When is star formation episodic? A delay differential equation "negative feedback" model

Alice C. Quillen[1] & Joss Bland-Hawthorn[2]
[1] *Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627*
[2] *Institute of Astronomy, School of Physics, University of Sydney, NSW 2006, Australia*
aquillen@pas.rochester.edu jbh@aao.gov.au

5 March 2008

ABSTRACT

We introduce a differential equation for star formation in galaxies that incorporates negative feedback with a delay. When the feedback is instantaneous, solutions approach a self-limiting equilibrium state. When there is a delay, even though the feedback is negative, the solutions can exhibit cyclic and episodic solutions. We find that periodic or episodic star formation only occurs when two conditions are satisfied. Firstly the delay timescale must exceed a cloud consumption timescale. Secondly the feedback must be strong. This statement is quantitatively equivalent to requiring that the timescale to approach equilibrium be greater than approximately twice the cloud consumption timescale. The period of oscillations predicted is approximately 4 times the delay timescale. The amplitude of the oscillations increases with both feedback strength and delay time.

We discuss applications of the delay differential equation (DDE) model to star formation in galaxies using the cloud density as a variable. The DDE model is most applicable to systems that recycle gas and only slowly remove gas from the system. We propose likely delay mechanisms based on the requirement that the delay time is related to the observationally estimated time between episodic events. The proposed delay timescale accounting for episodic star formation in galaxy centers on periods similar to $P \sim 10$ Myrs, irregular galaxies with $P \sim 100$ Myrs, and the Milky Way disk with $P \sim 2$ Gyr, could be that for exciting turbulence following creation of massive stars, that for gas pushed into the halo to return and interact with the disk and that for spiral density wave evolution, respectively.

1 INTRODUCTION

Gas present in a galaxy fuels star formation or nuclear black hole growth. However both star formation and active galactic nuclei then release energy and momentum into the interstellar medium (ISM). Consequently the activity can suppress subsequent star formation. The process in which part of the output of a system is returned to its input and influences its further output is termed "feedback." Early studies showed that when feedback by radiative heating is taken into account during gas accretion onto a central mass, steady solutions may not exist (Ostriker et al. 1976) and the feedback process can cause oscillations or periodic bursts of accretion (Cowie et al. 1978; Scalo & Struck-Marcell 1986; Parravano 1996; Kamaya 2005). Simulations taking into account feedback processes illustrate that gas flows and star formation in galaxies can exhibit episodic or cyclic behavior (Dong et al. 2003; Pelupessy et al. 2004; Ciotti & Ostriker 2007; Stinson et al. 2007) or alternatively can asymptotically converge onto a self-regulated equilibrium state (Andersen & Burkert 2000; Robertson & Kravtsov 2008).

Galaxies display complex star formation histories. Studies of irregular galaxy populations (e.g., Tosi et al. 1991; Smecker-Hane et al. 1994; Dohm-Palmer et al. 2002; Dolphin et al. 2003; Skillman 2005; Young et al. 2007; Dellenbusch et al. 2008), the Milky Way disk (Rocha-Pinto et al. 2000a), galaxy centers (Bland-Hawthorn & Cohen 2003; Walcher et al. 2006; Cecil et al. 2001) and the statistics of distant galaxies (Glazebrook et al. 1999) infer that multiple events of vigorous star formation, separated by millions to billions of years, can occur even in isolated galactic systems. Other studies (e.g., van Zee 2001; Skillman 2005) find little evidence for episodic star formation. However, theoretical work has primarily focused on self-regulated star formation (Struck & Smith 1999; Andersen & Burkert 2000; Silk 2001; Elmegreen 2002; Monaco 2004; Krumholz et al. 2006; Slyz et al. 2005; Li et al. 2006; Dib et al. 2006; Joung & Mac Low 2006; Elmegreen 2007; Booth et al. 2007; Wada & Norman 2007; Schaye & Dalla Vecchia 2007; Robertson & Kravtsov 2008) and has not explored when episodic rather than a steady rate of star formation is expected (though see Scalo & Struck-Marcell 1986; Parravano 1996; Dong et al. 2003; Kamaya 2005).

As gas flows involving energy input, heating and cooling are complex, there is no simple way to predict when behavior is episodic or cyclic. However it is possible that average quantities can be estimated for these flows and relations based on these quantities can be used to classify their behavior. Delay differential equations can exhibit solutions that asymptotically approach a self-limiting equilibrium state and those that are periodic, even when feedback is negative. Consequently these equations can be used to differentiate between these two behaviors. Delay differential equations have been used to model biological systems with delayed negative feedback (e.g., Wazewska-Czyzewska & Lasota 1988; Györi & Ladas 1991; Gurney et al. 1980; Kulenovic et al. 1989) but have seldom been applied to astrophysical systems. Previous work using coupled differential equations for stars and the different phases of the interstellar medium found that star formation is episodic only when the delay time between cloud formation and destruction exceeds the cloud collision or condensation timescale (Scalo & Struck-Marcell 1986; Parravano 1996). In this paper, using a delay differential equation, we determine when cyclic or periodic behavior is exhibited by the solutions rather than a smooth decay to a self-regulated steady state. We apply the theory to star forming galaxies, identifying delay mechanisms that could account for episodic accretion events inferred from observations.

2 ONE DIMENSIONAL FEEDBACK MODELS

We begin by considering a galactic disk model with cloud surface density $\Sigma(t)$ in units of mass per unit area that depends on time, t. This density could represent the total disk gas, or the gas in self gravitating clouds, or the gas in molecular form, depending upon the setting. The gas density available for star formation decreases when clouds are dispersed following star formation. Conversely, the gas density increases during accretion, coagulation or cooling, all of which can enhance star formation. We therefore write

$$\dot{\Sigma}(t) = g(\Sigma, t) - h(\Sigma, t)$$

where $\dot{\Sigma} = d\Sigma/dt$. Here the function $g(\Sigma, t)$ is the accretion or cloud formation rate. The star formation rate should depend on the cloud density variable, Σ and this should determine the rate that clouds are dispersed or disrupted. A Schmidt type law (Schmidt 1959; Kennicutt 1998) relates the cloud dispersal rate to the disk density variable with the function $h(\Sigma)$. In principle, the accretion rate also depends on time in a non-trivial manner. For example, it could depend on the previous cloud surface density and associated star formation rate. We do not expect feedback to be instantaneous as it takes millions of years for a burst of star formation to produce type II supernovae, and winds and supernova remnants require time to evacuate gas or induce turbulence in a gas disk. Following a burst of star formation, accretion onto the disk would not resume until heated, evacuated or dispersed gas has had time to cool and reform into clouds.

Before introducing complicated functions for the accretion rate, we first consider the simplest case, that lacking any feedback, $g(x, t) = A$, corresponding to a constant accretion or cloud formation rate. The above differential equation can be written

$$\dot{x} = f(x) = A - Bx^{\alpha} \tag{1}$$

where we have replaced Σ with the variable x, use a Schmidt type law (Schmidt 1959) for the cloud destruction or dispersal rate with positive power index α, and A and B are positive constants. By setting $dx/dt = 0$ and solving for x we find a fixed point, x_*, corresponding to the self-regulated steady state or equilibrium value at $x_* = (A/B)^{1/\alpha}$. We can assess the nature of solutions by taking the derivative of the right hand side with respect to x; or $\frac{df}{dx} = -B\alpha x^{\alpha-1}$. This derivative is always negative and is smoothly decreasing function implying that solutions always smoothly (asymptotically) approach the equilibrium state solution on a timescale determined by the inverse of this derivative. There are no oscillating or divergent solutions.

2.1 Instantaneous Feedback

The case of instantaneous feedback can be modeled with the assumption that the accretion rate is affected by the current star formation rate, which in turn is set by the density of the disk. We expect that feedback would occur by reducing the quantity of gas available for star formation in the disk when the star formation rate is high. Since the gas quantity available to form stars is reduced by the star formation process, the feedback is negative. We can describe this situation with an accretion rate $g(x) = AG(x)$, where $G(x)$ is function that approaches unity when x is small and there is no feedback, and drops to zero when x is large, star formation is vigorous and the energy arising from it has prevented further accretion or cloud formation. A simple form for the function G that satisfies our requirements is $G(x) = e^{-x/C}$ for which $C > 0$. The parameter C depends on the cloud density and associated star formation rate that is effective at cutting off accretion or cloud formation.

The evolution of the disk density is then described by

$$\dot{x} = f(x) = g(x) - h(x) = Ae^{-x/C} - Bx^{\alpha}. \tag{2}$$

The equilibrium state can be found by solving $\dot{x} = 0$ for x and satisfies

$$x_*^{\alpha} = \frac{A}{B} e^{-x_*/C}. \tag{3}$$

The derivative of f

$$\frac{df}{dx} = -B\alpha x^{\alpha-1} - AC^{-1}e^{-x/C}.$$

Since $A, B, C, \alpha > 0$ the derivative is always negative and solutions always smoothly approach the equilibrium state. Because the feedback is negative there is no instability, and no periodic or cyclic solutions exist. Solutions to this equation resemble those that do not oscillate shown in in Figure 1.

It is useful to define two timescales, a consumption timescale dependent only on the second term of equation 2 and evaluated at x_*

$$t_{con} \equiv \left| \frac{x_*}{h(x_*)} \right| = \frac{1}{Bx_*^{\alpha-1}}, \tag{4}$$

and the timescale to approach equilibrium, t_{eq}, that depends on the derivative of f evaluated at x_*;

$$t_{eq} \equiv \left| \frac{df}{dx} \right|_{x_*}^{-1} = t_{con} \left(\alpha + x_*/C \right)^{-1}. \tag{5}$$

It is also useful to quantify the strength of the feedback near the equilibrium point from the sensitivity of the accretion term $g(x)$ or

$$S \equiv \left| \frac{dg}{dx} \frac{x}{g(x)} \right|_{x_*} = \frac{x_*}{C}. \tag{6}$$

The above "strength parameter" is large when the feedback is strong.

2.2 Delayed feedback

When feedback is delayed the current accretion rate is reduced by the cloud density and associated star formation rate at an earlier time $t - \tau$ where τ is the delay timescale. The accretion rate is $g(x(t-\tau))$ and the model described by equation 2 becomes

$$\dot{x}(t) = f(x,t) = Ae^{-x(t-\tau)/C} - Bx(t)^{\alpha}. \tag{7}$$

Terms of this form were considered by Parravano (1996); Dong et al. (2003). In the limit of $\tau \to 0$ we recover equation 2 for instantaneous feedback. The above differential equation belongs to the class of one dimensional equations with delayed negative feedback which includes the delayed logistic equation, the model used by Gurney et al. (1980) to describe the dynamics of Nicholson's blowflies and the Lasota-Wazewska model for the survival of red blood cells (Wazewska-Czyzewska & Lasota 1988).

Even though the feedback is negative, the above differential equation has oscillating solutions but not for all values of the 4 positive parameters A, B, C, τ and index α. Figure 1 shows two solutions that converge to a periodic solution that oscillates about the equilibrium state forever and one that oscillates while decaying to the equilibrium state. The equilibrium state for this differential equation is identical to that for the equivalent model lacking delay and is a solution of equation 3. To display solutions we directly integrated Equation 7 with a first order or Euler method. We allow the delayed feedback to initiate only at times $t > t_0 + \tau$ for initial time t_0.

For non-extreme values of initial conditions and parameters, the solutions exhibit 3 types of behavior:

(i) The solutions lack oscillations. After some time period, solutions smoothly or asymptotically approach the stable equilibrium state.
(ii) The solutions exhibit oscillations about an equilibrium state but asymptotically approach that state.
(iii) The solutions oscillate and are attracted to a periodic function or cycle.

When oscillating solutions are present, the oscillation period is approximately four times the delay timescale or $P \sim 4\tau$. Roughly speaking, this follows by considering the equation $\dot{x}(t) = -\omega x(t-\tau)$ that has the solution $x(t) = \sin(\omega t)$ with a period of 4τ if $\omega\tau = \pi/2$. From figure 1 we see that the actual period displayed by the oscillating solutions is approximately 5τ. This is broadly consistent with the approximate estimate for the period of 4τ.

To facilitate classification of solutions, we transform equation 7 into dimensionless form. We let a dimensionless density variable $y = x/x_*$ and time variable $T = t/t_{con}$ with the depletion or consumption timescale defined in equation 4. Using these new variables, equation 7 becomes

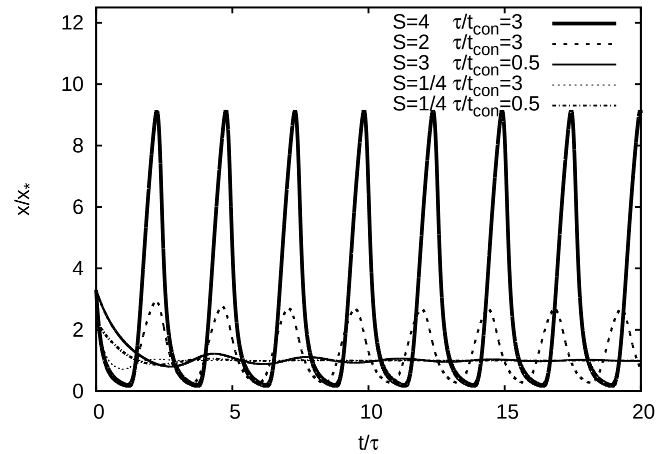

Figure 1. We show the results of integrating equation 7 for different dimensionless ratios $S = x_*/C$ and $\bar{\tau} = \tau/t_{con}$. The power index is $\alpha = 1.4$. The solution with high values of these $\bar{\tau}$ and S is strongly non-sinusoidal and approaches a high amplitude periodic solution (thick solid line). At lower values of these parameters a lower amplitude but periodic solution is approached (thick dashed line). For even lower values there is an oscillating solution that decays to the equilibrium value (solid thin line). At the lowest values of $\bar{\tau}$ and S the solution asymptotically approaches the equilibrium value (dotted thin line and dot dashed line). The ratios are listed in the plot key. The oscillation period is approximately five times the delay timescale. The y-axis is x divided by that of equilibrium state, x_*. The x-axis is time divided by the delay time, τ.

$$\frac{dy}{dT} = e^{(1-y(T-\bar{\tau}))S} - y^{\alpha} \tag{8}$$

where the dimensionless parameters are

$$\bar{\tau} \equiv \frac{\tau}{t_{con}} \qquad S \equiv x_*/C, \tag{9}$$

and we have used equation 6 for the feedback strength S. For a given exponent, α, equation 8 only depends on two parameters, $\bar{\tau}$ and S, thus solutions of this equation can be classified based on estimates for these two ratios alone.

In the case of power index $\alpha = 1$, the above DDE (equation 7) is the same as the model used to describe survival of red blood cells by Wazewska-Czyzewska & Lasota (1988). Initially positive solutions of the dimensionless version (equation 8) oscillate about the equilibrium value, $y_* = 1$, if and only if

$$\bar{\tau} S e^{(\bar{\tau}+1)} > 1, \tag{10}$$

(Kulenovic & Ladas 1987; Györi & Ladas 1991). The equilibrium value is a global attractor (solutions approach this value) when

$$S(1 - e^{-\bar{\tau}}) < \ln 2 \tag{11}$$

(Kulenovic et al. 1989; Györi & Ladas 1991). If this condition is not satisfied, a periodic oscillating attractor may exist.

A more generalized oscillation criterion that can be used when $\alpha \neq 1$ is

$$M_1 \equiv \bar{\tau} S e^{(\alpha\bar{\tau}+1)} > 1,$$

where we have defined M_1 as a parameter describing the nature of the DDE. We have derived this criterion in the appendix using the procedures rigorously described by Györi & Ladas (1991) and following the example given for the Lasota-Wazewska model. By analogy with equation 11 we guess that y_* is a global attractor when

$$M_2 \equiv S(1 - e^{-\alpha\bar{\tau}}) < \ln 2.$$

The parameters $M_1, M_2 > 1$ when

$$\bar{\tau} = \tau/t_{con} \gtrsim 1 \qquad \text{and} \qquad S \gtrsim 1. \tag{12}$$

Using equation 5 we find that the second of these conditions is equivalent to a constraint on the timescale to reach equilibrium

$$t_{con}/t_{eq} \gtrsim 1 + \alpha.$$

Thus when two conditions are satisfied we predict periodic solutions:

(i) The delay timescale exceeds the consumption timescale.
(ii) Feedback is strong and effective at shutting off accretion or cloud formation near the equilibrium density. Equivalently the timescale to approach equilibrium exceeds the consumption timescale by a factor similar to 2.

The first of these conditions is similar to that found by Scalo & Struck-Marcell (1986) for two coupled delay differential equations based on the Oort model for molecular cloud collisions. Previous work has not found an additional requirement for episodic behavior dependent on the feedback strength.

We consider how the amplitude of oscillations depends on the parameters. We describe the amplitude of oscillations as the ratio of the maximum x divided by the minimum in an oscillation period, after the system has converged to a cycle. Oscillation amplitudes are shown in Figure 2 as a function of $\bar{\tau}$ and strength S, for index $\alpha = 1$ and $\alpha = 1.5$. The conditions for periodic behavior estimated in equation 12 are consistent with the lowest contours in Figure 2 that separate between solutions that decay to the equilibrium state and those that are periodic. As expected from the form of $M1, M2$, this division occurs at a higher strength when $\alpha = 1.5$ than for $\alpha = 1.0$.

The further away from the line dividing asymptotically decaying solutions from those with periodic solutions, the larger the oscillations about the equilibrium value. The amplitude of the oscillations does not depend on the initial conditions but rather on the parameters defining the differential equation. Since x cannot cross zero when large oscillations are present, the periodic solutions are less symmetric or less like sinusoids but exhibit spikes followed by longer periods of low periods of accretion when the amplitudes are high (see figure 1). This follows as the accretion rate depends on the exponential of x so when x is high, it can take a long time for the system to recover from a previous episode of star formation.

3 APPLICATIONS OF THE ONE DIMENSIONAL MODEL TO GALAXIES

Our DDE model for delayed feedback is appropriate if the mean gas density averaged over long periods of time is nearly

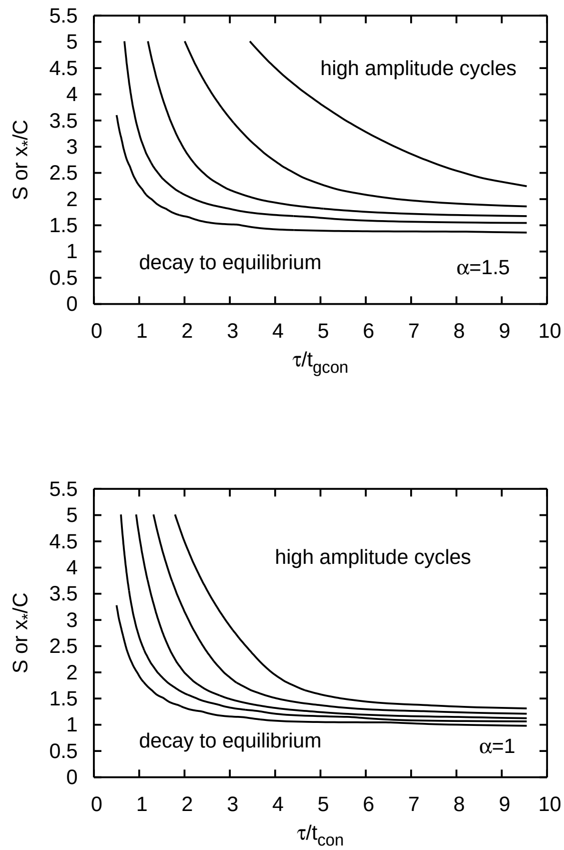

Figure 2. Amplitude contours are shown for solutions of equation 7 after the solution has decayed either to an equilibrium value or a periodic cycle. The amplitude is the maximum divided by the minimum value of x during the cycle. The lowest contour divides solutions that converge to a periodic function from those that decay to an equilibrium value. This contour has the amplitude value 1.05. The remaining contours are evenly spaced in the log with amplitude values 3.16, 10.0, 31.6, and 100 (second from bottom to top contour). For the solutions to exhibit periodic behavior we find that the ratio of the delay to consumption timescales $\tau/t_{con} \gtrsim 1$ and feedback strength $S = x_*/C \gtrsim 1$. The amplitude of the cycles increases with increasing S and τ. a) For index $\alpha = 1.5$ b) For index $\alpha = 1.0$

constant. This follows because the form we have for the accretion or cloud formation rate does not change, though it does depend on the past disk density. The DDE model is best applied to systems that recycle gas and only slowly remove gas from the system. Star formation laws illustrate that star formation is inefficient. For example, Kennicutt (1998) found that star formation rates in nearby galaxies could be described with $\dot{\Sigma} \sim \epsilon\Sigma\Omega$, where Ω is the angular rotation rate and the efficiency is low, $\epsilon \sim 0.017$ (Kennicutt 1998). This suggests that we should not adopt as our defining variable the total density in molecular and atomic gas but rather that in molecular clouds or self-gravitating clouds as adopted in explanations for the Schmidt-Kennicutt star formation law and observational studies of molecular gas in galaxies (Gao & Solomon 2004; Wu et al. 2005; Krumholz et al. 2006; Blitz

& Rosolowsky 2006; Robertson & Kravtsov 2008). In this case the DDE tracks cloud formation and cloud disruption following star formation. The accretion term of the DDE describes how cloud formation is depressed by star formation associated with the past cloud density.

Molecular clouds are estimated to last $t_C \sim 2-3 \times 10^7$yrs and are disrupted following star formation (Blitz et al. 2007). Theoretical work suggests that clouds disperse after a few times their free fall or dynamical timescale (Krumholz & McKee 2005) so lifetimes of star forming clouds could be shorter in denser environments (Wada & Norman 2007), such as circumnuclear disks. The depletion term in equation 7 has $B = t_C^{-1}$ and index $\alpha = 1$, so the consumption timescale in our model is the mean cloud lifetime, $t_{con} \sim t_C$.

3.1 Possible Delay Timescales

For star formation to be maintained in a disk, clouds must constantly reform. Enhanced turbulence in the disk should reduce the star formation rate (e.g., Struck & Smith 1999; Silk 2001; de Avillez, & Breitschwerdt 2004; Li et al. 2006). Turbulence increases the disk thickness reducing the mean density and increasing the mean free fall timescale (cf., Silk 2001; Krumholz & McKee 2005; Dib et al. 2006; Joung & Mac Low 2006; McKee & Ostriker 2007; Robertson & Kravtsov 2008). The primary energy source for the turbulence is expected to be from supernovae, though differential rotation, gravitational and magnetic instabilities and stellar outflows could also play a role (Silk 2001; Kim et al. 2003; Quillen et al. 2005; Piontek & Ostriker 2007; de Avillez & Breitschwerdt 2007). Thus a delay time for feedback is the sum of the time for massive stars to move off the main sequence and produce supernovae (a few times 10^6yr), the timescale for the supernova remnants to reach their maximum size (of order 10^7 yr but depending on the ambient pressure and density), and the timescale for them to be mixed into the disk (e.g., Dib et al. 2006). This last timescale is a turbulence mixing timescale, $t_{mix} \sim h/\sigma$, that depends on the gas disk thickness, h, and gas velocity dispersion, σ. The mixing timescale is similar to a few times 10^7 yrs in the solar neighborhood. Thus the delay time for a reduction in the rate of molecular cloud formation by turbulence in disks such as the Milky Way is a few times 10 Myrs and dominated by the timescale for mixing and supernova remnant expansion. (The timescale could be shorter if star formation is triggered by the rapid collapse of the evacuated region (~ 2 Myr) shortly after the hot gas escapes the disk.) Both turbulent mixing timescales and supernova remnant expansion timescales should be longer in the outskirts of galaxies and in irregular or dwarf galaxies where the densities and pressures are lower. In contrast, on the scales of circumnuclear disks (~ 10 pc), mixing and supernova remnant expansion timescales should be shorter due to the higher densities and pressures and larger velocity dispersions.

In spiral galaxies, molecular cloud formation occurs primarily in spiral arms so their formation is triggered on a timescale related to the spiral density wave pattern rather than on a timescale related to turbulent mixing of supernova remnants (e.g., Elmegreen 2007). A possible longer delay timescale is that for spiral density waves to evolve (e.g., Clarke & Gittins 2006). When the Toomre Q parameter is greater than 1.5, spiral structure is suppressed. Here $Q \equiv \frac{\sigma\kappa}{\pi G \Sigma_g}$ where κ is the epicyclic frequency and G the gravitational constant. The Q parameter is related to the gas freefall timescale (e.g., McKee & Ostriker 2007; Krumholz & McKee 2005) and so its value can be discussed in terms of a self-regulated star formation model. Spiral density waves are expected to grow on a timescale of a few rotation periods (Sellwood & Carlberg 1984; Vorobyov & Theis 2006; Clarke & Gittins 2006). Star formation not only influences the gaseous velocity dispersion but lowers the mean stellar velocity dispersion and increases the stellar mass density. Hence the current strength of spiral structure (set by Q) may depend on the star formation rate a few galactic rotation periods ago. In this setting the cloud formation rate would be forced by spiral arms sweeping through the disk with an oscillation period dependent on the spiral pattern speed and amplitude dependent on the strength of spiral structure. This amplitude would be the quantity that experiences the delayed feedback.

A third candidate for a delay timescale is that for material driven out of the disk to return and stir the disk. This could be influenced by a cooling timescale for hot and low density gas in the galactic halo. This timescale would be longer than the local disk turbulent mixing timescale and would be of order $10^8 - 10^9$ yrs. It may be related to the 100-200 Myr relaxation timescale exhibited by simulations (de Avillez, & Breitschwerdt 2004; Joung & Mac Low 2006; Stinson et al. 2007) but could also depend on the dark matter halo mass or density (as discussed in these works).

In summary, the relevant consumption timescale is the molecular cloud lifetime of order 10 Myrs but could be shorter in denser environments. For delay timescales we have three primary candidates: 1) The timescale for supernovae to enhance disk turbulence (a few times 10 Myrs but longer at lower densities and pressures). 2) The timescale for gas heated up and moved into the halo to cool back into and stir the disk (order $10^8 - 10^9$ yrs). 3) The timescale for spiral arms to evolve (a few times the rotation period). Future work may identify delay times associated with other processes such as magneto-gravitational instabilities, or internally generated stellar outflows. The delay timescale associated with disk turbulence may not exceed the cloud consumption timescale. However delay timescales associated with larger scale turbulence and cooling in the halo and spiral arm evolution are likely to exceed the cloud consumption timescale.

3.2 Delay mechanisms as suggested by observations

We now put these timescales in context with observations keeping in mind that the DDE (equation 7) displays episodic bursts only when the delay timescale is longer than the consumption timescale.

The survey by Rocha-Pinto et al. (2000a) reveals that star formation in the solar neighborhood experienced 3 bursts each separated by about 3 Gyrs. A delay timescale of one quarter of this or about 0.8 Gyr would be required to predict this periodicity with the DDE of equation 7. As spiral structure is responsible for molecular cloud formation in the solar neighborhood a possible delay mechanism is the timescale for spiral arms to evolve. The time 0.8 Gyr corresponds to 3 rotation periods at the solar circle. Clarke & Gittins (2006) have previously proposed that variations in spiral arm strength could affect the star formation rate. Here we couple the gas and stars, relying on feedback and a delay time but involving the same principle, that the spiral density waves are a strong trigger for star formation.

Surveys of galaxy centers have revealed that most late type and elliptical galaxies harbor circumnuclear star clusters (Böker et al. 2002; Koda et al. 2005; Coté et al. 2006; Christopher et al. 2005) and have experienced star formation in their nuclei in the past few to 100Myr (Veilleux et al. 1994; Bland-Hawthorn & Cohen 2003; Walcher et al. 2006; Quillen et al. 2006; Cecil et al. 2001). The sizes of these star clusters ranges from tens to a few hundred pc and gas densities of 10^3-$10^6 M_\odot$ pc^{-2}. Since the gas densities are high, cloud lifetimes should be shorter than that for molecular clouds in the Milky Way's disk or Local Group galaxies. Supernova remnant expansion and turbulent mixing timescales may be shorter than in the solar neighborhood due to higher pressures. However the timescale for stars to evolve must be similar in both settings. We expect episodic star formation with a period similar to a few times 10^7 years (set by stellar evolution of massive stars). This behavior would only occur when the timescale for excitation of turbulence in the disk, depending on the timescale for stars to produce winds, is longer than the lifetime of the star forming self-gravitating clouds.

Studies of irregular dwarf galaxies have revealed that they have complex star formation histories experiencing separated bursts of star formation separated by a hundred Myrs to Gyrs (e.g., Tosi et al. 1991; Dohm-Palmer et al. 2002; Dolphin et al. 2003; Skillman 2005; Young et al. 2007; Dellenbusch et al. 2008). Recent simulations (Pelupessy et al. 2004; Kamaya 2005; Stinson et al. 2007) have illustrated periodic bursts of star formation separated by 200-400 Myr. The simulations do not display strong spiral structure. The spiral structure mediated model proposed by Clarke & Gittins (2006) can account for bursts of star formation in dwarf galaxies, however this model cannot account for the bursts seen in these simulations as they lack spiral structure. According to our model, the delay timescale must be one quarter of the time between bursts or 50-100Myr. The supernova remnant expansion timescale for the galaxy simulated by Pelupessy et al. (2004) is similar to that of a supernova in the solar neighborhood as the interstellar medium pressures are similar. Likewise turbulent mixing timescales are similar. Hence the long inferred delay timescale must involve longer timescales such as for cooling of material in the halos of these galaxies and interactions between this cooling material and the disk.

In all three of these cases, it is likely that the delay timescale exceeds the cloud consumption timescale, one of the conditions for the DDE to exhibit cyclic solutions. We base our choices for the likely delay mechanism on the requirement that the delay time is related to the observational inferred timescale between episodic events. Thus we suspect that the relevant delay timescale accounting for episodic star formation in galaxy centers, irregular galaxies and the Milky Way disk could be that for exciting turbulence following creating of massive stars, that for gas pushed into the halo to return and interact with the disk and that for spiral density wave evolution, respectively. In all three cases, the total supply of gas is consumed only slowly leaving a reservoir for ongoing star formation. Since the feedback is delayed on a timescale that exceeds the cloud consumption timescale, recurrent and periodic star formation events could occur even though the feedback is negative.

3.3 Is the feedback strong enough?

We now discuss the second requirement for cyclic solutions, that feedback be effective at reducing the formation rate of molecular clouds. We have characterized the feedback strength, S, with a parameter defined in equation 6 that describes the change in cloud formation rate caused by a change in cloud density. Only when $S \gtrsim 1$ are the solutions to the DDE periodic in behavior. Consequently we need to estimate the change in the cloud formation rate (or star formation rate) caused by a small change in the mean gas density.

There are few studies that have considered the timescale for cloud formation (q.v. Padoan et al. 2006). More commonly, a density spectrum resulting from turbulence has been used to predict the number of clouds above a critical density. The star formation rate is estimated from this gas fraction divided by the dynamical timescale at that density (Elmegreen 2002; Kravtsov 2003; Krumholz & McKee 2005; Wada & Norman 2007). A nearly universal property of isothermal turbulent media in experimental and numerical simulation studies is that the cloud densities have a log normal density distribution (Warhaft 2000; Pumir 1994; Padoan & Nordlund 2002). We adopt this distribution[1] to estimate the strength parameter S in equation 6.

Stars are born primarily in the densest clumps that form as a result of turbulence within the interstellar medium. The disk velocity dispersion is predicted to be proportional to the square root of the supernova rate (Dib et al. 2006). So the mean gas density should depend on the square root of the star formation rate. The star formation rate is estimated from the fraction of material in the densest clumps or that above a critical density. (Padoan & Nordlund 2002; Elmegreen 2002; Krumholz & McKee 2005; Kravtsov 2003; Wada & Norman 2007). The fraction of the mass with a density, ρ, larger than a threshold, ρ_c

$$f_c = \frac{\int_{\rho_c}^{\infty} \rho p(\rho) d\rho}{\int_0^{\infty} \rho p(\rho) d\rho} \tag{13}$$

where the normalized probability density function

$$p(u) = (2\pi\Delta^2)^{-1/2} \exp\left(-0.5[\ln u - \ln u_0]^2/\Delta^2\right) \frac{d\ln u}{du}. \tag{14}$$

Here $u = \rho/\bar{\rho}$ is the density in units of the mean density and $\ln u_0$ is the mean of the normal distribution. The mean and dispersion of the normal distribution depend on the Mach number on the largest scale and are in the range 1-5 (Padoan & Nordlund 2002).

[1] While there is no theoretical basis for this distribution, R. Sutherland (personal communication, 2008) points out that it is a natural consequence of a turbulent cascade with multiplicative rather than additive random phases due to folding and stretching within the medium.

After integrating, we estimate $f_c \propto \mathrm{erfc}\left(\frac{2\ln u_{crit} - \Delta^2}{2^{3/2}\Delta}\right)$ (based on equation 20 by Krumholz & McKee 2005), where the critical density ratio $u_{crit} = \rho_c/\bar{\rho}$, we have used a complementary error function and assumed that the critical density ratio exceeds the mean by more than a few dispersion lengths Δ. In the large asymptotic limit this becomes $f_c \sim e^{-(\ln u_{crit}/\Delta)^2}$. A change in the density ratio u_{crit} leads to a change in the cloud fraction

$$S = \left|\frac{df_c}{du}\frac{u}{f_c}\right|_{u_{crit}} \sim \frac{2\ln u_{crit}}{\Delta^2}. \tag{15}$$

The above ratio, equivalent to the strength parameter defined in equation 6, tells us how large a change in the fraction of clouds above the critical density is caused by a fractional change in the mean density. The density ratio u_{crit} is estimated to be in the range of $10^4 - 10^6$ (Elmegreen 2002; Krumholz & McKee 2005). For $\Delta = 2.4$ (Elmegreen 2002; Padoan & Nordlund 2002) and $u_{crit} = 10^5$, the above fraction $S \sim 4$. We expect the condition strength $S \gtrsim 1$ for our model is satisfied but that the strength is also not extremely large. For delay times exceeding the gas consumption timescale by a moderate factor with $S \sim 4$ we would predict solutions with moderate amplitude oscillations (see Figure 2b).

The feedback strength estimate shown in equation 15 suggests that the feedback would be weaker at higher Mach number but stronger at lower mean density, if the critical density is similar in different environments. Stinson et al. (2007) found that oscillations were lower amplitude for larger simulated dwarf galaxies. Figure 2 showing the amplitude as a function of feedback strength and delay timescale implies that the feedback strength would be lower for the larger simulated dwarfs because they have longer delay times and because their mean gas density is higher.

Further examination of these simulations may test the hypothesis that equation 15 describes the feedback strength and is consistent with the relationship between oscillation amplitude and feedback strength predicted by the model. The above estimate for the feedback strength is indirect as we have used a steady-state star formation rate to estimate the cloud formation rate. Timescales displayed by simulations of the density evolution and molecular cloud formation (e.g., Glover & Mac Low 2007) might allow a better and more appropriate estimate for the feedback strength. The strength we estimate above was based on a local probability density distribution but when feedback delay is very long (such as suggested in the solar neighborhood) the cloud formation rate should be integrated azimuthally around the galaxy and across spiral arms.

4 SUMMARY AND CONCLUSION

It is now widely recognized that a detailed understanding of feedback and accretion processes is essential to progress in many fields of astrophysics and across the entire cosmological hierarchy, from galaxy clusters down to the scales of individual star forming regions. To progress, we will need improvements in analytic algorithms and computer power, as well as better conceptual tools for classifying complex behavior. Some processes may indeed be episodic or cyclic, while other instances may exhibit quasi-periodic cycles on the way to fully chaotic behavior. A deeper understanding requires that we should to some degree be able to distinguish between these very different dynamical manifestations.

Here we have introduced a simple differential equation model that captures some of the complexity exhibited by astrophysical star forming systems with feedback. We introduce a one dimensional DDE for the molecular cloud density that allows cloud formation to depend on the star formation rate but at a previous time. Thus current star formation only affects the cloud distribution at a future time, we denote the delay time. The feedback is negative, so in the absence of delay there are no cyclic solutions or instabilities and all solutions asymptotically approach a self-limiting value.

We illustrate that even when the feedback is negative a delay can cause cyclic or episodic behavior. The DDE captures phenomena exhibited by astrophysical simulations of this process, including periodic solutions in some cases but not in others. The DDE allows us to classify the solutions and predict when an astrophysical system is self-limiting or likely to exhibit periodic behavior based on timescales that are related to physical feedback and star formation processes.

We find that periodic behavior is likely when two conditions are met. First, the delay timescale must exceed the cloud consumption timescale. Secondly, the star formation must be effective at reducing the rate of formation at densities near the self-limiting or steady state value. This is equivalent to requiring strong feedback or to requiring that the timescale to approach equilibrium be larger than approximately twice the cloud consumption timescale. We find that the amplitude of the oscillations is sensitive to the feedback strength and to a lesser extent on the ratio of the delay time to the consumption timescale.

Previous studies found a condition for episodic star formation similar to our first one, that the delay time exceed a timescale representative of the non-delayed system (Scalo & Struck-Marcell 1986; Parravano 1996). However these works did not explore the sensitivity of the system to feedback strength, or relate the period and amplitude of oscillations to parameters describing the system. These systems used coupled DDEs of 2 or more variables and 6 or more free parameters. Our DDE is simpler than those previously adopted. To classify solutions we require only two dimensional estimates, x_* and t_{con}, that can be estimated from observations, and two dimensionless ratios, the dimensionless delay timescale $\bar{\tau}$ and feedback strength parameter, S, set by the feedback mechanism.

We focus on the molecular or self-gravitating cloud density in a galaxy as the most likely variable for the DDE. This allows recycling of gas over long periods of time as gas is recycled through clouds much faster than it is depleted by star formation. The consumption timescale is set by the lifetime of molecular clouds. When feedback delay times are longer than this timescale we predict episodic star formation events and with a period approximately 4 times the delay timescale.

At the present time, there are no compelling constraints on either the feedback strength or the delay time, i.e. the two key parameters of the DDE model. Thus, it is difficult to apply the model rigorously although we suggest avenues for further exploration.

There is more than one candidate for the delay time and associated feedback mechanisms, in particular, the timescale for supernovae to contribute to turbulence, the timescale for spiral density waves to evolve, and the timescale for material sent into the halo to return to interact with the disk. We associate these three candidate delay mechanisms with possible explanations for episodic star formation events in galaxy centers (on 10 Myr timescales), the solar neighborhood (on Gyr timescales) and dwarf galaxies (on 100 Myr timescales), respectively. Using a log normal density distribution we estimate that feedback is likely to be strong enough that the second condition for episodic solutions can be satisfied.

Lacking currently are simulations and observational programs that constrain the timescales and strengths of possible feedback mechanisms and their functional form. Observational studies relating star burst amplitudes and timescales, dynamical times, mean gas densities, turbulence and deviations from star formation laws can provide evidence for feedback induced variations in star formation rates and could be used to quantify feedback strength and form. Other forms for the feedback function could be used, such as that of the Mackey-Glass model which can exhibit chaotic behavior (Glass & Mackey 1988). By modeling with additional variables it may be possible to model these systems without delays.

We thank Adam Frank, Eric Blackman, Jason Nordhaus and Richard Edgar for helpful discussions. Support for this work was in part provided by by NASA through awards issued by JPL/Caltech, National Science Foundation grants AST-0406823 & PHY-0552695, the National Aeronautics and Space Administration under Grant No.~NNG04GM12G issued through the Origins of Solar Systems Program, and HST-AR-10972 to the Space Telescope Science Institute. JBH is funded by a Federation Fellowship from the Australian Research Council.

APPENDIX A: APPLICATION OF LINEARIZED OSCILLATION THEORY

We would like to know when the non-linear DDE given in equation 7 exhibits oscillating solutions. For $\alpha = 1$ this differential equation is the same as that of the Lasota-Wazewska model. In this appendix we search for a more general criterion for oscillation that allows non-unity values of the index α. This is desirable because star formation laws have non-unity values for this index.

Non-linear DDEs can have oscillating solutions when an associated delay linear equation does. The non-linear DDE

$$\dot{x} + \sum_{i=1}^{n} p_i f_i(x(t-\tau_i)) = 0 \tag{A1}$$

can be associated with the linearized equation

$$\dot{y} + \sum_{i=1}^{n} p_i y(t-\tau_i) = 0, \tag{A2}$$

(Kulenovic & Ladas 1987; Györi & Ladas 1991). Here $p_i > 0$, $\tau_i \geqslant 0$, and the functions f_i are well behaved continuous functions. Given additional conditions on the functions, f_i, Kulenovic & Ladas (1987); Györi & Ladas (1991) proved that every solution of the non-linear equation oscillates if and only if every solution of the associated linearized equation does. One condition is the requirement that

$$\lim_{u\to 0} \frac{f(u)}{u} = 1. \tag{A3}$$

By manipulating equation 7 and requiring the above condition, we find an associated linearized equation that is similar to that used by Kulenovic & Ladas (1987); Györi & Ladas (1991) to establish when solutions oscillate for the Lasota-Wazewska model. This associated linearized equation is in the form

$$\dot{x}(t) + p_1 x(t) + p_2 x(t-\tau) = 0. \tag{A4}$$

A necessary and sufficient condition for the oscillation of all solutions of this linear DDE is

$$p_2 \tau e^{(p_1\tau+1)} > 1, \tag{A5}$$

as proved by Györi & Ladas (1991) in section 2.2. Once we find the coefficients p_1 and p_2 of the associated linearized equation, we can use the above oscillation criterion to establish when oscillating solutions exist for the original non-linear DDE.

We wish to find an associated linearized equation for the differential equation 7 restated here

$$\dot{x}(t) = Ae^{x(t-\tau)/C} - Bx(t)^{\alpha}, \tag{A6}$$

with equilibrium solution, x_* given by equation 3. The change of variables

$$x(t) = x_* + Cu(t) \tag{A7}$$

leads to the delay equation

$$\dot{u}(t) + \frac{Bx_*^{\alpha}}{C}\left[\left(1+\frac{Cu(t)}{x_*}\right)^{\alpha} - 1\right] + \frac{Bx_*^{\alpha}}{C}\left(1-e^{u(t-\tau)}\right) = 0. \tag{A8}$$

This can be written in the form of the linearized equation A2 with

$$\begin{aligned} p_1 &= B\alpha x_*^{\alpha-1} \\ p_2 &= \frac{Bx_*^{\alpha}}{C} \\ f_2(u) &= \frac{x_*}{\alpha C}\left[\left(1+\frac{Cu}{x_*^{\alpha}}\right) - 1\right] \\ f_2(u) &= 1-e^{u} \end{aligned} \tag{A9}$$

where the functions f_1, f_2 satisfy the condition shown in equation A3. The linearized equation is then in the form of equation A4. Inserting p_1 and p_2 into equation A5 we find that the requirement for oscillating solutions is

$$\frac{Bx_*^{\alpha}\tau}{C} e^{(\alpha Bx_*^{\alpha-1}\tau+1)} > 1. \tag{A10}$$

This is dimensionally correct and reduces to equation 10 for the oscillation criterion for the Lasota-Wazewska model when $\alpha = 1$, as expected.

REFERENCES

Andersen, R.-P., & Burkert, A. 2000, ApJ, 531, 296-311.
Blitz, L., & Rosolowsky, E. 2006, ApJ, 650, 933-944.

Blitz, L., Fukui, Y., Kawamura, A., Leroy, A., Mizuno, N., & Rosolowsky, E. 2007, Protostars and Planets V, B. Reipurth, D. Jewitt, and K. Keil (eds.), University of Arizona Press, Tucson, 951 pp., 2007., p.81-96
Böker, T., Laine, S., van der Marel, R. P., Sarzi, M., Rix, H.-W., Ho, L.C., & Shields, J. C. 2002, AJ, 123, 1389-1410.
Bland-Hawthorn, & J., Cohen, M. 2003, ApJ, 582, 246-256.
Booth, C. M., Theuns, T., & Okamoto, T. 2007, MNRAS, 376, 1588-1610.
Cecil, G.N., Bland-Hawthorn, J., Veilleux, S. & Filippenko, A.V. 2001, ApJ, 555, 338-355.
Christopher, M. H., Scoville, N. Z., Stolovy, S. R., & Yun, M. S. 2005, ApJ, 622, 346-365.
Ciotti, L., & Ostriker, J. P. 2007, ApJ, 665, 1038-1056.
Clarke, C., & Gittins, D. 2006, MNRAS, 371, 530
Coté, P., Piatek, S., Ferrarese, L., Jordan, A., Merritt, D., Peng, E. W., Hasegan, M., Blakeslee, J.P., Mei, S., West, M.J., Milosavljevic, M., & Tonry, J. L. 2006, ApJS, 165, 57-94.
Cowie, L. L., Ostriker, J. P., & Stark, A. A. 1978, ApJ, 226, 1041-1062.
de Avillez, M. A., & Breitschwerdt, D. 2007, ApJ, 665, L35-L38.
de Avillez, M. A., & Breitschwerdt, D. 2004, A&A, 425, 899-911.
Dellenbusch, K. E., Gallagher, J. S., III, Knezek, P. M., & Noble, A. G. 2008, AJ, ,135, 326-332.
Dib, S., Bell, E., & Burkert, A. 2006, ApJ, 638, 797-810.
Dohm-Palmer, R. C., Skillman, E. D., Mateo, M., Saha, A., Dolphin, A., Tolstoy, E., Gallagher, J.S., Cole, A. A. 2002, AJ, 123, 813-831
Dolphin, A. E., Saha, A., Skillman, E. D., Dohm-Palmer, R. C., Tolstoy, E., Cole, A. A., Gallagher, J. S., Hoessel, J. G., Mateo, M. 2003, AJ, 126, 187-196
Dong, S., Lin, D. N. C., & Murray, S. D. 2003, ApJ, 596, 930-943.
Glass, L. and Mackey, M. C. 1988, From Clocks to Chaos, The Rhythms of Life, (Princeton University Press, Princeton, NJ.
Elmegreen, B. G. 2002, ApJ, 577, 206
Elmegreen, B. G. 2007, ApJ, 668, 1064-1082.
Gao, Y., & Solomon, P. M. 2004, ApJ, 606, 271-290.
Glazebrook, K., Blake, C., Economou, F., Lilly, S., & Colless, M. 1999, MNRAS, 306, 843-856.
Gurney, W. S., Blythe, S. P., & Nisbet R. M. 1980, Nature, 287, 17-21.
Györi, I. & Ladas, G. 1991, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford
Joung, M. K. R., & Mac Low, M.-M. 2006, ApJ, 653, 1266-1279.
Glover, S. C. O., & Mac Low, M.-M. 2007, ApJ, 659, 1317
Kamaya, H. 2005, A&A, 434, 163
Kennicutt, R. C., Jr. 1998, ApJ, 498, 541-552.
Kim, W.-T., Ostriker, E.C., & Stone, J. M. 2003, ApJ, 599, 1157
Koda, J., Okuda, T., Nakanishi, K., Kohno, K., Ishizuki, S., Kuno, N., & Okumura, S. K. 2005, A&A, 431, 887-891.
Kravtsov, A. V. 2003, ApJ, 590, L1-4.
Krumholz, M. R., & McKee, C. F. 2005, ApJ, 630, 250-268.
Krumholz, M. R., Matzner, C. D., & McKee, C. F. 2006, ApJ, 653, 361-382.
Kulenovic, M. R. S., & Ladas, G. 1987, Bulletin of Mathematical Biology, 49, 615-627.
Kulenovic, M. R. S., Ladas, G., & Sficas, Y. G. 1989, Comput. Math. Appl. 18, no. 10-11, 925-928.
Li, Y., Mac Low, M.-M., & Klessen, R. S. 2006, ApJ, 639, 879-896.
McKee, C., & Ostriker, E. C. 2007, ARA&A, 45, 565-687.
Monaco, P. 2004, MNRAS 352, 181-204.
Ostriker, J. P., Weaver, R., Yahil, A., & McCray, R. 1976, ApJ, 208, L61-L65.
Padoan, P., & Nordlund, A. 2002, ApJ, 576, 870
Padoan, P., Nordlund, A., Kritsuk, A.G., Norman, M.L. & Li, P.S. 2007, ApJ, 661, 972-981.
Parravano, A. 1996, ApJ, 462, 594
Pelupessy, F. I., van der Werf, P. P., & Icke, V. 2004, A&A, 422, 55-64 .
Piontek, R. A., Ostriker, E.C. 2007, ApJ, 663, 183
Pumir, A. 1994, Phys. Fluids, 6(12), 3974
Quillen, A. C., Thorndike, S.L., Cunningham, A., Frank, A., Gutermuth, R. A., Blackman, E. G., Pipher, J. L., & Ridge, N. 2005, ApJ, 632, 941
Quillen, A. C., Bland-Hawthorn, J., Brookes, M. H., Werner, M. W., Smith, J. D., Stern, D., Keene, J., & Lawrence, C. R. 2006, ApJ, 641, L29-L32.
Robertson, B., & Kravtsov, A. 2008, ApJ, in press, 2007arXiv0710.2102
Rocha-Pinto, H. J., Scalo, J., Maciel, W. J., & Flynn, C. 2000, ApJ, 531, L115-118.
Scalo, J. M., Struck-Marcell, C. 1986, ApJ, 301, 77
Schaye, J., & Dalla Vecchia, C. 2007 MNRAS in press
Schmidt, M. 1959, ApJ, 129, 243-258.
Skillman, E. D. 2005, NewAR, 49, 453-460.
Smecker-Hane, T. A., Stetson, P. B., Hesser, J. E., & Lehnert, M. D. 1994, AJ, 108, 507-513.
Stinson, G. S., Dalcanton, J. J., Quinn, T., Kaufmann, T., & Wadsley, J. 2007, ApJ, 667, 170-175.
Struck, C., & Smith, D. C. 1999, ApJ, 527, 673
Walcher, C. J., Böker, T., Charlot, S., Ho, L. C., Rix, H.-W., Rossa, J., Shields, J. C., & van der Marel, R. P. 2006, ApJ, 649, 692-708.
Tosi, M.., Greggio. L., Marconi., G., & Focardi, P. 1991, AJ, 102, 951-974.
Spiral instabilities provoked by accretion and star formation
Sellwood, J. A., & Carlberg, R. G. 1984, ApJ, 282, 61
Silk, J. 2001, MNRAS, 24, 313-318.
Slyz, A. D., Devriendt, J. E. G., Bryan, G., & Silk, J. 2005, MNRAS, 356, 737-752
van Zee, L. 2001, AJ, 121, 2003-2019
Veilleux, S., Cecil, G., Bland-Hawthorn, J., Tully, R. B., Filippenko, A. V., & Sargent, W. L. W. 1994, ApJ, 433, 48-64.
Wada, K., & Norman, C. A. 2007, ApJ, 660, 276-287.
Warhaft, Z. 2000, Annual Review of Fluid Mechanics, 32, 203
Wazewska-Czyzewska, M., & Lasota, A. 1988, Annals of the Polish Mathematical Society, Series III, Applied Mathematics, 17, 23-40.
Vorobyov, E. I., Theis, Ch. 2006, MNRAS, 273, 197
Wu, J., Evans, N. J., II, Gao, Y., Solomon, P. M., Shirley, Y. L., & Vanden Bout, P. A. 2005, ApJ, 635, L173-176.

Young, L. M., Skillman, E. D., Weisz, D. R., & Dolphin, A. E. 2007, ApJ, 659, 331-338.