

Example

Mass density at center of uniform sphere with mass $M$ and radius $R$:

$$\rho_0 = \frac{M}{V} = \frac{3}{4\pi} \frac{M}{R^3}$$
Scaling relations (continued)

Mass density at center of sphere with mass $M$ and density that varies according to $\rho(r) = \rho_0 e^{-r/R}$:

\[
M = \int \rho dV = \int \rho(r) 4\pi r^2 dr = 4\pi \rho_0 \int r^2 e^{-r/R} dr
\]

\[
= 4\pi \rho_0 R^3 \int_0^\infty u^2 e^{-u} du
\]

Integrate by parts twice...

\[
= 8\pi \rho_0 R^3 \quad ; \quad \text{that is,}
\]

\[
\rho_0 = \frac{1}{8\pi} \frac{M}{R^3}.
\]
Scaling relations

It looks as if the mass density at the center of the sphere would always have the form

\[ \rho_0 = \left[ \text{factor independent of mass and radius} \right] \times \frac{M}{R^3} \]

no matter what the details of the density. Common astrophysical nomenclature:

\[ \rho_0 \propto \frac{M}{R^3} \]

(“central density is proportional to, or scales with, \( M/R^3 \)”). Here the scaling factor is unitless and likely to be of order unity.
Scaling relations (concluded)

Why is this scaling relation useful?

- Consider that it means that, even if you don’t know how the density actually varies with radius in this sphere, you would know that the central density probably changes by a factor of 8 if the radius of the sphere changes by a factor of 2, and that the central density probably changes by a factor of 2 if the mass changes by a factor of 2.

Characteristic scales

- Note that the sphere doesn’t have a sharp edge in the exponential-density case. Thus \( R \) is not “the” radius of the sphere in this case; it is a radius characteristic of the material in the sphere.
Scaling relations
– in context with known quantities

It is often useful to peg a scaling relation with a quantity that is known. For example: the period of a planet at radius $r$ rotating around a star of mass $M$ is

$$P = 2\pi \sqrt{\frac{r^3}{GM}} \propto M^{-1/2} r^{3/2}$$

It is often useful to write it this like

$$P = \text{1 year} \left( \frac{M}{M_\odot} \right)^{-1/2} \left( \frac{r}{1\text{AU}} \right)^{3/2}$$

Only scaling information and a known value are needed to write a scaling relation in this form.
Order of Magnitude Estimates

Often in astronomy you to know how large is a physical quantity so you can estimate:

• What physical forces are going to be important?
• Is your effect detectable?
• Approximately what does it depend on? (can you test your physical model with a scaling law?)
Example 1: How fast do nearby stars move on the sky?

First we need to figure out how to describe angles on the sky.
Consider a star $D=10\text{pc}$ away

How big is 1" (1 arcsecond) in cm at this distance?

The angle on the sky (in radians) is how far it moved divided by the distance.

One arcsecond, $1'' = \pi/180\text{deg}/60\text{min}/60''$

$= 5 \times 10^{-6}$ radians.

For a star at $D=10\text{pc} = 3 \times 10^{19}\text{cm}$, 1" corresponds to

$r = D\theta = 3 \times 10^{19} \times 5 \times 10^{-6} = 15 \times 10^{13}\text{cm}$

$= 10\text{AU}$
Example 1: How fast do nearby stars move on the sky? (continued)

Consider the star $D=10\text{pc}$ away moving at $v=10\text{km/s}$ with respect to the sun. $1\text{year} \sim 3 \times 10^7 \text{seconds}$ so the star moves at velocity

$$v = \frac{10\text{km}}{s} \times \frac{10^5 \text{cm}}{\text{km}} \times \frac{3 \times 10^7 \text{s}}{\text{yr}} = 3 \times 10^{13} \text{cm/yr}$$

The angle on the sky (in radians) is how far it moved divided by the distance.

$$r = D \theta \text{ so } v = D \frac{d\theta}{dt} \text{ and } \frac{d\theta}{dt} = \frac{v}{D}$$

$D=10\text{pc} = 3 \times 10^{19} \text{cm}$.

$v=3 \times 10^{13} \text{ cm/yr}/3 \times 10^{19} \text{ cm} = 1 \times 10^{-6} \text{ radians/yr}$

The star moves a solid angle $1 \times 10^{-6} \text{ radians/yr}$

One arcsecond, $1'' = \pi/180\text{deg}/60\text{min}/60'' = 5 \times 10^{-6} \text{ radians}$.

So the star moves about $0.2''/\text{year}$

This is detectable in a few years.
Motions of nearby stars (continued)

\[ r = D \theta \quad \text{so} \quad v = D \frac{d\theta}{dt} \quad \rightarrow \quad \frac{d\theta}{dt} = \frac{v}{D} \]

The angular motion is proportional to the velocity and inversely proportional to the distance.

We refer to our calculation and write

\[ \frac{d\theta}{dt} \sim 0.2"/\text{year} \left( \frac{v}{10 \text{km/s}} \right) \left( \frac{10 \text{pc}}{D} \right) \]

High proper motion stars are either nearby, or are fast moving halo stars.

Searches for high proper motion stars tend to discover the nearest stars.
High Proper Motion Stars

Barnard’s Star
Dimensional Analysis

You guess at which physical parameters are important. Figure out all the ways to combine them to predict size or time scales.

For example: A supernova goes off with energy $E=10^{51}$ ergs into the interstellar medium (primarily Hydrogen) of particle density $n=1\text{cm}^{-3}$. What is the size of the remnant after $t$ million years?
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\[
E \text{ has units of g cm}^2 \text{ s}^{-2}
\]
\[
\frac{E}{\rho} \text{ has units of cm}^5 \text{ s}^{-2} \text{ where } \rho \text{ is the density}
\]
\[
R = \left( \frac{E}{\rho} \right)^{1/5} t^{2/5} = 70 \text{ pc} \left( \frac{E}{10^{51} \text{ erg}} \right)^{1/5} \left( \frac{n}{1 \text{ cm}^{-3}} \right)^{-1/5} \left( \frac{t}{\text{ Myr}} \right)^{2/5}
\]

the Sedov solution was applied to atomic bombs, but can also be used to estimate sizes of supernova remnants.
Averages

Astrophysicists frequently cannot measure certain important parameters of a system, but may know how those parameters are distributed in a population of such systems.

- In such cases they may use average values of those parameters.

- By “distribution of parameter $x$” we mean the probability $p(x)$ that a $x$ has a certain value, as a function of $x$.

  - That parameter has to have some value, so if $p$ is a continuous function of $x$, and $x$ can range over values from $a$ to $b > a$, it is normalized:

$$\int_{a}^{b} p(x)dx = 1 = 100\%$$
Averages (continued)

- Then the average value of \( x \) is given by

\[
\langle x \rangle = \int_a^b x p(x) \, dx
\]

- In general the average value of any function of \( x \), say \( f(x) \), is given similarly:

\[
\langle f(x) \rangle = \int_a^b f(x) p(x) \, dx
\]

- This is very similar to the sort of average about which you will learn in statistical mechanics, and will learn to call an ensemble average.
Example. In binary star systems, astronomers can measure the component of orbital velocity along the line of sight for each star.

If the orientation of such a system were known

The inclination angle $i$ of the system’s axis with respect to the line of sight – such measurements would enable the stars’ masses to be measured.

\[ v_r = v \cos \left( \frac{\pi}{2} - i \right) \]
\[ = v \sin i \]
Averages (continued)

- However, if the stars are too close together to observe separately, the orientation cannot be determined.

- Still, an estimate of $\sin i$ could lead to a useful estimate of the masses. What would be a reasonable estimate of $\sin i$? Let’s try an average value, assuming that binary-star orbits in general are uniformly distributed.

  • That is, all orientations are equally probable.

  $$p(\theta, \phi) = \frac{1}{4\pi}$$

  $$\int p(\theta, \phi) d\Omega = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta p(\theta, \phi) = 1$$

  $$\langle \sin i \rangle = \int p(\theta, \phi) \sin \theta d\Omega = \int_{0}^{\pi} \frac{d\Omega}{4\pi} \sin \theta = \int_{0}^{\pi} \frac{1}{2} \sin^2 \theta d\theta$$

  $$= \int_{0}^{\pi} \frac{1}{2} \left[ 1 - \cos 2\theta \right] d\theta = \frac{\pi}{4} = 0.785$$
Averages (continued)

\[ \langle \sin i \rangle = 0.78 \]

Typical value of \( \sin i \) measured is only 0.8 lower than an edge on system. Planetary masses estimated through Doppler surveys are low. But face on systems are actually unlikely.
Units and notation

1 pc is \( \sim 3 \times 10^{18} \) cm\( \sim 3 \) light years
1 AU is \( \sim 1.5 \times 10^{13} \) cm
1” is about \( \sim 5 \times 10^{-6} \) radians
MAS is often used to denote milli-arcsecond
Proper motion is how an object moves on the sky.

Concepts:
Scaling laws
First order approximations
Order of magnitude estimates
Dimensional analysis
Averaging