Problem Set #9, AST111
Solutions

1) *PS 7.1. Consider a comet on an elliptical orbit about the Sun. The perihelion distance is 0.3AU and the aphelion distance is 15AU. Assume a solar wind velocity of 400 km/s.*

a) *Compute the tangential (non radial) component of the velocity v_θ at perihelion and at aphelion. Hint: see problem set #2 for derivation of the velocity at perihelion and aphelion.*

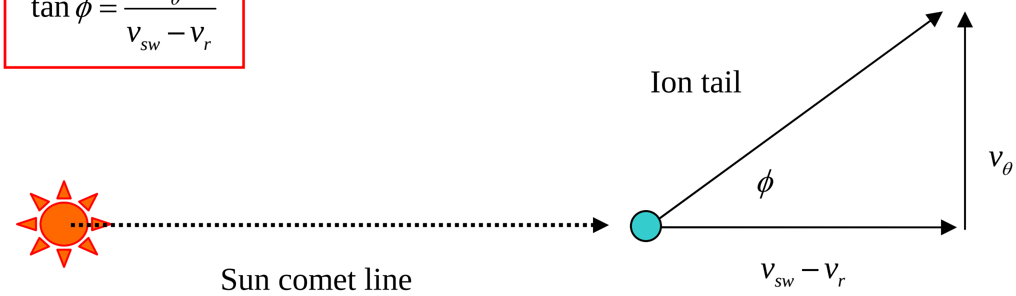
b) *Calculate the angle between the ion tail and the Sun-comet line at perihelion and at aphelion.*

a) At aphelion $v_\theta^2 = \left(\frac{GM}{a}\right)\left(\frac{1-e}{1+e}\right)$ (see problem set #2 for the derivation of this). and at perihelion $v_\theta^2 = \left(\frac{GM}{a}\right)\left(\frac{1+e}{1-e}\right)$. We scale both from the circular velocity $v_c^2 = \left(\frac{GM}{a}\right)$. The distance of perihelion $a(1-e) = 0.3\text{AU}$ and that of aphelion is $a(1+e) = 15\text{AU}$ so the semi-major axis is $a = \frac{1}{2}(0.3+15) = 7.65\text{AU}$ and the comet's eccentricity is $e = \frac{r_{ap} - r_{peri}}{2a} = \frac{15-0.3}{2\times 7.65} = 0.961$. Now we plug in our values for *a*,*e* into the velocity estimates. $\sqrt{\frac{1+e}{1-e}} = 7.1$

$v_c = \sqrt{\frac{GM}{a}} = 30\text{km/s}\left(\frac{1\text{AU}}{a}\right)^{1/2} = 10.8\text{km/s}$. The velocity at perihelion is then 7.1 times this or 77 km/s and that at aphelion is 1/7.1 times this or 1.5km/s.

b) We use equation 7.1 from the book; $\tan\phi = \frac{v_\theta}{v_{sw} - v_r}$. At perihelion and aphelion the radial velocity component is zero, so the angle can be estimated from the ratio of the solar wind speed and the tangential component. We find $\tan\phi = 77/400 = 0.19$ or $\phi = 11^o$ at perihelion and $\tan\phi = 0.004$ or 2^o at aphelion.

$$\tan\phi = \frac{v_\theta}{v_{sw} - v_r}$$



2) *On the size of magnetospheres. (See Problem 7.4 of PS).*

a) *Estimate the radius of the Earth's magnetopause (in units of Earth's radius), assuming a quiescent solar wind near Earth of number density* n_{sw}~10 protons cm^{-3}*, and speed* v_{sw} ~ 300 km/s, . *Approximate the Earth's magnetic field is approximately a dipole with an equatorial surface magnetic field of 0.3Gauss. A balance in magnetic pressure versus ram pressure from the solar wind leads to the approximate equality* $\rho_{sw} v_{sw}^{\ 2} \sim \frac{B(r)^2}{8\pi}$.

b) *Compare the solar wind velocity to the escape velocity from the solar system at the position of the Earth. Is the solar wind slowing down significantly as it flows to larger distances from the Sun? Hint: consider the energy of a particle with a velocity of the Solar wind.*

c) *Using conservation of mass in the solar wind, what typical number density would the quiescent solar wind have at the radius of Jupiter (5.2AU)? Hint: Consider the mass flow out of a sphere of radius r.*

d) *Jupiter's dipole equatorial magnetic field at its surface is 4.3Gaus. Estimate the size of Jupiter's magnetopause. Hint: You can do this by scaling from your result in part a).*

a) For a dipole $B \propto r^{-3}$. So we can assume that $B(r) \sim 0.3\text{Gaus}\left(\frac{r_\oplus}{r}\right)^3$ where 0.3Gaus is the equatorial magnetic field on the surface. If I keep all units in cgs then B is in Gauss. To convert the number density into a density I need to multiply the number density by the mass of a proton or $1.7\times10^{-24}\text{g}$. Inserting the magnetic field as a function of radius from the Earth's center into the pressure balance equation leads to

$$\left(\frac{r}{r_\oplus}\right) = \left[\frac{B_{eq}^{\ 2}}{8\pi\rho_{sw} v_{sw}^{\ 2}}\right]^{1/6} = \left[\frac{0.3^2}{8\pi\times10\times1.7\times10^{-24}\times(300\times10^5)^2}\right]^{1/6} = 8$$

. Typical size of Earth's magnetosphere is 10 Earth radii.

b) The escape velocity from the location of Earth's orbit is $v_{escape} = \sqrt{\frac{2GM_\odot}{r = 1\text{AU}}} = 42\text{km/s}$. This is far below the solar wind speed. Consider the energy of a particle with the velocity of the solar wind. $E/m = \frac{v_r^{\ 2}}{2} - \frac{GM}{r} = \frac{v_r^{\ 2}}{2} + \frac{v^2_{\ escape}(r)}{2}$. The square of the escape velocity is much much less than the velocity of the particle. As the particle moves outward the change in potential energy is insignificant. So the particle will not slow down much as it moves outward.

c) The mass flow out of a sphere of radius *r* is $\frac{dM}{dt} = 4\pi r^2 \rho v$. This is the mass flux through the sphere of area $4\pi r^2$. If the mass flux is constant with radius (no new sources of mass) and the wind is spherically symmetric then we find that $\rho \propto r^{-2}$. Scaling from the quiescent solar wind value the number density at Jupiter would be

$$n_{jupiter} = 10\text{protons cm}^{-3} \times \left(\frac{1\text{AU}}{5.2\text{AU}}\right)^2 = 0.4 \text{ protons cm}^{-3}.$$

d) The size of the magnetosphere depends on the equatorial magnetic field to the 1/3rd power and the density to the minus 1/6th power. $\frac{r}{r_p} = 8\left(\frac{4.3}{0.3}\right)^{1/3}\left(\frac{0.4}{10}\right)^{-1/6} = 33$. Jupiter's magnetosphere is about 30 times its radius.

3) *On the gyroradius. The gyro or Larmor radius of a charge particle with charge q, mass m, is* $R_L = \frac{cp_\perp}{qB}$ *where B is the magnitude of the magnetic field and* $p_\perp$ *is the momentum component perpendicular to the field. Taking into account relativistic effects the energy of a particle* $E = \gamma mc^2$ *and momentum* $p = \gamma vm$ *where* $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. *Here v is the velocity in our frame, m is the rest mass and c is the speed of light. The convention for specifying energies of non-relativistic particles is that they are often given as energy minus the rest mass energy (*mc^2*). This gives you the kinetic energy in the non-relativistic limit. Note energies are often given in eV, where* $1\text{eV}=1.6\times10^{-12}\text{erg}$. *The rest mass energy (*mc^2*) of an electron is about 0.5MeV and that of a proton is about 2000 times larger or 1GeV. The charge of the electron is* $q = 4.8\times10^{-10}\text{esu}$.

a) *What is the gyro or Larmor radius for a 100keV electron in the Earth's inner radiation belt (where the magnetic field strength is about 0.15 Gaus)? Hint: this particle is not relativistic so the energy is the kinetic energy.*

b) *When the gyro radius of a charged particle exceeds the size of the system, the particle is no longer trapped by the magnetic field. Above what energy in eV is a cosmic ray proton not trapped by the Earth's magnetosphere? This particle will be relativistic, so you will need to use the relativistic formula for energy. The Earth's radius is* $R_\oplus = 6.4\times10^8\text{cm}$.

a) Since 100kev < 0.5 Mev (the rest mass of an electron) the energy given is the kinetic energy. $E_k = \frac{p^2}{2m}$ so the momentum $p = \sqrt{2mE_k}$. We insert this into the equation for the gyro radius

$$R_L = \frac{cp}{qB} = \frac{3\times10^{10}\text{cm/s}\sqrt{2\times1.7\times10^{-24}\text{g}/2000\times100\times10^{3}\text{ev}\times1.6\times10^{-12}\text{erg/ev}}}{4.8\times10^{-10}\text{esu}\times0.15\text{Gaus}} = 0.07\text{km}$$

b) If the particle is free then $R_L > 10R_\oplus$ where the size of the magnetosphere is about 10 Earth radii. For relativistic particles $v \sim c$ so $p \sim E/c$. We can write $R_L = \frac{cp}{qB} \sim \frac{E}{qB}$. Particles with energies above

$$E > 10R_\oplus qB \sim 10\times6.4\times10^{8}\text{cm}\times4.8\times10^{-10}\text{esu}\times0.15G \sim 0.46\text{erg}/1.6\times10^{-12}\text{erg/ev}\sim300\text{GeV}$$

will not be trapped.